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# BOUNDEDNESS AND CONTINUITY OF THE MILD SOLUTIONS OF SEMILINEAR STOCHASTIC FUNCTIONAL EVOLUTION EQUATIONS

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*Abstract.* In this paper, we study the boundedness and stability with respect to the parameters for the mild solutions of stochastic functional evolution equations in which the nonlinearity satisfies a monotone type condition. Our main tool is a version of the Itô-type inequality by means of which we can obtain an appropriate bound for $p$-th moment, $p \geqslant 2$, of the mild solutions.

## 1. Introduction

Over the last years, many authors attempted to extend the existence, uniqueness and stability results for classical ordinary [3] and functional differential equations [10] to equations in infinite dimensional spaces [1, 2, 4, 9, 16, 19, 20]. In particular, based on the general results on monotone nonlinear operator equations in Hilbert and reflexive Banach spaces, Browder [1, 2] proved the existence and uniqueness of generalized and also mild solutions for the initial value problem $x'(t) = f(t, x(t))$ with $x(0) = v$ on a real Hilbert space $H$, in the case $f : [0,T] \times H \to H$ is continuous, carries bounded sets of $[0,T] \times H$ into bounded subsets of $H$ and has a semi-monotone property. In [20], Vainberg dropped coercivity condition that was essential in the abstract machinery of Browder and assuming only that $f$ is demicontinuous and bounded, established the existence and uniqueness of the generalized and mild solutions. Note that the results become very important specifically when the evolution with respect to the time of the dynamical system modeled by an abstract differential equation, undergoes some sort of randomness. In this case, we should take also into account the measurability of the solutions. Appealing to a version of random Schauder's fixed point theorem, Jahanipur [14] established the existence, uniqueness and measurability of the generalized and mild solutions of functional evolution equations on a Hilbert space in which the nonlinearity is demicontinuous and satisfies a condition of monotone type.

One of the most important problems concerning the solution of a differential equation modeling the behaviour of a dynamic phenomenon is its continuous dependence on the initial data and the various parameters existing on the right hand side of the equation which are the mathematical counterparts of the quantities involved in the system.

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The boundedness, continuity and smoothness of the solution of a stochastic differential equation depending on a parameter, have been well-studied by several authors (see e.g. [7, 11]). Métivier [17] proved the continuity and smoothness of an $H$-valued stochastic differential equation of Lipschitz type with respect to parameters. In [8], Faris and Jona-Lasinio proved that the solution of the integral equation

$$X(t) = U(t,0)X_0 + \int_0^t U(t,s)f(s,X(s))ds + V(t)$$

is continuous function of $V$ in the special case when the generator of $U$ is $\frac{d^2}{dx^2}$ and $f(x) = -\lambda x^3 - \mu x$. Da Prato and Zabczyk [5] generalized this result to the case when $U$ is an analytic semigroup and $f$ satisfies a locally Lipschitz condition on a Banach space. Zangeneh [22] proved the continuity of the solution when all of $V$, $f$ and $X_0$ are varied. In this way, he could first extend the results in [5, 8] to a more general setting where $U$ is an evolution operator and $f$ satisfies the semi-monotone (which is weaker than Lipschitz) condition, a problem that was open after Faris and Jona-Lasinio [8], and then generalized the results obtained by Métivier [17], to the semilinear stochastic evolution equations with monotone nonlinearity.

In this paper, we study the continuous dependence of the mild solutions of semilinear stochastic functional evolution equations with respect to the parameters. Our novelty is that the nonlinear part of the equation is demicontinuous and satisfies a special kind of monotone condition instead of Lipschitz one. In addition to this, the linear part involves a time-dependent family of unbounded linear operators generating an exponentially bounded evolution operator. The existence, uniqueness and moment stability of the mild solutions of such a class of stochastic functional equations along with several interesting examples have been studied in [13, 14, 15]. A version of the Itô-type inequality established in [12] has an essential role in obtaining the continuity and boundedness of the mild solutions.

## 2. Preliminaries

In this section, we give some definitions and preliminaries including a number of notions from semigroup theory, Brownian motion and stochastic integration on Hilbert spaces. Let $H$ be a real separable Hilbert space with the norm and inner product denoted by $\|\,.\,\|$ and $\langle .\,,.\rangle$, respectively. A family $\{U(t,s):\ 0 \leqslant s \leqslant t \leqslant T\}$ of bounded linear operators on $H$ is said to be an *evolution operator* if

- $U(t,t) = I, \quad U(t,r)U(r,s) = U(t,s), \quad$ for $\quad 0 \leqslant s \leqslant r \leqslant t \leqslant T$, where $I$ is the identity operator;
- The mapping $(t,s) \mapsto U(t,s)$ is strongly continuous for $0 \leqslant s \leqslant t \leqslant T$.

Let $\{A(t):\ 0 \leqslant t \leqslant T\}$ be a family of closed densely defined linear operators on $H$ whose domain $D$ is independent of $t \in [0,T]$. We say that $U(t,s)$ is a *strong evolution operator* with generator $A(t)$ if the following hold:

**(a)** For all $s \leqslant t$ and for each $x \in D$,

$$U(t,s)x - x = \int_s^t U(t,r)A(r)x\,dr;$$

**(b)** Let $x \in D$ and $s \in [0,T]$. For all $t > s$, we have $U(t,s)D \subseteq D$ and

$$\int_s^t A(r)U(r,s)x\,dr = (U(t,s) - I)x.$$

The following are the relevant hypotheses concerning the family $\{A(t) : 0 \leqslant t \leqslant T\}$ in order to be the generator of an evolution operator.

HYPOTHESIS 2.1. *There exists a $\lambda \in \mathbb{R}$ such that*

**(a)** *For each $t \in [0,T]$, $A(t) - \lambda I$ is the generator of a strongly continuous contraction semigroup; that is, $A(t) - \lambda I$ is a closed maximal monotone operator with dense domain;*

**(b)** *For each $\mu > \lambda$, the operator-valued function $(-A(t) + \mu I)^{-1}$ is strongly continuously differentiable with respect to $t \in [0,T]$;*

**(c)** *$B(t,s) = A(t)[\mu I - A(s)]^{-1}$ is uniformly bounded in $(t,s)$ for $\mu > \lambda$ with a bound depending on $\mu$.*

It is turned out (see [1, 2, 16, 18, 19]) that under the above conditions, the family $U(t,s)$ is a strong evolution operator with generator $A(t)$ which is *exponentially bounded with parameter* $\lambda$ on $[0,T]$; i.e., $\|U(t,s)\| \leqslant e^{\lambda(t-s)}$ for all $0 \leqslant s \leqslant t \leqslant T$. These conditions apply to a large class of parabolic, hyperbolic and functional evolution equations (see e.g. [4]).

Let $K$ be another real separable Hilbert space. We use the same notations $\|\,.\,\|$ and $\langle .,. \rangle$ for the norm and inner product in $K$ as well as in $H$. Suppose that $(\Omega, \mathscr{F}, \mathscr{F}_t, P)$ is a complete stochastic basis with a right continuous filtration.

DEFINITION 2.2. A family of random linear functionals $\{W_t : t \geqslant 0\}$ on $K$ is called a cylindrical Brownian motion on $K$, if

**(a)** $W_0 = 0$ and $W_t(x)$ is $\mathscr{F}_t$-adapted for every $x \in K$;

**(b)** For every $x \in K$ such that $x \neq 0$, $W_t(x)/\|x\|$ is a one-dimensional Brownian motion.

For the properties of cylindrical Brownian motion and its relation to other definitions of Brownian motion in infinite dimensions, see [21].

DEFINITION 2.3. Let $\xi : [0,\infty) \to K$ be an $\mathscr{F}_t$-adapted, predictable process such that $E[\int_0^t \|\xi(s)\|^2 ds] < \infty$ for all $t \geqslant 0$. The stochastic integral of $\xi$ with respect to the cylindrical Brownian motion $\{W_t : t \geqslant 0\}$ is a real-valued continuous martingale given by

$$\int_0^t \langle \xi(s), dW_s \rangle = \sum_{n=1}^{\infty} \int_0^t \langle \xi(s), e_n \rangle \, dW_s(e_n),$$

where $\{e_n\}_{n=1}^{\infty}$ is a complete orthonormal basis of $K$.

Assume that $L_2(K,H)$ is the space of Hilbert-Schmidt operators from $K$ to $H$ with the familiar Hilbert-Schmidt norm $\|\,.\,\|_2$. Now, we define the $H$-valued stochastic integral for $L_2(K,H)$-valued processes.

DEFINITION 2.4. Let $\Phi : [0,\infty) \to L_2(K,H)$ be an $\mathscr{F}_t$-adapted, predictable process such that $E[\int_0^t \|\Phi(s)\|_2^2 \, ds] < \infty$ for all $t \geqslant 0$. The stochastic integral of $\Phi$ is an $H$-valued continuous martingale given by

$$\langle h, \int_0^t \Phi(s)\, dW_s \rangle = \int_0^t \langle \Phi^*(s)h, dW_s \rangle, \qquad \forall h \in H,$$

where $\Phi^*$ is the adjoint operator of $\Phi$.

For a fixed real $r > 0$, let $C_H = C(-r,0;H)$ be the Banach space of all continuous $H$-valued functions $\psi : [-r,0] \to H$ defined on the finite delay interval $[-r,0]$ with the usual sup-norm $\|\psi\|_{C_H} = \sup_{\theta \in [-r,0]} \|\psi(\theta)\|$. Also, whenever a problem of measurability is in concern, we will equip $C_H$ with the $\sigma$-algebra of its Borel sets. Given any $p \geqslant 2$, denote by $C^p_{\mathscr{F}_0}$ the space of all continuous processes $\phi : [-r,0] \times \Omega \to H$ such that $\phi(\theta\,,.)$ is $\mathscr{F}_0$-measurable for each $\theta \in [-r,0]$ and $E(\sup_{-r \leqslant \theta \leqslant 0} \|\phi(\theta)\|^p) < \infty$. To any adapted continuous stochastic process

$$X : [-r,T] \times \Omega \to H \quad \text{such that} \quad E(\sup_{-r \leqslant t \leqslant T} \|X(t)\|^p) < \infty,$$

(with $\mathscr{F}_t = \mathscr{F}_0$ for $t \in [-r,0]$) there corresponds a $C_H$-valued adapted process $X_t \in C^p_{\mathscr{F}_0}$ defined on $\theta \in [-r,0]$ by

$$X_t(\theta) = X(t+\theta), \quad t \in [0,T].$$

Consider on $H$ a semilinear stochastic functional evolution equation of the form

$$\begin{cases} dX(t) = [A(t)X(t) + f(t,X_t)]dt + g(t,X_t)dW_t, \; t \in [0,T] \\ X(\theta) = \phi(\theta), \qquad \theta \in [-r,0], \end{cases} \tag{2.1}$$

where the initial data $\phi \in C^p_{\mathscr{F}_0}$.

DEFINITION 2.5. An $H$-valued, $\mathscr{F}_t$-adapted predictable process $X(t)$, $t \in [-r,T]$, is called a mild solution of (2.1) if $X(t)$ satisfies the integral equation

$$X(t) = U(t,0)\phi(0) + \int_0^t U(t,s)f(s,X_s)\, ds + \int_0^t U(t,s)g(s,X_s)\, dW_s, \quad t \in [0,T], \tag{2.2}$$

with $X(\theta) = \phi(\theta)$ for $\theta \in [-r,0]$.

Let us consider a slightly more general integral equation

$$X(t) = U(t,0)\phi(0) + \int_0^t U(t,s)f(s,X_s)\,ds + \int_0^t U(t,s)g(s,X_s)\,dW_s + V(t), \qquad (2.3)$$

on the time interval $[0,T]$ where the given continuous process $V : [-r,T] \times \Omega \to H$ satisfies $V(\theta) = 0$ for all $\theta \in [-r,0]$. The following are the relevant hypotheses concerning $A, U, g$ and the nonlinear part $f$.

HYPOTHESIS 2.6. **(a)** *The function $f : [0,T] \times \Omega \times C_H \to H$ is jointly measurable;*

**(b)** *For each $t \in [0,T]$ and $\omega \in \Omega$, the mapping $\psi \mapsto f(t,\omega,\psi)$ is demicontinuous; i.e., whenever $\{\psi_n\}$ is a sequence which is strongly convergent to $\psi$ in $C_H$, then $f(t,\omega,\psi_n)$ converges weakly to $f(t,\omega,\psi)$ in $H$;*

**(c)** *There exists a nonnegative number $M$ such that for each $\omega \in \Omega$, the function $(t,\psi) \mapsto f(t,\omega,\psi)$ is semimonotone with parameter $M$. By this, we mean that*

$$\langle f(t,\omega,\psi_1) - f(t,\omega,\psi_2),\, \psi_1(0) - \psi_2(0)\rangle \leqslant M\|\psi_1(0) - \psi_2(0)\|^2,$$

*for all $t \in [0,T]$ and $\psi_1, \psi_2 \in C_H$;*

**(d)** *There exists a constant $C > 0$ such that*

$$\|f(t,\omega,\psi)\| \leqslant C(1 + \|\psi\|_{C_H}),$$

*for all $t \in [0,T]$, $\omega \in \Omega$ and $\psi \in C_H$;*

**(e)** *$g : [0,T] \times \Omega \times C_H \to L_2(K,H)$ is a predictable process on $H$ such that*

$$\|g(t,\omega,\psi_1) - g(t,\omega,\psi_2)\|_2 \leqslant C\|\psi_1 - \psi_2\|_{C_H},$$

*for all $t \in [0,T]$ and $\psi_1, \psi_2 \in C_H$;*

**(f)** *$A$ and $U$ satisfy Hypothesis* 2.1.

Now, we state the existence result of a unique continuous solution for (2.3). Recall that if $X : [-r,T] \times \Omega \to H$ is an $H$-valued stochastic process, then the notation $X^*(t), t \in [0,T]$, always means $\sup_{-r \leqslant s \leqslant t} \|X(s)\|$. Also we denote by $\|X\|_\infty$ the usual sup-norm of $X$ on $[-r,T]$.

THEOREM 2.7. ([15]) *Let $p \geqslant 2$. If $E[\sup_{0 \leqslant s \leqslant t} \|g(s,0)\|_2^p]$ and $E(V^*(t))^p$ are finite for all $t \in [0,T]$ and hypotheses* 2.6 *hold, then the integral equation* (2.3) *has a unique continuous adapted solution $X$ with $E(X^*(t))^p < \infty$ for all $t \in [0,T]$.*

Finally, we give the Itô-type inequality [12] which is our main tool in this paper to prove continuity with respect to the parameters and $p$-th mean boundedness of the mild solutions. Let $\{W_t : t \geqslant 0\}$ be the cylindrical Brownian motion with respect to

$(\Omega,\mathscr{F},\mathscr{F}_t,P)$. Assume that $p\geqslant 2$ and $f,g$ are two processes defined on $[0,T]$ with values in $H$ and $L_2(K,H)$, respectively and satisfy

$$\int_0^T E\|f(t)\|^p\,dt<\infty,\qquad \int_0^T E\|g(t)\|_2^p\,dt<\infty.$$

THEOREM 2.8. (Itô-type inequality) *Let $\xi$ be an $H$-valued, $\mathscr{F}_0$-measurable random variable. Suppose that $U$ and $A$ satisfy Hypothesis* 2.1. *If*

$$X(t)=U(t,0)\xi+\int_0^t U(t,s)f(s)\,ds+\int_0^t U(t,s)g(s)\,dW_s,\quad t\in[0,T],$$

*then*

$$\begin{aligned}\|X(t)\|^p\leqslant e^{p\lambda t}\|\xi\|^p&+p\int_0^t e^{p\lambda(t-s)}\|X(s)\|^{p-2}\langle X(s)\,,\,f(s)\rangle\,ds\\&+p\int_0^t e^{p\lambda(t-s)}\|X(s)\|^{p-2}\langle X(s)\,,\,g(s)dW_s\rangle\\&+\frac{p(p-1)}{2}\int_0^t e^{p\lambda(t-s)}\|X(s)\|^{p-2}\|g(s)\|_2^2\,ds,\end{aligned}$$

*for all $t\in[0,T]$.*

## 3. Boundedness of the mild solutions

In this section, we apply the Itô-type inequality of the previous section to prove a boundedness theorem for the $p$-th mean of the mild solutions for stochastic functional evolution equations under hypotheses 2.6. Two useful inequalities help us to obtain our main result in this section and we state them in the next two lemmas. The first is an elementary inequality due to Young and the second is a simple consequence of the Burkholder-Davis-Gundy inequality [6] and its proof can be found in [23].

LEMMA 3.1. *For all nonnegative real numbers $u$ and $v$ and all $0\leqslant\alpha\leqslant 1$, we have $u^{1-\alpha}v^{\alpha}\leqslant(1-\alpha)u+\alpha v$.*

LEMMA 3.2. *Let $X(t),t\geqslant 0$, be an $H$-valued continuous process. If $M(t)$ is an $H$-valued continuous martingale, then for any constant $K>0$ we have*

$$E\left(\sup_{0\leqslant\rho\leqslant t}\left|\int_0^\rho\langle X(s)\,,\,dM(s)\rangle\right|\right)\leqslant\frac{3}{2K}E(X^*(t))^2+\frac{3K}{2}E([M](t)),$$

*where $[\,]$ stands for the quadratic variation process.*

THEOREM 3.3. *Suppose that all conditions of Theorem* 2.7 *hold. If $X(t),t\in[-r,T]$, is the solution of* (2.3), *then there exists a constant $\bar{C}>0$ such that*

$$E(X^*(t))^p\leqslant\bar{C}\left\{1+E(\|\phi(0)\|^p)+E\left(\int_0^t\|g(s,0)\|_2^p\,ds\right)+E(V^*(t))^p\right\}.$$

*In particular, $X^*(t)\in L^p$ for all $t\in[-r,T]$.*

*Proof.* First of all, note that by Lemmas 1 and 5 of [23], we may assume without loss of generality that $g(s,0)=0$ for all $s\in[0,T]$ and by a simple transformation [14], we may reduce the problem to the case when $\lambda=0$. Define $Y(t)=X(t)-V(t)$ for all $t\in[-r,T]$. Then we can rewrite (2.3) as

$$Y(t)=U(t,0)\phi(0)+\int_0^t U(t,s)f(s,X_s)\,ds+\int_0^t U(t,s)g(s,X_s)\,dW_s.$$

By the Itô-type inequality of Section 2, one can derive that

$$\begin{aligned}\|Y(t)\|^p \leqslant \|\phi(0)\|^p &+ p\int_0^t \|Y(s)\|^{p-2}\langle Y(s)\,,\,f(s,X_s)\rangle\,ds\\ &+p\int_0^t \|Y(s)\|^{p-2}\langle Y(s)\,,\,g(s,X_s)dW_s\rangle\\ &+\frac{p(p-1)}{2}\int_0^t \|Y(s)\|^{p-2}\|g(s,X_s)\|_2^2\,ds.\end{aligned}\tag{3.1}$$

Since $X_s=Y_s+V_s$ for all $s\in[0,T]$, we have

$$\begin{aligned}p\int_0^t \|Y(s)\|^{p-2}\langle Y(s)\,,\,f(s,X_s)\rangle\,ds &= p\int_0^t \|Y(s)\|^{p-2}\langle Y(s)\,,\,f(s,Y_s+V_s)-f(s,V_s)\rangle\,ds\\ &\quad+p\int_0^t \|Y(s)\|^{p-2}\langle Y(s)\,,\,f(s,V_s)\rangle\,ds.\end{aligned}$$

Now, we use Lemma 3.1 and Hypothesis 2.6**(c)** and **(d)** to conclude that the right hand side of the above equation is

$$\begin{aligned}&\leqslant pM\int_0^t \|Y(s)\|^p\,ds+pCT(Y^*(t))^{p-1}(1+V^*(t)) \qquad (3.2)\\ &\leqslant pM\int_0^t (Y^*(s))^p\,ds+\frac{1}{2}(Y^*(t))^p+CT[4pCT(1-\frac{1}{p})]^{p-1}\{1+(V^*(t))^p\}.\end{aligned}$$

Substituting (3.2) in (3.1), it then follows that

$$\begin{aligned}\frac{1}{2}(Y^*(t))^p \leqslant \|\phi(0)\|^p &+pM\int_0^t (Y^*(s))^p\,ds+CT[4pCT(1-\frac{1}{p})]^{p-1}\{1+(V^*(t))^p\}\\ &+p\sup_{0\leqslant\rho\leqslant t}\left|\int_0^\rho \|Y(s)\|^{p-2}\langle Y(s)\,,\,g(s,X_s)dW_s\rangle\right|\\ &+\frac{p(p-1)}{2}\int_0^t \|Y(s)\|^{p-2}\|g(s,X_s)\|_2^2\,ds\end{aligned}\tag{3.3}$$

Multiplying both sides of (3.3) by 2, taking the mathematical expectation and using Lemma 3.2 on the next to the last term, yield

$$\begin{aligned}E(Y^*(t))^p \leqslant 2E(\|\phi(0)\|^p) &+2pM\int_0^t E(Y^*(s))^p\,ds\\ &+2CT[4pCT(1-\frac{1}{p})]^{p-1}\{1+E(V^*(t))^p\}+\frac{3p}{K}E(Y^*(t))^p\\ &+[p(p-1)+3pK]\int_0^t E(\|Y(s)\|^{p-2}\|g(s,X_s)\|_2^2)\,ds,\end{aligned}$$

where $K$ is an arbitrary positive constant. Choose $K = 6p$ and note that by Theorem 2.7, $E(Y^*(t))^p < \infty$. Then

$$\begin{aligned} E(Y^*(t))^p \leqslant\ & 4E(\|\phi(0)\|^p) + 4pM \int_0^t E(Y^*(s))^p\, ds \\ & + 4CT[4pCT(1-\frac{1}{p})]^{p-1}\{1 + E(V^*(t))^p\} \\ & + 2[p(p-1) + 2(3p)^2] \int_0^t E(\|Y(s)\|^{p-2}\|g(s,X_s)\|_2^2)\, ds. \end{aligned} \tag{3.4}$$

By Hypothesis 2.6**(e)** and the fact that $g(s,0) = 0$, we obtain that the last integral on the right of (3.4) is

$$\begin{aligned} & \leqslant C^2 \int_0^t E(\|Y(s)\|^{p-2}\|X_s\|_{C_H}^2)\, ds \\ & \leqslant 2C^2 \int_0^t E(\|Y(s)\|^{p-2}\|V_s\|_{C_H}^2)\, ds + 2C^2 \int_0^t E(\|Y_s\|_{C_H}^p)\, ds \\ & \leqslant 4C^2(1-\frac{1}{p}) \int_0^t E(Y^*(s))^p\, ds + \frac{4C^2T}{p}\{1 + E(V^*(t))^p\}. \end{aligned}$$

Therefore, we can rewrite (3.4) as

$$\begin{aligned} E(Y^*(t))^p \leqslant\ & 4E(\|\phi(0)\|^p) + \{4CT[4pCT(1-\frac{1}{p})]^{p-1} + \frac{4C^2T}{p}\delta\}\{1 + E(V^*(t))^p\} \\ & + \{4pM + 4C^2\delta(1-\frac{1}{p})\} \int_0^t E(Y^*(s))^p\, ds, \end{aligned}$$

in which $\delta = 2[p(p-1) + 2(3p)^2]$. Now, by the well-known Gronwall inequality, we have

$$E(Y^*(t))^p \leqslant \alpha[1 + E(\|\phi(0)\|^p) + E(V^*(t))^p]e^{\gamma t},$$

where $\alpha = \max\{4, 4CT[4pCT(1-\frac{1}{p})]^{p-1} + \frac{4C^2T}{p}\delta\}$ and $\gamma = 4pM + 4C^2\delta(1-\frac{1}{p})$. Finally, since

$$(X^*(t))^p \leqslant 2^{p-1}\{(Y^*(t))^p + (V^*(t))^p\},$$

we conclude that

$$E(X^*(t))^p \leqslant \bar{C}\{1 + E(\|\phi(0)\|^p) + E(V^*(t))^p\},$$

where the constant $\bar{C} = 2^{p-1}\alpha e^{\gamma T}$. This completes the proof of the theorem. □

## 4. Continuity with respect to the parameters

In this section, we give a proof for the continuity of the mild solution or more generally, the solution of equation (2.3), with respect to the parameters; i.e., we prove that the solution changes continuously when any or all of $V$, $f$, $g$ and the initial data $\phi$ are varied. Before that, we are going to prove a lemma.

LEMMA 4.1. *Let $f^i$, $g^i$, $V^i$ and $\phi^i$, $i=1,2$ satisfy the conditions of Theorem 2.7 and $X^i(t)$, $i=1,2$ are solutions of the integral equations*

$$X^i(t)=U(t,0)\phi^i(0)+\int_0^t U(t,s)f^i(s,X^i(s))\,ds+\int_0^t U(t,s)g^i(s,X^i(s))\,dW_s+V^i(t).$$

*If $Y^i=X^i-V^i$ for $i=1,2$, then*

$$\begin{aligned}&p\int_0^t \|Y^2(s)-Y^1(s)\|^{p-2}\langle Y^2(s)-Y^1(s)\,,f^2(s,X_s^2)-f^1(s,X_s^1)\rangle\,ds\\ &\leqslant [4(p-1)M+(2p-3)]\int_0^t \|Y^2(s)-Y^1(s)\|^p\,ds\\ &\quad+\int_0^t \|f^2(s,X_s^1)-f^1(s,X_s^1)\|^p\,ds+I\left(\int_0^t \|V^2(s)-V^1(s)\|^p\,ds\right)^{\frac{1}{2}},\end{aligned}$$

*in which*

$$I=4M\left(\int_0^t \|V^2(s)-V^1(s)\|^p\,ds\right)^{\frac{1}{2}}+2\left(\int_0^t \|f^2(s,X_s^2)-f^2(s,X_s^1)\|^p\,ds\right)^{\frac{1}{2}},$$

*for all $t\in[0,T]$.*

*Proof.* We denote by $J(t)$ the left hand side of the asserted inequality:

$$J(t)=p\int_0^t \|Y^2(s)-Y^1(s)\|^{p-2}\langle Y^2(s)-Y^1(s)\,,f^2(s,X_s^2)-f^1(s,X_s^1)\rangle\,ds,$$

and decompose it into two parts:

$$\begin{aligned}J(t)&=p\int_0^t \|Y^2(s)-Y^1(s)\|^{p-2}\langle Y^2(s)-Y^1(s)\,,f^2(s,X_s^2)-f^2(s,X_s^1)\rangle\,ds\\ &\quad+p\int_0^t \|Y^2(s)-Y^1(s)\|^{p-2}\langle Y^2(s)-Y^1(s)\,,f^2(s,X_s^1)-f^1(s,X_s^1)\rangle\,ds\\ &=J_1(t)+J_2(t).\end{aligned}$$

Then we estimate each part separately. Since, $Y^i=X^i-V^i$, for $i=1,2$ and $f^2$ satisfies Hypothesis 2.6(**c**), we get

$$\begin{aligned}J_1(t)&\leqslant p\int_0^t \|Y^2(s)-Y^1(s)\|^{p-2}\|V^2(s)-V^1(s)\|\,\|f^2(s,X_s^2)-f^2(s,X_s^1)\|\,ds\\ &\quad+pM\int_0^t \|Y^2(s)-Y^1(s)\|^{p-2}\|X^2(s)-X^1(s)\|^2\,ds. \qquad (4.1)\end{aligned}$$

Using the Hölder inequality to the first term on the right hand side of (4.1) and since $X_s^i=Y_s^i+V_s^i$ for $i=1,2$, we obtain

$$
\begin{aligned}
J_1(t) \leqslant p\left(\int_0^t \|V^2(s)-V^1(s)\|^{\frac{p}{2}} \|f^2(s,X_s^2)-f^2(s,X_s^1)\|^{\frac{p}{2}}\, ds\right)^{\frac{2}{p}} \\
\times\left(\int_0^t \|Y^2(s)-Y^1(s)\|^p\, ds\right)^{1-\frac{2}{p}} \\
+2pM\int_0^t \|Y^2(s)-Y^1(s)\|^{p-2}\|V^2(s)-V^1(s)\|^2\, ds \\
+2pM\int_0^t \|Y^2(s)-Y^1(s)\|^p\, ds.
\end{aligned} \tag{4.2}
$$

Applying Lemma 3.1 with $\alpha = \frac{2}{p}$ to the first and second term on the right of (4.2), one can see that

$$
\begin{aligned}
J_1(t) \leqslant{} & [2pM + 2p(1-\frac{2}{p})M]\int_0^t \|Y^2(s)-Y^1(s)\|^p\, ds + 4M\int_0^t \|V^2(s)-V^1(s)\|^p\, ds \\
& +2\left(\int_0^t \|V^2(s)-V^1(s)\|^p\, ds\right)^{\frac{1}{2}}\left(\int_0^t \|f^2(s,X_s^2)-f^2(s,X_s^1)\|^p\, ds\right)^{\frac{1}{2}} \\
& +(p-2)\int_0^t \|Y^2(s)-Y^1(s)\|^p\, ds.
\end{aligned} \tag{4.3}
$$

On the other hand, again by Lemma 3.1, we have

$$
\begin{aligned}
J_2(t) &\leqslant p\int_0^t \|Y^2(s)-Y^1(s)\|^{p-1}\|f^2(s,X_s^1)-f^1(s,X_s^1)\|\, ds \\
&\leqslant p(1-\frac{1}{p})\int_0^t \|Y^2(s)-Y^1(s)\|^p\, ds + \int_0^t \|f^2(s,X_s^1)-f^1(s,X_s^1)\|^p\, ds.
\end{aligned} \tag{4.4}
$$

From (4.3) and (4.4) the desired result follows. □

Now, we are ready to state and prove our main theorem on the continuity with respect to data for the mild solutions of (2.1) and generally the solutions of (2.3).

THEOREM 4.2. *Let $f^i$, $g^i$, $V^i$ and $\phi^i$, $i=1,2$ satisfy the conditions of Theorem 2.7 and $X^i(t)$, $i=1,2$ are solutions of the integral equations*

$$
X^i(t) = U(t,0)\phi^i(0) + \int_0^t U(t,s)f^i(s,X_s^i)\, ds + \int_0^t U(t,s)g^i(s,X_s^i)\, dW_s + V^i(t).
$$

*Then there exist positive constants $\alpha$, $\gamma$ and $C$ such that*

$$
\begin{aligned}
E\{(X^2-X^1)^*(t)\}^p \leqslant 2^{p-1}\alpha e^{\gamma T}[E(\|\phi^2(0)-\phi^1(0)\|^p) + K\{E(\|V^2-V^1\|_\infty^{\frac{p}{2}})\}^{\frac{1}{2}} \\
+\int_0^T E(\|g^2(s,X_s^1)-g^1(s,X_s^1)\|_2^{\frac{p}{2}})\, ds \\
+\int_0^T E(\|f^2(s,X_s^1)-f^1(s,X_s^1)\|^p\, ds],
\end{aligned}
$$

*where*

$$
K \leqslant C\left(1 + E(\|\phi^2(0)\|^p) + E(\|\phi^1(0)\|^p) + E(\|V^2\|_\infty^{\frac{p}{2}}) + E(\|V^1\|_\infty^{\frac{p}{2}})\right). \tag{4.5}
$$

*Proof.* As we said in the previous section, by a simple transformation [14] we may assume without lose of generality that $\lambda = 0$. Setting $Y^i(t) = X^i(t) - V^i(t)$, we have

$$Y^i(t) = U(t,0)\phi^i(0) + \int_0^t U(t,s) f^i(s, X^i_s)\, ds + \int_0^t U(t,s) g^i(s, X^i_s)\, dW_s.$$

Hence,

$$\begin{aligned} Y^2(t) - Y^1(t) = U(t,0)(\phi^2(0) - \phi^1(0)) + \int_0^t U(t,s)[f^2(s,X^2_s) - f^1(s,X^1_s)]\, ds \\ + \int_0^t U(t,s)[g^2(s,X^2_s) - g^1(s,X^1_s)] dW_s. \end{aligned}$$

Now, define

$$Y = Y^2 - Y^1, \quad V = V^2 - V^1, \quad \phi = \phi^2 - \phi^1.$$

Using the Itô-type inequality of Section 2, we obtain

$$\begin{aligned} \|Y(t)\|^p \leqslant \|\phi(0)\|^p + p \int_0^t \|Y(s)\|^{p-2} \langle Y(s), f^2(s,X^2_s) - f^1(s,X^1_s) \rangle\, ds \\ + p \int_0^t \|Y(s)\|^{p-2} \langle Y(s), (g^2(s,X^2_s) - g^1(s,X^1_s)) dW_s \rangle \\ + \frac{p(p-1)}{2} \int_0^t \|Y(s)\|^{p-2} \|g^2(s,X^2_s) - g^1(s,X^1_s)\|_2^2\, ds. \end{aligned} \tag{4.6}$$

There are three integrals on the right hand side of the above inequality which we denote them by $I_1(t)$, $I_2(t)$ and $I_3(t)$, respectively. In fact,

$$I_1(t) = \int_0^t \|Y(s)\|^{p-2} \langle Y(s), f^2(s,X^2_s) - f^1(s,X^1_s) \rangle\, ds,$$

$$I_2(t) = \int_0^t \|Y(s)\|^{p-2} \langle Y(s), (g^2(s,X^2_s) - g^1(s,X^1_s)) dW_s \rangle,$$

and

$$I_3(t) = \int_0^t \|Y(s)\|^{p-2} \|g^2(s,X^2_s) - g^1(s,X^1_s)\|_2^2\, ds.$$

We try to estimate each of the foregoing integrals to obtain an ultimate bound for the $p$-th moment of $Y^*(t)$. By Lemma 4.1, we get

$$\begin{aligned} I_1(t) \leqslant [4(p-1)M + (2p-3)] \int_0^t \|Y(s)\|^p ds + \sqrt{T} I \|V\|_\infty^{\frac{p}{2}} \\ + \int_0^t \|f^2(s,X^1_s) - f^1(s,X^1_s)\|^p\, ds, \end{aligned} \tag{4.7}$$

where

$$I \leqslant 4M\sqrt{T} \|V\|_\infty^{\frac{p}{2}} + 2 \left( \int_0^t \|f^2(s,X^2_s) - f^2(s,X^1_s)\|^p\, ds \right)^{\frac{1}{2}}.$$

But, since $f^2$ satisfies the linear growth condition; i.e., it is bounded by the function $\psi(r) = C(1+r)$ on $\mathbb{R}^+$, we have

$$\|f^2(s,X_s^2) - f^2(s,X_s^1)\| \leqslant C(1+\|X^2\|_\infty) + C(1+\|X^1\|_\infty),$$

for all $s \in [0,T]$. Consequently,

$$I \leqslant 4M\sqrt{T}\|V\|_\infty^{\frac{p}{2}} + 2^{p+1}\sqrt{T}C^{\frac{p}{2}}(2+\|X^2\|_\infty^{\frac{p}{2}} + \|X^1\|_\infty^{\frac{p}{2}}).$$

Moreover, by Lemma 3.2, we deduce that

$$E\{\sup_{0\leqslant\rho\leqslant t} I_2(\rho)\} \leqslant \frac{3L}{2}E(I_3^*(t)) + \frac{3}{2L}E(Y^*(t))^p, \tag{4.8}$$

for arbitrary positive constant $L$. Choose $L = 3p$ in (4.8). Since by Theorem 2.7, $E(Y^*(t))^p < \infty$, we obtain that

$$E(Y^*(t))^p \leqslant 2E(\|\phi(0)\|^p) + 2pE(I_1^*(t)) + [(3p)^2 + p(p-1)]E(I_3^*(t)). \tag{4.9}$$

On the other hand, according to Lemma 3.1 with $\alpha = \frac{2}{p}$ and Hypothesis 2.6**(e)**, we find for the third integral that

$$\begin{aligned} E(I_3^*(t)) &\leqslant 2C^2\int_0^t E(\|Y(s)\|^{p-2}\|X_s^2 - X_s^1\|_{C_H}^2)\,ds \\ &\quad +2\int_0^t E(\|Y(s)\|^{p-2}\|g^2(s,X_s^1) - g^1(s,X_s^1)\|_2^2)\,ds \\ &\leqslant [8C^2(1-\frac{1}{p}) + 2(1-\frac{2}{p})]\int_0^t E(Y^*(s))^p\,ds + \frac{4C^2T}{p}E(\|V\|_\infty^{\frac{p}{2}}) \\ &\quad +\frac{4}{p}\int_0^t E(\|g^2(s,X_s^1) - g^1(s,X_s^1)\|_2^{\frac{p}{2}})\,ds. \end{aligned} \tag{4.10}$$

Combining (4.7), (4.9) and (4.10) and using the Cauchy-Schwartz inequality, one can derive that

$$\begin{aligned} E(Y^*(t))^p &\leqslant 2E(\|\phi(0)\|^p) + 2\{[4(p-1)M + (2p-3)] \\ &\quad +[(3p)^2 + p(p-1)][8C^2(1-\frac{1}{p}) + 2(1-\frac{2}{p})]\}\int_0^t E(Y^*(s))^p\,ds \\ &\quad +2\sqrt{T}\{E(I^2)\}^{\frac{1}{2}}\{E(\|V\|_\infty^{\frac{p}{2}})\}^{\frac{1}{2}} + \frac{4C^2T}{p}[(3p)^2 + p(p-1)]E(\|V\|_\infty^{\frac{p}{2}}) \\ &\quad +2\int_0^t E(\|f^2(s,X_s^1) - f^1(s,X_s^1)\|^p)\,ds \\ &\quad +\frac{4}{p}[(3p)^2 + p(p-1)]\int_0^t E(\|g^2(s,X_s^1) - g^1(s,X_s^1)\|_2^{\frac{p}{2}})\,ds. \end{aligned}$$

Now, by the Gronwall inequality

$$\begin{aligned} E(Y^*(t))^p &\leqslant \alpha[E(\|\phi(0)\|^p) + \int_0^T E(\|f^2(s,X_s^1) - f^1(s,X_s^1)\|^p)\,ds \\ &\quad + \int_0^T E(\|g^2(s,X_s^1) - g^1(s,X_s^1)\|_2^{\frac{p}{2}})\,ds + E(\|V\|_\infty^{\frac{p}{2}}) \\ &\quad +\{E(I^2)\}^{\frac{1}{2}}\{E(\|V\|_\infty^{\frac{p}{2}})\}^{\frac{1}{2}}]e^{\gamma t}, \end{aligned} \tag{4.11}$$

in which $\alpha = \max\{2, 2\sqrt{T}, 4C^2T(10p-1), 4(10p-1)\}$ and

$$\gamma = 2[4(p-1)M + (2p-3)] + (10p-1)[8C^2(p-1) + 2(p-2)].$$

Since $X^*(t) \leqslant Y^*(t) + \|V\|_\infty$, (4.11) implies that

$$E(X^*(t))^p \leqslant 2^{p-1}\alpha e^{\gamma T}[E(\|\phi(0)\|^p) + \int_0^T E(\|f^2(s,X_s^1) - f^1(s,X_s^1)\|^p)\,ds + \int_0^T E(\|g^2(s,X_s^1) - g^1(s,X_s^1)\|_2^p)\,ds + K\{E(\|V\|_\infty^p)\}^{\frac{1}{2}}],$$

where $K = \{E(I^2)\}^{\frac{1}{2}} + \{E(\|V\|_\infty^p)\}^{\frac{1}{2}}$. To complete the proof of the theorem, we only need to show that the constant $K$ satisfies (4.5). Indeed, by Theorem 3.3, there exists a constant $\bar{C}$ depending on $p$ and $T$ such that

$$E(\|X^i\|_\infty^p) \leqslant \bar{C}(1 + E(\|\phi^i(0)\|^p) + E(\|V^i\|_\infty^p)),$$

for $i = 1, 2$. Therefore,

$$E(I^2) \leqslant C_1\left(1 + E(\|\phi^1(0)\|^p) + E(\|\phi^2(0)\|^p) + E(\|V^1\|_\infty^p) + E(\|V^2\|_\infty^p)\right), \tag{4.12}$$

for some positive constant $C_1$. Moreover,

$$\|V\|_\infty^p \leqslant 2^{p-1}(\|V^1\|_\infty^p + \|V^2\|_\infty^p). \tag{4.13}$$

Thus, from (4.12) and (4.13) we conclude that there exists a positive constant $C$ such that

$$K \leqslant C\left(1 + E(\|\phi^1(0)\|^p) + E(\|\phi^2(0)\|^p) + E(\|V^1\|_\infty^p) + E(\|V^2\|_\infty^p)\right).$$

This completes the proof of the theorem. □

COROLLARY 4.3. (continuity with respect to the initial data) *If $X^1(t)$ and $X^2(t)$ are two mild solutions of* (2.1) *with the initial data $\phi^1$ and $\phi^2$, respectively, then there exists a positive constant $C$ such that*

$$E\{(X^2 - X^1)^*(t)\}^p \leqslant CE(\|\phi^2(0) - \phi^1(0)\|^p),$$

*for all $t \in [0,T]$.*

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