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BEST POSSIBLE INEQUALITIES AMONG HARMONIC, GEOMETRIC, LOGARITHMIC AND SEIFFERT MEANS

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Abstract. In this paper, we find the greatest value α and the least values β, p, q and r in $(0,1/2)$ such that the inequalities $L(\alpha a+(1-\alpha)b,\alpha b+(1-\alpha)a)<P(a,b)<L(\beta a+(1-\beta)b,\beta b+(1-\beta)a)$, $H(pa+(1-p)b,pb+(1-p)a)>G(a,b)$, $H(qa+(1-q)b,qb+(1-q)a)>L(a,b)$, and $G(ra+(1-r)b,rb+(1-r)a)>L(a,b)$ hold for all $a,b>0$ with $a\neq b$. Here, $H(a,b)$, $G(a,b)$, $L(a,b)$ and $P(a,b)$ denote the harmonic, geometric, logarithmic and Seiffert means of two positive numbers a and b, respectively.

1. Introduction

For $a,b>0$ with $a\neq b$, the harmonic mean $H(a,b)$, geometric mean $G(a,b)$, logarithmic mean $L(a,b)$ and Seiffert mean $P(a,b)$ are defined by

$$H(a,b)=\frac{2ab}{a+b},\quad G(a,b)=\sqrt{ab},\quad L(a,b)=\frac{a-b}{\log a-\log b} \tag{1.1}$$

and

$$P(a,b)=\frac{a-b}{4\arctan\sqrt{a/b}-\pi}, \tag{1.2}$$

respectively.

Recently, the inequalities for means have been the subject of intensive research. In particular, many remarkable inequalities for the harmonic, geometric, logarithmic and Seiffert means can be found in the literature [1–18]. Let $I(a,b)=1/e(b^b/a^a)^{1/(b-a)}$, $A(a,b)=(a+b)/2$ and $M_p(a,b)=[(a^p+b^p)/2]^{1/p}(p\neq 0)$ and $M_0(a,b)=\sqrt{ab}$ be the identric, arithmetic and $p-$th power means of two positive numbers a and b with $a\neq b$, respectively. Then it is well-known that

$$\begin{aligned}\min\{a,b\}<H(a,b)=M_{-1}(a,b)<G(a,b)=M_0(a,b)<L(a,b)\\ <P(a,b)<I(a,b)<A(a,b)=M_1(a,b)<\max\{a,b\}\end{aligned}$$

for all $a,b>0$ with $a\neq b$.

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Lin [10] proved that the double inequality

$$M_0(a,b) < L(a,b) < M_{1/3}(a,b)$$

holds for all $a,b>0$ with $a \neq b$.

In [9], the authors presented that

$$L(a,b) < \frac{1}{3}A(a,b) + \frac{2}{3}G(a,b)$$

for all $a,b>0$ with $a \neq b$.

The following bounds for the Seiffert mean $P(a,b)$ were established in [5, 7, 8, 14, 16]:

$$P(a,b) > \frac{2}{\pi}A(a,b),$$

$$M_{\log 2/\log \pi}(a,b) < P(a,b) < M_{2/3}(a,b),$$

$$A^{2/3}(a,b)G^{1/3}(a,b) < P(a,b) < \frac{2}{3}A(a,b) + \frac{1}{3}G(a,b)$$

and

$$\frac{2}{\pi}A(a,b) + \frac{\pi-2}{\pi}H(a,b) < P(a,b) < \frac{5}{6}A(a,b) + \frac{1}{6}H(a,b)$$

for all $a,b>0$ with $a \neq b$.

For fixed $a,b>0$ with $a \neq b$ and $x \in [0,1/2]$, let

$$f_1(x) = H(xa+(1-x)b, xb+(1-x)a),$$

$$f_2(x) = G(xa+(1-x)b, xb+(1-x)a)$$

and

$$f_3(x) = L(xa+(1-x)b, xb+(1-x)a).$$

Then it is not difficult to verify that $f_1(x)$, $f_2(x)$ and $f_3(x)$ are continuous and strictly increasing from $[0,1/2]$ onto $[H(a,b),A(a,b)]$, $[G(a,b),A(a,b)]$ and $[L(a,b),A(a,b)]$, respectively.

Therefore, it is natural to ask what are the greatest value α and the least values β, p, q, and r in $(0,1/2)$ such that the inequalities $L(\alpha a+(1-\alpha)b, \alpha b+(1-\alpha)a) < P(a,b) < L(\beta a+(1-\beta)b, \beta b+(1-\beta)a)$, $H(pa+(1-p)b, pb+(1-p)a) > G(a,b)$, $H(qa+(1-q)b, qb+(1-q)a) > L(a,b)$, and $G(ra+(1-r)b, rb+(1-r)a) > L(a,b)$ hold for all $a,b>0$ with $a \neq b$. The main purpose of this paper is to answer these questions.

2. Main Results

LEMMA 2.1. *The equation*

$$\log[(1-\lambda)/\lambda] = \pi(1-2\lambda)$$

has a unique solution $\lambda = \lambda_0$ *in the interval* $(0,1/2)$ *and* $\lambda_0 < (2-\sqrt{2})/4$.

Proof. Let $f(\lambda) = \log[(1-\lambda)/\lambda]$ and $g(\lambda) = \pi(1-2\lambda)$. Then simple computations lead to

$$f(1/2) = g(1/2) = 0, \tag{2.1}$$

$$f(0^+) = +\infty, \tag{2.2}$$

$$f'(\lambda) = -\frac{1}{\lambda(1-\lambda)} \tag{2.3}$$

and

$$f''(\lambda) = \frac{1-2\lambda}{\lambda^2(1-\lambda)^2}. \tag{2.4}$$

It follows from equations (2.3) and (2.4) that $f(\lambda)$ is strictly decreasing and convex in $(0,1/2]$. Note that

$$f\left(\frac{2-\sqrt{2}}{4}\right) = 2\log(2+\sqrt{2}) - \log 2 < \frac{\sqrt{2}}{2}\pi = g\left(\frac{2-\sqrt{2}}{4}\right). \tag{2.5}$$

Therefore, Lemma 2.1 follows from (2.1), (2.2) and (2.5) together with the monotonicity and convexity of $f(\lambda)$ in $(0,1/2]$. □

THEOREM 2.2. *If* $\alpha,\beta \in (0,1/2)$, *then the double inequality*

$$L(\alpha a+(1-\alpha)b, \alpha b+(1-\alpha)a) < P(a,b) < L(\beta a+(1-\beta)b, \beta b+(1-\beta)a)$$

holds for all $a,b > 0$ *with* $a \neq b$ *if and only if* $\alpha \leqslant \lambda_0$ *and* $\beta \geqslant (2-\sqrt{2})/4$, *where* λ_0 *is defined as in Lemma 2.1.*

Proof. Let $\mu = (2-\sqrt{2})/4$. We first prove that inequalities

$$P(a,b) > L(\lambda_0 a+(1-\lambda_0)b, \lambda_0 b+(1-\lambda_0)a) \tag{2.6}$$

and

$$P(a,b) < L(\mu a+(1-\mu)b, \mu b+(1-\mu)a) \tag{2.7}$$

hold for all $a,b > 0$ with $a \neq b$.

Without loss of generality, we assume that $a > b$. Let $t = \sqrt{a/b} > 1$ and $p \in (0,1/2)$, then from (1.1) and (1.2) one has

$$\begin{aligned} &L(pa+(1-p)b, pb+(1-p)a) - P(a,b) \\ =&\frac{b(t^2-1)}{(4\arctan t-\pi)[\log((1-p)t^2+p) - \log(pt^2+(1-p))]}h(t), \end{aligned} \tag{2.8}$$

where

$$h(t) = (1-2p)(4\arctan t - \pi) - \log[(1-p)t^2 + p] + \log[pt^2 + (1-p)]. \tag{2.9}$$

Simple computations lead to

$$h(1) = 0, \tag{2.10}$$

$$\lim_{t\to+\infty} h(t) = (1-2p)\pi - \log\frac{1-p}{p}, \tag{2.11}$$

$$h'(t) = \frac{2(1-2p)t(t-1)^2}{(t^2+1)[pt^2+(1-p)][(1-p)t^2+p]} h_1(t), \tag{2.12}$$

where

$$h_1(t) = 2p(1-p)(t+\frac{1}{t}) + 4p(1-p) - 1. \tag{2.13}$$

We clearly see that $h_1(t)$ is strictly increasing in $(1,\infty)$. Note that

$$h_1(1) = -8p^2 + 8p - 1, \tag{2.14}$$

$$\lim_{t\to+\infty} h_1(t) = +\infty. \tag{2.15}$$

We divide the proof into two cases.

Case A. $p = \lambda_0$. Then from equations (2.11) and (2.14) together with Lemma 2.1 we have

$$\lim_{t\to+\infty} h(t) = (1-2p)\pi - \log\frac{1-p}{p} = 0 \tag{2.16}$$

and

$$h_1(1) = -8p^2 + 8p - 1 < 0. \tag{2.17}$$

It follows from (2.12), (2.15) and (2.17) together with the monotonicity of $h_1(t)$ that there exists $t_0 > 1$, such that $h(t)$ is strictly decreasing in $(1,t_0)$ and strictly increasing in (t_0,∞).

Therefore, inequality (2.6) follows from (2.8)-(2.10) and (2.16) together with the piecewise monotonicity of $h(t)$.

Case B. $p = \mu = (2-\sqrt{2})/4$. Then $h_1(1) = -8p^2 + 8p - 1 = 0$, and $h_1(t) > 0$ for $t \in (1,\infty)$. Thus inequality (2.7) follows from (2.8)-(2.10) and (2.12).

Next, we prove that λ_0 is the best possible parameter such that inequality (2.6) holds for all $a,b > 0$ with $a \neq b$. In fact, if $\lambda_0 < p < 1/2$, then from the proof of Lemma 2.1 and equation (2.11) we conclude that

$$\lim_{t\to+\infty} h(t) = (1-2p)\pi - \log\frac{1-p}{p} > 0. \tag{2.18}$$

Inequality (2.18) implies that there exists $T = T(p) > 1$ such that

$$h(t) > 0 \tag{2.19}$$

for $t \in (T,+\infty)$.

Therefore, $P(a,b) < L(pa+(1-p)b, pb+(1-p)a)$ for $a/b \in (T^2, +\infty)$ follows from equations (2.8) and (2.9) together with inequality (2.19).

Finally, we prove that $p = \mu = (2-\sqrt{2})/4$ is the best possible parameter such that inequality (2.7) holds for all $a,b > 0$ with $a \neq b$. In fact, if $0 < p < \mu = (2-\sqrt{2})/4$, then from equation (2.14) we get

$$h_1(1) = -8p^2 + 8p - 1 < 0. \tag{2.20}$$

Inequality (2.20) implies that there exists $\delta > 0$ such that

$$h_1(t) < 0 \tag{2.21}$$

for $t \in (1, 1+\delta)$.

Therefore, $P(a,b) > L(pa+(1-p)b, pb+(1-p)a)$ for $a/b \in (1, (1+\delta)^2)$ follows from equations (2.8)-(2.10) and (2.12) together with inequality (2.21). □

THEOREM 2.3. *If $p \in (0, 1/2)$, then inequality*

$$H(pa+(1-p)b, pb+(1-p)a) > G(a,b)$$

holds for all $a,b > 0$ with $a \neq b$ if and only if $p \geqslant (2-\sqrt{2})/4$.

Proof. Without loss of generality, we assume that $a > b$. Let $t = \sqrt{a/b} > 1$, then from (1.1) one has

$$H(pa+(1-p)b, pb+(1-p)a) - G(a,b) = \frac{bt(t-1)^2}{t^2+1} h_1(t), \tag{2.22}$$

where $h_1(t)$ is defined as in (2.13).

Therefore, Theorem 2.3 follows from (2.14), (2.15) and (2.22) together with the monotonicity of $h_1(t)$. □

REMARK 2.4. From (2.22) and the range of $h_1(t)$ we know that there exist $a,b > 0$ such that $H(pa+(1-p)b, pb+(1-p)a) > G(a,b)$ for any $p \in (0, 1/2)$.

THEOREM 2.5. *If $q \in (0, 1/2)$, then inequality*

$$H(qa+(1-q)b, qb+(1-q)a) > L(a,b) \tag{2.23}$$

holds for all $a,b > 0$ with $a \neq b$ if and only if $q \geqslant (3-\sqrt{3})/6$.

Proof. Without loss of generality, we assume that $a > b$. Let $t = a/b > 1$, then from (1.1) one has

$$H(qa+(1-q)b, qb+(1-q)a) - L(a,b) = \frac{2b[qt+(1-q)][(1-q)t+q]}{(t+1)\log t} g(t), \tag{2.24}$$

where

$$g(t) = \log t - \frac{t^2-1}{2[qt+(1-q)][(1-q)t+q]}. \tag{2.25}$$

Simple computations lead to

$$g(1)=0, \tag{2.26}$$

$$g'(t)=\frac{(t-1)^2}{2[qt+(1-q)]^2[(1-q)t+q]^2}g_1(t), \tag{2.27}$$

where

$$g_1(t)=2q^2(1-q)^2(t+\frac{1}{t})-4q^4+8q^3-10q^2+6q-1. \tag{2.28}$$

Clearly $g_1(t)$ is strictly increasing in $(1,\infty)$. Note that

$$g_1(1)=-6q^2+6q-1, \tag{2.29}$$

$$\lim_{t\to+\infty} g_1(t)=+\infty. \tag{2.30}$$

Making use of (2.25)-(2.30) and the monotonicity of $g_1(t)$ together with the similar argument as in the proof of Theorem 2.2 we know that $g(t)>0$ for all $t>1$ if and only if $q\geqslant(3-\sqrt{3})/6$. Then equation (2.24) leads to Theorem 2.5. □

REMARK 2.6. From (2.25) we clearly see that $\lim_{t\to+\infty} g(t)=+\infty$ for any $q\in(0,1/2)$. Therefore, there does not exist $q\in(0,(3-\sqrt{3})/6)$ such that inequality $H(qa+(1-q)b,qb+(1-q)a)<L(a,b)$ holds for all $a,b>0$ with $a\neq b$.

THEOREM 2.7. *If $r\in(0,1/2)$, then inequality*

$$G(ra+(1-r)b,rb+(1-r)a)>L(a,b) \tag{2.31}$$

holds for all $a,b>0$ with $a\neq b$ if and only if $r\geqslant(3-\sqrt{6})/6$.

Proof. Without loss of generality, we assume that $a>b$. Let $t=\log(a/b)>0$, then from (1.1) one has

$$[G(ra+(1-r)b,rb+(1-r)a)]^2-L^2(a,b)=\left(\frac{b}{t}\right)^2 J(t), \tag{2.32}$$

where

$$J(t)=[r(1-r)t^2-1]e^{2t}+[(2r^2-2r+1)t^2+2]e^t+r(1-r)t^2-1. \tag{2.33}$$

Making use of Taylor series expansion, (2.33) becomes

$$\begin{aligned} J(t)=&\sum_{n=3}^{\infty}\frac{(n+1)(n+2)[2^n r(1-r)+2r^2-2r+1]-2^{n+2}+2}{(n+2)!}t^{n+2}\\ &-(r^2-r+\frac{1}{12})t^4. \end{aligned} \tag{2.34}$$

We divide the proof into three cases.

Case 1. $r=(3-\sqrt{6})/6$. Then equation (2.34) reduces to

$$J(t)=\frac{1}{12}\sum_{n=6}^{\infty}\frac{2^n[(n+1)(n+2)-48]+10(n+1)(n+2)+24}{(n+2)!}t^{n+2}$$
$$+\frac{t^6}{240}+\frac{t^7}{240}>0. \tag{2.35}$$

Therefore, inequality (2.31) follows from (2.32) and (2.35).

Case 2. $(3-\sqrt{6})/6<r<1/2$. Then from Case 1 and the monotonicity of the function $f_2(x)=G(xa+(1-x)b,xb+(1-x)a)$ in $[0,1/2]$ we know that inequality (2.31) holds.

Case 3. $0<r<(3-\sqrt{6})/6$. Then $-(r^2-r+1/12)<0$, and (2.34) implies that there exists $\delta_1>0$ such that

$$J(t)<0 \tag{2.36}$$

for $t\in(0,\delta_1)$.

Therefore, $G(ra+(1-r)b,rb+(1-r)a)<L(a,b)$ for all $a,b>0$ with $a/b\in(1,e^{\delta_1})$ follows from (2.32) and (2.36). □

REMARK 2.8. If $r\in[0,(3-\sqrt{6})/6)$, then inequality $G(ra+(1-r)b,rb+(1-r)a)<L(a,b)$ holds for all $a,b>0$ with $a\neq b$ if and only if $r=0$. In fact, equation (2.33) implies that $\lim_{t\to+\infty}J(t)=+\infty$ for any $q\in(0,(3-\sqrt{6})/6)$.

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