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A SHARP UNCERTAINTY PRINCIPLE AND HARDY–POINCARÉ INEQUALITIES ON SUB–RIEMANNIAN MANIFOLDS

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Abstract. We prove a sharp Heisenberg uncertainty principle inequality and Hardy-Poincaré inequality on the sub-Riemannian manifold $\mathbb{R}^{2n+1}=\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}$ defined by the vector fields:

$$X_j=\frac{\partial}{\partial x_j}+2ky_j|z|^{2k-2}\frac{\partial}{\partial l}\,,\quad Y_j=\frac{\partial}{\partial y_j}-2kx_j|z|^{2k-2}\frac{\partial}{\partial l}\,,\qquad j=1,2,\ldots,n$$

where $|z|=(|x|^2+|y|^2)^{1/2}$ and $k\geqslant 1$.

1. Introduction

The present work is concerned with the Heisenberg uncertainty principle inequality and a new (sharp) form of the weighed Hardy-Poincaré type inequality on the sub-Riemannian manifold $\mathbb{R}^{2n+1}=\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}$ defined by the vector fields:

$$X_j=\frac{\partial}{\partial x_j}+2ky_j|z|^{2k-2}\frac{\partial}{\partial l}\,,\quad Y_j=\frac{\partial}{\partial y_j}-2kx_j|z|^{2k-2}\frac{\partial}{\partial l}\,,\qquad j=1,2,\ldots,n \tag{1.1}$$

where $|z|=(|x|^2+|y|^2)^{1/2}$ and $k\geqslant 1$. The vector fields (1.1) satisfy Hörmander's condition for any $k\in\mathbb{N}$, i.e, X_j,Y_j and their iterated Lie brackets span the tangent space of $\mathbb{R}^{2n+1}$ at each point [19]. The number of brackets plus 1 is called the step of the sub-Riemannian manifold, and in our case is $2k$. If $k=1$ then we have the vector fields :

$$X_j=\frac{\partial}{\partial x_j}+2y_j\frac{\partial}{\partial l}\,,\quad Y_j=\frac{\partial}{\partial y_j}-2x_j\frac{\partial}{\partial l}\,,\qquad j=1,2,\ldots,n \tag{1.2}$$

that are left invariant with respect to the following Lie group law on $\mathbb{R}^{2n+1}$

$$(x,y,l)\circ(x',y',l')=(x+x',y+y',l+l'-2(xy'-yx')).$$

These vector fields satisfy the "Heisenberg commutation relations"

$$[X_j,Y_j]=-4\frac{\partial}{\partial l},\qquad [X_j,X_i]=[Y_j,Y_i]=[X_j,\frac{\partial}{\partial l}]=[Y_j,\frac{\partial}{\partial l}]=0 \tag{1.3}$$

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and the Lie group $\mathbb{H}^n = (\mathbb{R}^{2n+1}, \circ)$ is called the n-dimensional Heisenberg group. The non-trivial commutator $[X_j, Y_j] = -4\frac{\partial}{\partial l}$ is known as the analogue of the Heisenberg uncertainty principle on the Heisenberg group $\mathbb{H}^n$. The first and most famous uncertainty principle goes back to Heisenberg's seminal work which was developed in the context of quantum mechanics [18]. It says that the position and momentum of a particle cannot be determined exactly at the same time but only with an "uncertainty". There are various forms of the uncertainty principle inequality and the most well known form on the Euclidean space $\mathbb{R}^n$ is the following:

$$\Big(\int_{\mathbb{R}^n} |x|^2 |f(x)|^2 dx\Big)\Big(\int_{\mathbb{R}^n} |\nabla f(x)|^2 dx\Big) \geqslant \frac{n^2}{4}\Big(\int_{\mathbb{R}^n} |f(x)|^2 dx\Big)^2 \tag{1.4}$$

for all $f \in L^2(\mathbb{R}^n)$. Here the constant $\frac{n^2}{4}$ is sharp and also it is well-known that equality is attained in (1.3) if and only if f is a Gaussian (i.e. $f(x) = Ae^{-\alpha|x|^2}$ for some $A \in \mathbb{R}, \alpha > 0$). The Heisenberg uncertainty principle is a fundamental concept in harmonic analysis, signal and information theory as well as in quantum mechanics and has been extensively studied in Euclidean space and generalized to various settings [14]. In this direction our first goal is to obtain an analogue of the classical Heisenberg uncertainty principle inequality (1.4) on sub-Riemannian manifold $\mathbb{R}^{2n+1}$ defined by the vector fields (1.1). In addition to this we shall also prove a higher order uncertainty principle inequality.

Another important inequality in mathematical analysis is the following Hardy inequality:

$$\int_{\mathbb{R}^n} |\nabla \phi(x)|^2 dx \geqslant \Big(\frac{n-2}{2}\Big)^2 \int_{\mathbb{R}^n} \frac{\phi^2}{|x|^2} dx \tag{1.5}$$

where $\phi \in C_0^\infty(\mathbb{R}^n)$ and $n \geqslant 3$. Here the constant $\left(\frac{n-2}{2}\right)^2$ is sharp in the sense that

$$\Big(\frac{n-2}{2}\Big)^2 = \inf_{0 \neq \phi \in C_0^\infty(\mathbb{R}^n)} \frac{\int_{\mathbb{R}^n} |\nabla \phi(x)|^2 dx}{\int_{\mathbb{R}^n} \frac{|\phi(x)|^2}{|x|^2} dx}.$$

It is well known that the Hardy inequality and its improved versions (non-negative terms are added to right hand-side of (1.5)) play important role in partial differential equations with singular potentials [5], [4], [10], [15], [16], [20] and references therein. Note that an immediate application of the Hardy and Cauchy-Schwarz inequalities yields the non sharp version of the Heisenberg uncertainty principle inequality.

An analogue of the Hardy inequality (1.5) for the vector fields (1.1) has been proved by Niu, Zang and Wang [23]

$$\int_{\mathbb{R}^{2n+1}} |\nabla_k \phi|^p dz dl \geqslant \Big(\frac{2n+2k-p}{p}\Big)^p \int_{\mathbb{R}^{2n+1}} \frac{|z|^{(2k-1)p}}{(|z|^{4k} + l^2)^{\frac{p}{2}}} |\phi|^p dz dl \tag{1.6}$$

where $\phi \in C_0^\infty(\mathbb{R}^{2n+1}) \setminus \{O\}$ and $1 < p < 2n+2k$. Here $\nabla_k \phi = (X_1\phi, ..., X_n\phi, Y_1\phi, ..., Y_n\phi)$ denotes the horizontal gradient of ϕ, $(z,l) = (x,y,l) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ and $O = (0,0,0)$.

On the other hand Niu, Ou and Han [24] applied the method of Goldstein and Kombe [17] and obtained (among other inequalities) a weighted version of the Hardy type inequality (1.6):

$$\int_{\mathbb{R}^{2n+1}} \rho^{\alpha}|\nabla_k\phi|^p dzdl \geqslant \Big(\frac{2n+2k+\alpha-p}{p}\Big)^p \int_{\mathbb{R}^{2n+1}} \rho^{\alpha}\frac{|\nabla_k\rho|^p}{\rho^p}|\phi|^p dzdl \qquad (1.7)$$

where $\phi \in C_0^{\infty}(\mathbb{R}^{2n+1})\setminus\{O\}$, $1<p<2n+2k$, $2n+2k+\alpha-p>0$, $\rho=(|z|^{4k}+l^2)^{1/4k}$ and $|\nabla_k\rho| = \frac{|z|^{2k-1}}{\rho^{2k-1}}$.

Recently, Ahmetolan and Kombe [3] obtained a sharp extension of (1.7) (involving two radial weight functions):

$$\int_{\Omega} \rho^{\alpha}|\nabla_k\rho|^{pt}|\nabla_k\phi|^p dzdl \geqslant \Big(\frac{2n+2k+\alpha-p}{p}\Big)^p \int_{\Omega} \rho^{\alpha}|\nabla_k\rho|^{pt}\frac{|\nabla_k\rho|^p}{\rho^p}|\phi|^p dzdl \quad (1.8)$$

where $\phi \in C_0^{\infty}(\mathbb{R}^{2n+1})$, $t\in\mathbb{R}$, $1<p<2n+2k$ and $2n+2k+\alpha-p>0$.

Since the above inequalities are strict unless ϕ is identical equal to zero, it is natural expect some extra term might be added on the right hand side of (1.6), (1.7) and (1.8). In this direction Ahmetolan and Kombe [3] obtained various remainder terms for the inequality (1.8). These inequalities play important roles in the study of linear and nonlinear partial differential equations involving singular potential [20], [21], [1], [2]. Another important fact is that the weighted Hardy-type inequalities are the main tools for proving weighted Rellich type inequalities (see [3]).

Our second goal is to prove a new form of the weighted Hardy-Poincaré type inequality on sub-Riemannian manifold $\mathbb{R}^{2n+1}$ defined by the vector fields (1.1). We should point out that one of the advantage of our Hardy-Poincaré inequality (4.1) in this paper is that it implies and thus provides another shorter proof of weighted Hardy type inequality (1.6), (1.7) and (1.8). Furthermore, Hardy-Poincaré inequality (4.1) and Cauchy-Schwarz inequality yields the sharp form of the Heisenberg uncertainty principle inequality (3.1).

The plan of the paper is as follows. In Section 2 we introduce fundamental notations, generalized Greiner vector fields, basic facts about the horizontal gradient ∇_k, sub-Laplacian Δ_k. In Section 3 we prove a sharp Heisenberg uncertainty inequality associated vector fields (1.1). In Section 4 we prove various sharp weighted Hardy-Poincaré type inequalities.

2. Preliminary and Notations

In this section, we will introduce some notations, definitions, and preliminary facts which will be used throughout the article. A generic point is $w=(z,l)=(x,y,l)\in\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}$ with $n\geqslant 1$. The sub-elliptic gradient is the $2n$ dimensional vector field given by

$$\nabla_k := (X_1,...,X_n,Y_1,...,Y_n) \qquad (2.1)$$

where

$$X_j = \frac{\partial}{\partial x_j} + 2ky_j|z|^{2k-2}\frac{\partial}{\partial l}, \quad Y_j = \frac{\partial}{\partial y_j} - 2kx_j|z|^{2k-2}\frac{\partial}{\partial l}, \qquad j=1,2,...,n. \qquad (2.2)$$

The sub-Laplacian associated with the vector fields (1.1) is the second order partial differential operator on $\mathbb{R}^{2n+1}$ given by

$$\Delta_k = \sum_{j=1}^{n}(X_j^2 + Y_j^2) = \Delta_z + 4k^2|z|^{4k-2}\frac{\partial^2}{\partial l^2} + 4k|z|^{2k-2}\frac{\partial}{\partial l}T \tag{2.3}$$

where $\Delta_z = \sum_{j=1}^{n}(\frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2})$ is the Laplacian in the variable $z = (x,y) \in \mathbb{R}^{2n}$ and T denotes the vector field as $T = \sum_{j=1}^{n}(y_j\frac{\partial}{\partial x_j} - x_j\frac{\partial}{\partial y_j})$. The sub-Laplacian arises in diverse areas of mathematics including boundary value problems in several complex variables, harmonic analysis and quantum mechanics of anharmonic oscillators. We refer the reader to the articles [6], [12] and the book [11] for further details.

There is a natural norm:

$$\rho = |(z,l)| = (|z|^{4k} + l^2)^{1/4k} \quad ,k \geqslant 1, \tag{2.4}$$

and a notion of dilation $\delta_\lambda(z,l) = (\lambda z, \lambda^{2k} l)$, where $\lambda > 0$, on this model. Notice that norm function is homogeneous of degree one with respect to the natural dilation δ_λ. The function ρ is related to the fundamental solution of sub-Laplacian Δ_k at the origin (see, [13], [7], [8], [9], [23]). The change of variable formula for the Lebesgue measure gives that

$$d\delta_\lambda(z,l) = \lambda^Q dw = \lambda^Q dzdl \tag{2.5}$$

where

$$Q = 2(n+k) \tag{2.6}$$

is the homogeneous dimension with respect to dilation δ_λ and $dw = dzdl$ denotes the Lebesgue measure on $\mathbb{R}^{2n+1}$.

A direct computation shows that

$$X_j\rho = \frac{|z|^{2(k-1)}}{\rho^{4k-1}}(x_j|z|^{2k} + y_j l)\ , \quad Y_j\rho = \frac{|z|^{2(k-1)}}{\rho^{4k-1}}(y_j|z|^{2k} - x_j l). \tag{2.7}$$

Let $\phi = \phi(\rho)$ be a smooth radial function (i.e., ϕ only depends on the function ρ) then we have:

$$|\nabla_k\phi(\rho)| = \frac{|z|^{2k-1}}{\rho^{2k-1}}|\phi'(\rho)| \tag{2.8}$$

and

$$\Delta_k\phi(\rho) = \frac{|z|^{4k-2}}{\rho^{4k-2}}\Big(\phi'' + \frac{Q-1}{\rho}\phi'\Big) = |\nabla_k\rho|^2\Big(\phi'' + \frac{Q-1}{\rho}\phi'\Big). \tag{2.9}$$

In particular

$$\Delta_k\rho = \frac{|\nabla_k\rho|^2}{\rho}(Q-1). \tag{2.10}$$

We now use the above formulas and obtain:

$$\nabla_k(|\nabla_k\rho|^2)\cdot\nabla_k\rho = 0 \tag{2.11}$$

which shows that the norm function (2.4) is infinite harmonic. An immediate consequence of the equation (2.10) is the following formula:

$$\nabla_k\Big(\frac{\rho}{|\nabla_k\rho|^2}\Big)\cdot\nabla_k\rho = 1. \tag{2.12}$$

A straightforward computation shows that

$$\nabla_k\cdot\Big(\frac{\rho}{|\nabla_k\rho|^2}\nabla_k\rho\Big) = \nabla_k\Big(\frac{\rho}{|\nabla_k\rho|^2}\Big)\cdot\nabla_k\rho + \frac{\rho}{|\nabla_k\rho|^2}\Delta_k\rho. \tag{2.13}$$

Substituting (2.10) and (2.12) into (2.13) we obtain the following formula:

$$\nabla_k\cdot\Big(\frac{\rho}{|\nabla_k\rho|^2}\nabla_k\rho\Big) = Q \tag{2.14}$$

which plays a fundamental role in this paper.

In order to compute some radial integrals we use the following spherical transformation in [22]:

Let $w=(x,y,l)$ and

$$\begin{aligned}
x_1 &= \rho(\sin\varphi)^{1/2k}\cos\psi_1\cos\theta_1,\\
y_1 &= \rho(\sin\varphi)^{1/2k}\cos\psi_1\sin\theta_1,\\
&\dots\\
x_{n-1} &= \rho(\sin\varphi)^{1/2k}\sin\psi_1\dots\sin\psi_{n-2}\cos\psi_{n-1}\cos\theta_{n-1},\\
y_{n-1} &= \rho(\sin\varphi)^{1/2k}\sin\psi_1\dots\sin\psi_{n-2}\cos\psi_{n-1}\sin\theta_{n-1},\\
x_n &= \rho(\sin\varphi)^{1/2k}\sin\psi_1\dots\sin\psi_{n-2}\sin\psi_{n-1}\cos\theta_n,\\
y_n &= \rho(\sin\varphi)^{1/2k}\sin\psi_1\dots\sin\psi_{n-2}\sin\psi_{n-1}\sin\theta_n,\\
l &= \rho^{2k}\cos\varphi,
\end{aligned} \tag{2.15}$$

for $R_1<\rho<R_2$, $0\leqslant\varphi\leqslant\pi$, $0\leqslant\psi_j\leqslant\frac{\pi}{2}$, $j=1,\dots,n-1$, and $0\leqslant\theta_j\leqslant 2\pi$, $j=1,\dots,n$. Then the volume element satisfies the following relation

$$dw = dzdl = dxdydl = \rho^{Q-1}d\rho(\sin\varphi)^{\frac{n-k}{k}}d\varphi\prod_{j=1}^{n-1}\Big[\cos\psi_j(\sin\psi_j)^{2(n-j)}d\psi_j\Big]\prod_{j=1}^{n}d\theta_j \tag{2.16}$$

and

$$|z|^2 = \rho^2\sin^{\frac{1}{k}}\varphi\,. \tag{2.17}$$

3. Uncertainty Principle Inequalities

The following inequality is the sharp analogue of the Heisenberg uncertainty principle inequality (1.4).

THEOREM 3.1. *Let $\phi\in C_0^\infty(\mathbb{R}^{2n+1})$, $k\geqslant 1$ and $Q=2(n+k)$. Then the following inequality is valid :*

$$\Big(\int_{\mathbb{R}^{2n+1}}\rho^2\phi^2dw\Big)\Big(\int_{\mathbb{R}^{2n+1}}\Big(\frac{\rho}{|z|}\Big)^{4k-2}|\nabla_k\phi|^2dw\Big)\geqslant\frac{Q^2}{4}\Big(\int_{\mathbb{R}^{2n+1}}\phi^2dw\Big)^2. \tag{3.1}$$

Moreover, the constant $\frac{Q^2}{4}$ *is sharp.*

Proof. Multiplying (2.14) by ϕ^2 and applying integration by parts gives

$$\int_{\mathbb{R}^{2n+1}} \frac{Q}{2}\phi^2 dw = -\int_{\mathbb{R}^{2n+1}} \phi\rho^{4k-1}|z|^{2(1-2k)}\nabla_k\rho\cdot\nabla_k\phi dw. \tag{3.2}$$

We now apply the Cauchy-Schwarz inequality to

$$R_H = -\int_{\mathbb{R}^{2n+1}} \phi\rho^{4k-1}|z|^{2(1-2k)}\nabla_k\rho\cdot\nabla_k\phi dw$$

and get

$$R_H \leqslant \Big(\int_{\mathbb{R}^{2n+1}} \rho^2\phi^2 dw\Big)^{1/2}\Big(\int_{\mathbb{R}^{2n+1}} |z|^{2(1-2k)}\rho^{2(2k-1)}|\nabla_k\phi|^2 dw\Big)^{1/2}. \tag{3.3}$$

Substituting (3.3) into the equation (3.2) gives the desired inequality.

To show that the constant $\frac{Q^2}{4}$ is sharp, we will use the Gaussian function $f(\rho) = Ae^{-B\rho^2}$, $A,B\in\mathbb{R}$ and $B>0$. After a straightforward computation, we have

$$\begin{aligned}
&\int_{\mathbb{R}^{2n+1}} \phi^2 dw = A^2B^{(-\frac{Q}{2})}2^{-(1+\frac{Q}{2})}\Gamma(\frac{Q}{2})\alpha_1,\\
&\int_{\mathbb{R}^{2n+1}} \rho^2\phi^2 dw = A^2B^{-(1+\frac{Q}{2})}2^{-(2+\frac{Q}{2})}\Gamma(1+\frac{Q}{2})\alpha_1\ ,\\
&\int_{\mathbb{R}^{2n+1}} |z|^{2(1-2k)}\rho^{2(2k-1)}|\nabla_k\phi|^2 dw = 4A^2B^{(1-\frac{Q}{2})}2^{-(2+\frac{Q}{2})}\Gamma(1+\frac{Q}{2})\alpha_1\ .
\end{aligned} \tag{3.4}$$

where Γ is the gamma function and

$$\alpha_1 = \int_0^\pi (\sin\varphi)^{(n-k)/k}d\varphi \times \int_0^{\pi/2}\prod_{j=1}^{n-1}\cos\psi_j(\sin\psi_j)^{2(n-j)}d\psi_j \times \int_0^{2\pi}\prod_{j=1}^{n} d\theta_j\ .$$

Substituting the integrals in (3.4) into (3.1) then, the case of equality in the Theorem is attained and so the proof is completed. □

REMARK. Note that even though $\phi(\rho) = Ae^{-B\rho^2}$ does not have a compact support, it can be approximated by such functions yielding that (3.1) is sharp.

There is natural link between second order Uncertainty principle and Rellich type inequalities. In our previous work we obtained the following Rellich type inequality [3]:

THEOREM 3.2. *Let* $\alpha\in\mathbb{R}$, $Q\geqslant 3$, *and* $\frac{8-Q}{3}<\alpha<Q$. *Then the following inequality holds;*

$$\int_{\mathbb{R}^{2n+1}} \rho^{\alpha+4k-2}|z|^{2(1-2k)}|\Delta_k\phi|^2 dw \geqslant \frac{(Q-\alpha)^2}{4}\int_{\mathbb{R}^{2n+1}} \rho^{\alpha-2}|\nabla_k\phi|^2 dw \tag{3.5}$$

for all $\phi\in C_0^\infty(\Omega)$. *Furthermore, the constant* $\frac{(Q-\alpha)^2}{4}$ *is sharp.*

We now have the following higher order Uncertainty principle type inequality.

THEOREM 3.3. *Let $\phi \in C_0^\infty(\mathbb{R}^{2n+1})$ and $Q \geqslant 8$. Then we have:*

$$\Big(\int_{\mathbb{R}^{2n+1}} \rho^4 (\frac{\rho}{|z|}\Big)^{4k-2} \phi^2 dw\Big)\Big(\int_{\mathbb{R}^{2n+1}} \Big(\frac{\rho}{|z|}\Big)^{4k-2} |\Delta_k \phi|^2 dw\Big) \geqslant \frac{Q^4}{16}\Big(\int_{\mathbb{R}^{2n+1}} \phi^2 dw\Big)^2 . \tag{3.6}$$

Proof. To prove the theorem, we use the inequality in (3.2). Applying the Cauchy-Schwarz inequality to the righthand side of (3.2), we obtain,

$$\frac{Q}{2}\int_{\mathbb{R}^{2n+1}} \phi^2 dw \leqslant \Big(\int_{\mathbb{R}^{2n+1}} \rho^4 (\frac{\rho}{|z|}\Big)^{4k-2} \phi^2 dw\Big)^{\frac{1}{2}} \Big(\int_{\mathbb{R}^{2n+1}} \frac{|\nabla_k \phi|^2}{\rho^2} dz dl\Big)^{\frac{1}{2}} . \tag{3.7}$$

Now, we apply the weighted Rellich type inequality (3.5) for $\alpha = 0$, then we have the desired inequality

$$\Big(\int_{\mathbb{R}^{2n+1}} \rho^4 (\frac{\rho}{|z|}\Big)^{4k-2} \phi^2 dw\Big)\Big(\int_{\mathbb{R}^{2n+1}} \Big(\frac{\rho}{|z|}\Big)^{4k-2} |\Delta_k \phi|^2 dw\Big) \geqslant \frac{Q^4}{16}\Big(\int_{\mathbb{R}^{2n+1}} \phi^2 dw\Big)^2 . \qquad \square$$

4. Sharp weighted Hardy-Poincaré type inequalities

In this section, we first prove a new form of the weighted Hardy-Poincaré type inequality with a sharp constant.

THEOREM 4.1. *Let $Q \geqslant 3$, $1 < p < Q$ and $Q + \alpha > 0$. Then the following inequality is valid for all compactly supported smooth functions $\phi \in C_0^\infty(\mathbb{R}^{2n+1})$:*

$$\int_{\mathbb{R}^{2n+1}} \rho^{\alpha+4kp-p} |z|^{p(2-4k)} |\nabla_k \rho \cdot \nabla_k \phi|^p dw \geqslant \Big(\frac{Q+\alpha}{p}\Big)^p \int_{\mathbb{R}^{2n+1}} \rho^\alpha |\phi|^p dw \tag{4.1}$$

where $\rho = (|z|^{4k} + l^2)^{\frac{1}{4k}}$. Furthermore, the constant $(\frac{Q+\alpha}{p})^p$ is sharp.

Proof. We will use the formula in (2.14) to prove the theorem. Multiply both sides of (2.14) by the function $\rho^\alpha |\phi|^p$ and integrate over $\mathbb{R}^{2n+1}$ to get

$$\int_{\mathbb{R}^{2n+1}} Q\rho^\alpha |\phi|^p dw = \int_{\mathbb{R}^{2n+1}} \rho^\alpha |\phi|^p dw + \int_{\mathbb{R}^{2n+1}} \rho^{\alpha+4k-1} |z|^{2(1-2k)} |\phi|^p \Delta_k \rho dw . \tag{4.2}$$

Applying integration by parts to the second integral in the righthand side, we have,

$$\begin{aligned}&\int_{\mathbb{R}^{2n+1}} \rho^{\alpha+4k-1} |z|^{2(1-2k)} |\phi|^p \Delta_k \rho dw \\ &= -(\alpha+1)\int_{\mathbb{R}^{2n+1}} \rho^\alpha |\phi|^p dw - \int_{\mathbb{R}^{2n+1}} \rho^{\alpha+4k-1} |z|^{2(1-2k)} \nabla_k |\phi|^p \cdot \nabla_k \rho dw .\end{aligned} \tag{4.3}$$

Substituting (4.3) into (4.2), we get

$$(Q+\alpha)\int_{\mathbb{R}^{2n+1}} \rho^{\alpha}|\phi|^{p}dw = -p\int_{\mathbb{R}^{2n+1}} \rho^{\alpha+4k-1}|z|^{2(1-2k)}\phi|\phi|^{p-2}\nabla_k\phi\cdot\nabla_k\rho\, dw. \quad (4.4)$$

An application of Hölder's inequality yields

$$\begin{aligned}&\Big(\int_{\mathbb{R}^{2n+1}} \rho^{\alpha+4kp-p}|z|^{2p(1-2k)}|\nabla_k\rho\cdot\nabla_k\phi|^{p}dw\Big)^{1/p}\Big(\int_{\mathbb{R}^{2n+1}} \rho^{\alpha}|\phi|^{p}dw\Big)^{(p-1)/p}\\ &\geqslant \int_{\mathbb{R}^{2n+1}} \rho^{\alpha+4k-1}|z|^{2(1-2k)}|\phi|^{p-1}|\nabla_k\rho\cdot\nabla_k\phi|dw\,.\end{aligned} \quad (4.5)$$

To continue the proof, we use the Young inequality: Let $p>1$, and $a\neq b$ be two positive real numbers then,

$$ab\leqslant \frac{a^{p}}{p}+\frac{(p-1)}{p}b^{\frac{p}{(p-1)}}\,. \quad (4.6)$$

For any $\varepsilon>0$, assuming that

$$\begin{aligned}a&=\varepsilon\Big(\int_{\mathbb{R}^{2n+1}} \rho^{\alpha+4kp-p}|z|^{2p(1-2k)}|\nabla_k\rho\cdot\nabla_k\phi|^{p}dw\Big)^{1/p},\\ b&=\varepsilon^{-1}\Big(\int_{\mathbb{R}^{2n+1}} \rho^{\alpha}|\phi|^{p}dw\Big)^{(p-1)/p},\end{aligned} \quad (4.7)$$

we have,

$$\begin{aligned}&\Big(\int_{\mathbb{R}^{2n+1}} \rho^{\alpha+4kp-p}|z|^{2p(1-2k)}|\nabla_k\rho\cdot\nabla_k\phi|^{p}dw\Big)^{1/p}\Big(\int_{\mathbb{R}^{2n+1}} \rho^{\alpha}|\phi|^{p}dw\Big)^{(p-1)/p}\\ &\leqslant \frac{\varepsilon^{p}}{p}\int_{\mathbb{R}^{2n+1}} \rho^{\alpha+4kp-p}|z|^{2p(1-2k)}|\nabla_k\rho\cdot\nabla_k\phi|^{p}dw+\frac{(p-1)}{p}\varepsilon^{-\frac{p}{p-1}}\int_{\mathbb{R}^{2n+1}} \rho^{\alpha}|\phi|^{p}dw\,.\end{aligned} \quad (4.8)$$

First substituting (4.8) into (4.5) and then rearranging the resulting inequality, we get,

$$\int_{\mathbb{R}^{2n+1}} \rho^{\alpha+4kp-p}|z|^{2p(1-2k)}|\nabla_k\rho\cdot\nabla_k\phi|^{p}dw\geqslant f(\varepsilon,Q,\alpha,p)\int_{\mathbb{R}^{2n+1}} \rho^{\alpha}|\phi|^{p}dw \quad (4.9)$$

where $f(\varepsilon,Q,\alpha,p)=\varepsilon^{-1}[(Q+\alpha)-(p-1)\varepsilon^{-\frac{p}{(p-1)}}]$. Here note that, the function f attains the maximum for $\varepsilon=(\frac{p}{Q+\alpha})^{(p-1)/p}$ and this maximum is equal to $(\frac{Q+\alpha}{p})^{p}$. Thus, we obtain the desired inequality as follows;

$$\int_{\mathbb{R}^{2n+1}} \rho^{\alpha+4kp-p}|z|^{2p(1-2k)}|\nabla_k\rho\cdot\nabla_k\phi|^{p}dw\geqslant \Big(\frac{Q+\alpha}{p}\Big)^{p}\int_{\mathbb{R}^{2n+1}} \rho^{\alpha}|\phi|^{p}dw\,. \quad (4.10)$$

Now, we need to show that $\left(\frac{Q+\alpha}{p}\right)^{p}$ is the best constant. Let C_H be the best constant in (4.10);

$$C_H:=\inf_{0\neq f\in C_0^{\infty}(\mathbb{R}^{2n+1})}\frac{\int_{\mathbb{R}^{2n+1}} \rho^{\alpha+4kp-p}|z|^{2p(1-2k)}|\nabla_k\rho\cdot\nabla_k\phi|^{p}dw}{\int_{\mathbb{R}^{2n+1}} \rho^{\alpha}|\phi|^{p}dw}\,. \quad (4.11)$$

It is clear from (4.10) that

$$\Big(\frac{Q+\alpha}{p}\Big)^p \leqslant \frac{\int_{\mathbb{R}^{2n+1}} \rho^{\alpha+4kp-p}|z|^{2p(1-2k)}|\nabla_k\rho\cdot\nabla_k\phi|^p dw}{\int_{\mathbb{R}^{2n+1}} \rho^{\alpha}|\phi|^p dw} \tag{4.12}$$

holds for all $\phi \in C_0^\infty(\mathbb{R}^{2n+1})$. By taking the infimum in (4.12), we have that

$$\Big(\frac{Q+\alpha}{p}\Big)^p \leqslant C_H \tag{4.13}$$

To prove that $C_H = \Big(\frac{Q+\alpha}{p}\Big)^p$, we only need to show that $C_H \leqslant \Big(\frac{Q+\alpha}{p}\Big)^p$. Hence, for a given $\varepsilon > 0$, we define the radial function: $\mathbb{R}^{2n+1}$.

$$\phi_\varepsilon(\rho) = \begin{cases} \rho^{\frac{Q+\alpha}{p}+\varepsilon} & \text{if} \quad 0 \leqslant \rho \leqslant 1, \\ \rho^{-(\frac{Q+\alpha}{p}+\varepsilon)} & \text{if} \quad \rho > 1\ . \end{cases} \tag{4.14}$$

Note that $\phi_\varepsilon(\rho)$ that can be approximated by smooth functions with compact support in $\mathbb{R}^{2n+1}$. By a direct computation, the integrands in (4.10) are determined as follows;

$$\rho^{\alpha+4kp-p}|z|^{2p(1-2k)}|\nabla_k\rho\cdot\nabla_k\phi|^p = \begin{cases} (\frac{Q+\alpha}{p}+\varepsilon)^p\rho^{Q+2\alpha+p\varepsilon} & \text{if} \quad 0 \leqslant \rho \leqslant 1, \\ (\frac{Q+\alpha}{p}+\varepsilon)^p\rho^{-Q-p\varepsilon} & \text{if} \quad \rho > 1\ . \end{cases} \tag{4.15}$$

and

$$\rho^\alpha|\phi_\varepsilon|^p = \begin{cases} \rho^{Q+2\alpha+p\varepsilon} & \text{if} \quad 0 \leqslant \rho \leqslant 1, \\ \rho^{-Q-p\varepsilon} & \text{if} \quad \rho > 1\ . \end{cases} \tag{4.16}$$

Defining a unit ball with respect to the homogeneous norm, ρ, denoted by $B_1 = \{w \in \mathbb{R}^{2n+1}, 0 \leqslant \rho \leqslant 1\}$, then integrating the functions in (4.15) and (4.16) over $\mathbb{R}^{2n+1}$, we can obtain the following relation;

$$\Big(\frac{Q+\alpha}{p}+\varepsilon\Big)^p \int_{\mathbb{R}^{2n+1}} \rho^\alpha|\phi_\varepsilon|^p dw = \Big(\frac{Q+\alpha}{p}+\varepsilon\Big)^p \Big(\int_{B_1} \rho^{Q+2\alpha+p\varepsilon}dw + \int_{\mathbb{R}^{2n+1}\setminus B_1} \rho^{-Q-p\varepsilon}dw\Big) \tag{4.17}$$

Note that, for every $\varepsilon > 0$, the weights $\rho^{Q+2\alpha+p\varepsilon}$ and $\rho^{-Q-p\varepsilon}$ are integrable at 0 and ∞. This implies that $\int_{\mathbb{R}^{2n+1}} \rho^\alpha|\phi_\varepsilon|^p dw$ is finite. Thus we have we have

$$\begin{aligned} &\Big(\frac{Q+\alpha}{p}+\varepsilon\Big)^p \int_{\mathbb{R}^{2n+1}} \rho^\alpha|\phi_\varepsilon|^p dw \\ &= \Big(\frac{Q+\alpha}{p}+\varepsilon\Big)^p \Big[\int_{B_1} \rho^{Q+2\alpha+p\varepsilon}dz + \int_{\mathbb{R}^{2n+1}\setminus B_1} \rho^{-Q-p\varepsilon}dw\Big] \\ &= \int_{\mathbb{R}^{2n+1}} \rho^{\alpha+4kp-p}|z|^{2p(1-2k)}|\nabla_k\rho\cdot\nabla_k\phi|^p dw. \end{aligned}$$

On the other hand

$$\frac{\left(\frac{Q+\alpha}{p}+\varepsilon\right)^p}{C_H}\int_{\mathbb{R}^{2n+1}}\rho^{\alpha+4kp-p}|z|^{2p(1-2k)}|\nabla_k\rho\cdot\nabla_k\phi|^p dw$$
$$\geqslant\left(\frac{Q+\alpha}{p}+\varepsilon\right)^p\int_{\mathbb{R}^{2n+1}}\rho^{\alpha}|\phi_\varepsilon|^p dw=\int_{\mathbb{R}^{2n+1}}\rho^{\alpha+4kp-p}|z|^{2p(1-2k)}|\nabla_k\rho\cdot\nabla_k\phi|^p dw \tag{4.18}$$

As ε approaches 0, $\varepsilon\to 0$, (4.18) yields

$$\left(\frac{Q+\alpha}{p}+\varepsilon\right)^p\geqslant C_H\ . \tag{4.19}$$

From (4.13) and (4.19), we have, $C_H=\left(\frac{Q+\alpha}{p}\right)^p$. □

As we said in the introduction, Hardy-Poincaré inequality (4.1) gives us a short proof of the inequalities (1.6), (1.7) and (1.8) as it is shown in the following theorem:

THEOREM 4.2. ([3]) *Let $Q\geqslant 3$, $1<p<\infty$, $t\in\mathbb{R}$, $\alpha\in\mathbb{R}$ and $Q+\alpha-p>0$. Then the following inequality is valid for all compactly supported smooth functions $\phi\in C_0^\infty(\mathbb{R}^{2n+1})$:*

$$\int_\Omega\rho^\alpha|\nabla_k\rho|^{pt}|\nabla_k\phi|^p dw\geqslant\left(\frac{Q+\alpha-p}{p}\right)^p\int_\Omega\rho^\alpha|\nabla_k\rho|^{pt}\frac{|\nabla_k\rho|^p}{\rho^p}|\phi|^p dw. \tag{4.20}$$

Furthermore, the constant $(\frac{Q+\alpha-p}{p})^p$ is sharp.

Proof. To prove the theorem, we replace $|\nabla\rho|^{t+1}\phi$ instead of ϕ in the inequality (4.1) where $t\in\mathbb{R}$. Using the identity (2.11) and $|\nabla_k\rho\cdot\nabla_k(|\nabla_k\rho|^{t+1}\phi)|\leqslant|\nabla_k\rho|^{t+2}|\nabla_k\phi|$ we have,

$$\int_\Omega\rho^{\alpha+p}|\nabla_k\rho|^{pt}|\nabla_k\phi|^p dw\geqslant\left(\frac{Q+\alpha}{p}\right)^p\int_\Omega\rho^\alpha|\nabla_k\rho|^{pt+p}|\phi|^p dw\ . \tag{4.21}$$

Replacing $\alpha-p$ instead of α gives the desired inequality:

$$\int_\Omega\rho^\alpha|\nabla_k\rho|^{pt}|\nabla_k\phi|^p dw\geqslant\left(\frac{Q+\alpha-p}{p}\right)^p\int_\Omega\rho^\alpha|\nabla_k\rho|^{pt}\frac{|\nabla_k\rho|^p}{\rho^p}|\phi|^p dw\ . \tag{4.22}$$

□

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