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TWO SHARP INEQUALITIES FOR TRIGONOMETRIC AND HYPERBOLIC FUNCTIONS

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Abstract. We determine the best positive constants p and q such that

$$(\sinh x/x)^p < x/\sin x < (\sinh x/x)^q.$$

Some applications for Wilker's type inequalities are given.

1. Introduction

The trigonometric and hyperbolic inequalities have been in the last years in the focus of many researchers. For many results and a long list of references we quote the papers [1], [2]. In paper [1], the following interesting inequalities have been proved:

THEOREM 1.1. *For any* $x \in \left(0, \frac{\pi}{2}\right)$,

$$\frac{\sinh x}{x} < \frac{x}{\sin x} < \left(\frac{\sinh x}{x}\right)^2. \tag{1.1}$$

The aim of this paper is to find the best possible form of inequalities (1.1) in the sense of determination of greatest $p > 0$ and least $q > 0$ such that

$$\left(\frac{\sinh x}{x}\right)^p < \frac{x}{\sin x} < \left(\frac{\sinh x}{x}\right)^q. \tag{1.2}$$

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2. Main results

First we state two auxiliary results:

LEMMA 2.1. *For any* $x > 0$ *one has*

$$\ln \frac{\sinh x}{x} > \frac{x \cosh x - \sinh x}{2 \sinh x}. \tag{2.1}$$

Proof. Let

$$L(a,b) = \frac{a-b}{\ln a - \ln b}, \quad G(a,b) = \sqrt{ab}, \quad I(a,b) = \frac{1}{e}(b^b/a^a)^{1/(b-a)}$$

for $a \neq b$ positive numbers. These are called as the logarithmic, geometric and identric means of a and b (see e.g. [3] for L, I and related means). The inequality

$$L^2 > G \cdot I \tag{2.2}$$

is due to H. Alzer [4]. Apply (2.2) for $a = e^x$, $b = e^{-x}$. Then it is easy to see that

$$L = L(a,b) = \frac{\sinh x}{x}, \quad G = G(a,b) = 1, \quad I = I(a,b) = e^{x \coth x - 1}.$$

After these substitutions, inequality (2.1) follows at once. $\square$

LEMMA 2.2. *For any* $x \in \left(0, \frac{\pi}{2}\right)$ *one has*

$$\ln \frac{x}{\sin x} < \frac{\sin x - x \cos x}{2 \sin x}. \tag{2.3}$$

Proof. Let $a(x) = \frac{\sin x - x \cos x}{2 \sin x} - \ln \frac{x}{\sin x}$, $0 < x < \frac{\pi}{2}$.
A simple computation gives

$$a'(x) = \frac{x^2 + x \sin x \cos x - 2 \sin^2 x}{2x \sin^2 x} > 0,$$

if $2 \sin^2 x < x^2 + x \sin x \cos x$.

As this may be written also as

$$2t^2 - t \cos x - 1 < 0,$$

where $t = \frac{\sin x}{x}$, by resolving this inequality of second variable in t, as $t > 0$, this is equivalent with

$$\frac{\sin x}{x} < \frac{\cos x + \sqrt{\cos^2 x + 8}}{4}. \tag{2.4}$$

But this follows by the "Cusa-Huygens inequality" (see [2])

$$\frac{\sin x}{x} < \frac{2+\cos x}{3} \tag{2.5}$$

as $\frac{2+\cos x}{3} < \frac{\cos x + \sqrt{\cos^2 x + 8}}{4}$, as a simple verification shows. Therefore, the function $a(x)$ is strictly increasing, and as $\lim_{x\to 0_+} a(x) = 0$, relation (2.3) follows. □

REMARK 2.1. Relation (2.1) could have been proved via auxiliary functions, too, but we opted for a proof with means, as this was the first method of discovery of (2.1) by the author.

We note that, by considering Seiffert's mean P (see [5]) defined by

$$P(a,b) = \frac{a-b}{2\arcsin\left(\dfrac{a-b}{a+b}\right)}, \quad a \neq b$$

and using the substitution $a = A(1+\sin x)$, $b = A(1-\sin x)$, where $A = \frac{a+b}{2}$ and $x = \arcsin\frac{a-b}{a+b}$ for $0 < a < b$; inequality (2.3) could be written as an inequality for means:

$$P^2 > A\cdot X \tag{2.6}$$

where $X = Ae^{G/P-1}$, with $X = X(a,b)$ a new mean (introduced for the first time here).

Indeed, (2.6) follows by remarking that $P(a,b) = A\cdot\frac{\sin x}{x}$, etc. and introducing in (2.3).

We state the main result of this paper:

THEOREM 2.1. *The function*

$$h(x) = \frac{\ln\dfrac{x}{\sin x}}{\ln\dfrac{\sinh x}{x}}, \quad x \in \left(0, \frac{\pi}{2}\right) \tag{2.7}$$

is strictly increasing.

As a corollary, the best constants in inequality (1.2) are

$$p = 1 \quad \textit{and} \quad q = \frac{\ln(\pi/2)}{\ln((\sinh\pi/2)/\pi/2)} \approx 1.18\ldots$$

Proof. Put $f(x) = \ln\frac{x}{\sin x}$, $g(x) = \ln\frac{\sinh x}{x}$. Simple computations give

$$f'(x) = \frac{\sin x - x\cos x}{x\sin x}, \quad g'(x) = \frac{x\cosh x - \sinh x}{x\sinh x}$$

and

$$xg^2(x)h'(x) = \left(\frac{\sin x - x\cos x}{\sin x}\right)\ln\frac{\sinh x}{x} - \left(\frac{x\cosh x - \sinh x}{x\sinh x}\right)\ln\frac{x}{\sin x}. \qquad (2.8)$$

By (2.1) and (2.3) the right side of (2.8) is strictly positive, implying $h'(x) > 0$. Thus h is strictly increasing. This implies

$$\lim_{x\to 0_+} h(x) < h(x) < h\left(\frac{\pi}{2}\right) = \frac{\ln \pi/2}{\ln\left(\frac{\sin\pi/2}{\pi/2}\right)}.$$

A simple computation shows $\lim_{x\to 0_+} h(x) = 1$, thus

$$p = 1 < h(x) < \frac{\ln\pi/2}{\ln\frac{(\sinh\pi/2)}{\pi/2}} = q. \qquad (2.9)$$

Clearly, the inequality (1.2) may be written also as

$$p < h(x) < q,$$

so (2.9) offer the best possible constants p and q. □

REMARK 2.2. A computer computation shows $q \approx 1.18\ldots < 2$, thus inequality (1.1) is improved on the right-hand side.

3. An application

A famous inequality of Wilker (see [2] for many connections with other inequalities) states that

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2, \quad 0 < x < \frac{\pi}{2}. \qquad (3.1)$$

In what follows, we will obtain a Wilker type inequality for the functions $\sin x$ and $\sinh x$:

THEOREM 3.1. *Let* $q = \dfrac{\ln\pi/2}{\ln\frac{(\sinh\pi/2)}{\pi/2}} \approx 1.18\ldots$

Then

$$\frac{\sin x}{x} + q\left(\frac{\sinh x}{x}\right) > q + 1. \qquad (3.2)$$

Proof. Apply the weighted arithmetic-geometric inequality

$$\lambda a + (1-\lambda)b > a^{\lambda} b^{1-\lambda}$$

for $a>0,\ b>0,\ a\neq b,\ \lambda\in(0,1)$. Put $\lambda=\dfrac{1}{q+1}$ and $a=\dfrac{\sin x}{x}$, $b=\dfrac{\sinh x}{x}$. Then, as the right side of (1.2) may be written as $ab^q>1$, and as

$$a+qb>(ab^q)^{1/(q+1)}(q+1)>q+1. \qquad \square$$

The following result will be a Wilker type inequality for the functions $\dfrac{\sin x}{x}$ and $\dfrac{\sinh x}{x}$:

THEOREM 3.2.

$$\left(\frac{\sinh x}{x}\right)^q+\frac{\sin x}{x}>2 \text{ for } x\in\left(0,\frac{\pi}{2}\right), \tag{3.3}$$

where q is the same as in Theorem 3.1.

Proof. The Bernoulli inequality states that (see e.g. [6])

$$(1+t)^\alpha>1+\alpha t, \text{ for } t>0,\ \alpha>0. \tag{3.4}$$

Put $t=\dfrac{\sinh x}{x}-1>0$, $\alpha=q$ in (3.4). Then this inequality, combined with inequality (3.2) implies (3.3). $\square$

REMARK 3.1. As $\left(\dfrac{\sinh x}{x}\right)^2>\left(\dfrac{\sinh x}{x}\right)^q$, we get from (3.3) a much weaker, but more familiar form of Wilker's inequality:

$$\left(\frac{\sinh x}{x}\right)^2+\frac{\sin x}{x}>2, \quad x\in\left(0,\frac{\pi}{2}\right). \tag{3.5}$$

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