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STECHKIN–MARCHAUD–TYPE INEQUALITIES OF WEIGHTED APPROXIMATION FOR BASKAKOV OPERATORS

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Abstract. In this paper, we introduce $\omega^2_{\varphi^\lambda}(f;t)_{w,\lambda}$ and prove a generalization of weighed Stechkin-Marchaud-type inequality for Baskakov operators, from which the inverse result of Baskakov operators with $\omega^2_{\varphi^\lambda}(f;t)_{w,\lambda}$ is obtained.

1. Introduction and main results

Let f be a function defined on the interval $[0,\infty)$. The Baskakov operator $V_n(f;x)$ is defined as follows

$$V_n(f;x) = \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) v_{n,k}(x), \tag{1}$$

where

$$v_{n,k}(x) = \binom{n+k-1}{k} x^k (1+x)^{-n-k},$$

which was first introduced by V. A. Baskakov in 1957 [1]. Z. Ditzian [3] studied the operators and verified the direct and converse theorems concerning them. Also by using a weighted norm, in [2, 9] M. Becker, P. C. Xun and D. X. Zhou obtained the direct theorem and a characterization for $V_n(f;x)$.

Due to the importance of Stechkin-Marchaud-type inequalities and strong converse inequalities for positive linear operators, in the past twenty years, many people have done their best to generalize them. E. van Wickeren [8] obtained the Stechkin-Marchaud-type inequalities for Bernstein polynomials. After that, Z. Ditzian and K. Ivanov [4] gave the strong converse inequality, and V. Toitik [6] using $\omega_\varphi(f;t)$ extended the Ditzian-Ivanov result to a large family of operators, for example, Szasz-Mirakjan, Baskakov operators. Recently, the Stechkin-Marchaud-type inequalities for Baskakov operators were given in [7]. However, it can be found that the previous consequences were proved with the help of modulus of smoothness $\omega^2(f;t)$ or $\omega^2_\varphi(f;t)$

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and $\omega_{\varphi}^2(f;t)_w$. Naturally, these motivate us to deal with the Stechkin-Marchaud-type inequalities based on the broadly-sensed weighted modulus of smoothness.

In what follows, the generalized weighted Stechkin-Marchaud-type inequality for Baskakov operators is established. Moreover, the inverse result of Baskakove operators with $\omega_{\varphi^\lambda}^2(f;t)_{w,\lambda}$ is obtained. It is easy to see that our result is more extensive. It unifies the results for $\omega_{\varphi}^2(f;t)$ and $\omega_{\varphi}^2(f;t)_w$.

First, we introduce some useful definitions and notations.

DEFINITION 1.1. Let $C[0,\infty)$ denote the set of bounded continuous function on the interval $[0,\infty)$, and let

$$C_{a,b,\lambda} = \left\{ f \mid f \in C[0,\infty), \varphi^{2(2-\lambda)} w f \in L_\infty[0,\infty) \right\},$$

$$C_{a,b,\lambda}^0 = \left\{ f \mid f \in C_{a,b,\lambda}, f(0) = 0 \right\},$$

where $\varphi(x) = \sqrt{x(1+x)}, w(x) = x^a(1+x)^{-b}, 0 \leqslant a < \lambda \leqslant 1, b \geqslant 0$.

Moreover, the weighted norm is given by $\|f\|_w = \|wf\|_\infty$, and the weighted modulus of smoothness by

$$\omega_{\varphi^\lambda}^2(f;t)_{w,\lambda} = \sup_{0 \leqslant h < t} \|\varphi^{2(1-\lambda)} \Delta_{h\varphi^\lambda}^2 f\|_w,$$

where $\Delta_h^2 f(x) = f(x+2h) - 2f(x+h) + f(x)$.

The K-functional is given by

$$K_{\varphi^\lambda}(f;t)_{w,\lambda} = \inf_{g \in D} \left\{ \|\varphi^{2(1-\lambda)}(f-g)\|_w + t\|\varphi^{2(2-\lambda)} g''\|_w \right\},$$

where $D = \left\{ g \mid g \in C_{a,b,\lambda}^0, g' \in A.C._{loc}, \|\varphi^{2(2-\lambda)} g''\|_w < \infty \right\}$.

We are now in a position to state our main results.

THEOREM 1.2. *Let $f \in C_{a,b,\lambda}^0$. Then*

$$K_{\varphi^\lambda}(f;n^{-1})_{w,\lambda} \leqslant Mn^{-1} \left(\sum_{k=1}^n \|\varphi^{2(1-\lambda)}(V_k(f)-f)\|_w + \|\varphi^{2(1-\lambda)} f\|_w \right). \tag{2}$$

THEOREM 1.3. *Let $f \in C_{a,b,\lambda}^0$. Then*

$$\omega_{\varphi^\lambda}^2(f;n^{-\frac{1}{2}}\varphi^{1-\lambda}(x))_{w,\lambda} \leqslant Mn^{-1} \left(\sum_{k=1}^n \|\varphi^{2(1-\lambda)}(V_k(f)-f)\|_w + \|\varphi^{2(1-\lambda)} f\|_w \right). \tag{3}$$

THEOREM 1.4. *Let $f \in C_{a,b,\lambda}^0, 0 < \alpha < 2$. Then*

$$\|\varphi^{2(1-\lambda)}(V_n(f)-f)\|_w = O(n^{-\alpha/2}) \Rightarrow \omega_{\varphi^\lambda}^2(f;t)_{w,\lambda} = O\left((t\varphi^{\lambda-1}(x))^\alpha \right). \tag{4}$$

Throughout this paper, M denotes a positive constant independent of x, n and f which may be different in different places.

2. Auxiliary Lemmas

To prove the theorems, we need the following Lemmas. By simple computation, we have

$$V_n''(f;x) = n(n+1)\sum_{k=0}^{\infty} v_{n+2,k}(x)\left(f\left(\frac{k+2}{n}\right) - 2f\left(\frac{k+1}{n}\right) + f\left(\frac{k}{n}\right)\right), \quad (5)$$

or

$$V_n''(f;x) = \sum_{k=0}^{\infty} v_{n,k}(x) f\left(\frac{k}{n}\right)\left(\frac{k(k-1)}{x^2} - \frac{2k(n+k)}{x(1+x)} + \frac{(n+k)(n+k+1)}{(1+x)^2}\right). \quad (6)$$

LEMMA 2.1. *Let* $c \geqslant 0, d \in \mathbb{R}$. *Then*

$$\sum_{k=1}^{\infty} v_{n,k}(x)\left(\frac{k}{n}\right)^{-c}\left(1+\frac{k}{n}\right)^{-d} \leqslant Mx^{-c}(1+x)^{-d}, \quad \textit{for } x > 0. \quad (7)$$

Proof. We notice [5]

$$\sum_{k=0}^{\infty} v_{n,k}(x)\left(\frac{n}{k+1}\right)^{l} \leqslant Mx^{-l}, \quad \text{for } l \in \mathbb{N},$$
$$\sum_{k=0}^{\infty} v_{n,k}(x)\left(1+\frac{k}{n}\right)^{m} \leqslant M(1+x)^{m}, \quad \text{for } m \in \mathbb{Z}.$$

For $c = 0,\ d = 0$, the result of (7) is obvious. For $c > 0,\ d \neq 0$, there exists $m \in \mathbb{Z}$, such that $0 < -\frac{2d}{m} < 1$, using the Hölder's inequality, we have

$$\begin{aligned}
&\sum_{k=1}^{\infty} v_{n,k}(x)\left(\frac{k}{n}\right)^{-c}\left(1+\frac{k}{n}\right)^{-d} \\
&\leqslant \left(\sum_{k=1}^{\infty} v_{n,k}(x)\left(\frac{k}{n}\right)^{-2c}\right)^{\frac{1}{2}}\left(\sum_{k=1}^{\infty} v_{n,k}(x)\left(1+\frac{k}{n}\right)^{-2d}\right)^{\frac{1}{2}} \\
&\leqslant \left(\sum_{k=1}^{\infty} v_{n,k}(x)\left(\frac{n}{k}\right)^{[2c]+1}\right)^{\frac{c}{[2c]+1}}\left(\sum_{k=1}^{\infty} v_{n,k}(x)\left(1+\frac{k}{n}\right)^{m}\right)^{-\frac{d}{m}} \\
&\leqslant M\left(x^{-([2c]+1)}\right)^{\frac{c}{[2c]+1}}\left((1+x)^{m}\right)^{-\frac{d}{m}} \leqslant Mx^{-c}(1+x)^{-d}. \quad (8)
\end{aligned}$$

For $c > 0,\ d = 0$ or $c = 0,\ d \neq 0$, the proof is similar to that of (8). Thus, this proof is complete. □

LEMMA 2.2. *Let* $f \in C^0_{a,b,\lambda}$, $n \in \mathbb{N}$. *Then*

$$\|\varphi^{2(2-\lambda)} V_n''(f)\|_w \leqslant Mn\|\varphi^{2(1-\lambda)} f\|_w. \quad (9)$$

Proof. For $x \in E_n^c = (0, 1/n]$, $(n+1)x(x+1) \leqslant 2n \cdot 2x \leqslant 4$, using (5) and Lemma 2.1 , we have

$$\begin{aligned}
&|w(x)\varphi^{2(2-\lambda)}(x)V_n''(f;x)| \\
\leqslant\ & w(x)\varphi^{2(1-\lambda)}(x)n(n+1)x(1+x)\left(\sum_{k=0}^{\infty} v_{n+2,k}(x)w^{-1}\left(\frac{k+2}{n}\right)\varphi^{-2(1-\lambda)}\left(\frac{k+2}{n}\right)\right. \\
&+2\sum_{k=0}^{\infty} v_{n+2,k}(x)w^{-1}\left(\frac{k+1}{n}\right)\varphi^{-2(1-\lambda)}\left(\frac{k+1}{n}\right) \\
&\left.+\sum_{k=1}^{\infty} v_{n+2,k}(x)w^{-1}\left(\frac{k}{n}\right)\varphi^{-2(1-\lambda)}\left(\frac{k}{n}\right)\right)\|\varphi^{2(1-\lambda)}f\|_w \\
\leqslant\ & Mnw(x)\varphi^{2(1-\lambda)}(x)w^{-1}(x)\varphi^{-2(1-\lambda)}(x)\|\varphi^{2(1-\lambda)}f\|_w \leqslant Mn\|\varphi^{2(1-\lambda)}f\|_w. \qquad (10)
\end{aligned}$$

For $x \in E_n = (1/n, \infty)$, by (6), we get

$$\begin{aligned}
&|w(x)\varphi^{2(2-\lambda)}(x)V_n''(f;x)| \\
=\ & \left|n^2 w(x)\varphi^{-2\lambda}(x)\sum_{k=1}^{\infty} v_{n,k}(x)f\left(\frac{k}{n}\right)\left(\left(\frac{k}{n}-x\right)^2 - \frac{1+2x}{n}\left(\frac{k}{n}-x\right) - \frac{x(1+x)}{n}\right)\right| \\
\leqslant\ & n^2 w(x)\varphi^{-2\lambda}(x)\|\varphi^{2(1-\lambda)}f\|_w \sum_{k=1}^{\infty} v_{n,k}(x)w^{-1}\left(\frac{k}{n}\right)\varphi^{-2(1-\lambda)}\left(\frac{k}{n}\right) \\
&\cdot\left(\left(\frac{k}{n}-x\right)^2 + \frac{1+2x}{n}\left|\frac{k}{n}-x\right| + \frac{x(1+x)}{n}\right) \\
\triangleq\ & n^2 w(x)\varphi^{-2\lambda}(x)\|\varphi^{2(1-\lambda)}f\|_w \big(I_1(n,x)+I_2(n,x)+I_3(n,x)\big). \qquad (11)
\end{aligned}$$

Note that for $x \in E_n$, one has the inequality [5]

$$n^{2m}V_n\big((t-x)^{2m};x\big) \leqslant Mn^m(\varphi(x))^{2m}, m \in \mathbb{N}.$$

Applying the Hölder's inequality and Lemma 2.1, we have

$$\begin{aligned}
I_1(n,x) &= \sum_{k=1}^{\infty} v_{n,k}(x)w^{-1}\left(\frac{k}{n}\right)\varphi^{-2(1-\lambda)}\left(\frac{k}{n}\right)\left(\frac{k}{n}-x\right)^2 \\
&\leqslant \left(\sum_{k=1}^{\infty} v_{n,k}(x)w^{-2}\left(\frac{k}{n}\right)\varphi^{-4(1-\lambda)}\left(\frac{k}{n}\right)\right)^{1/2}\left(\sum_{k=1}^{\infty} v_{n,k}(x)\left(\frac{k}{n}-x\right)^4\right)^{1/2} \\
&\leqslant Mx^{-a-1+\lambda}(1+x)^{b+\lambda-1}\frac{x(1+x)}{n} \leqslant Mn^{-1}w^{-1}(x)\varphi^{2\lambda}(x), \qquad (12)
\end{aligned}$$

and

$$I_2(n,x) = \sum_{k=1}^{\infty} v_{n,k}(x) w^{-1}\left(\frac{k}{n}\right)\varphi^{-2(1-\lambda)}\left(\frac{k}{n}\right)\left|\frac{k}{n}-x\right|\frac{1+2x}{n}$$
$$\leqslant \frac{1+2x}{n}\left(\sum_{k=1}^{\infty} v_{n,k}(x) w^{-2}\left(\frac{k}{n}\right)\varphi^{-4(1-\lambda)}\left(\frac{k}{n}\right)\right)^{1/2}\left(\sum_{k=1}^{\infty} v_{n,k}(x)\left(\frac{k}{n}-x\right)^2\right)^{1/2}$$
$$\leqslant Mw^{-1}(x)\varphi^{2\lambda}(x)n^{-3/2}(1+1/x)^{1/2}.$$

Note that for $x>1/n$, one has $1+1/x<2n$. Hence,

$$I_2(n,x) \leqslant Mn^{-1}w^{-1}(x)\varphi^{2\lambda}(x). \tag{13}$$

$$I_3(n,x) = \sum_{k=1}^{\infty} v_{n,k}(x) w^{-1}\left(\frac{k}{n}\right)\varphi^{-2(1-\lambda)}\left(\frac{k}{n}\right)\frac{x(1+x)}{n}$$
$$\leqslant Mn^{-1}x(1+x)x^{-a-1+\lambda}(1+x)^{b-1+\lambda} = Mn^{-1}w^{-1}(x)\varphi^{2\lambda}(x). \tag{14}$$

Combining (10)-(13), we get

$$|w(x)\varphi^{2(2-\lambda)}(x)V_n''(f;x)|_w \leqslant Mn\|\varphi^{2(1-\lambda)}f\|_w.$$

Thus,

$$\|\varphi^{2(2-\lambda)}V_n''(f)\|_w \leqslant Mn\|\varphi^{2(1-\lambda)}f\|_w.$$

The proof is complete. □

LEMMA 2.3. *Let* $f\in D$, $n\in\mathbb{N}$, $n\geqslant 2$. *Then*

$$\|\varphi^{2(2-\lambda)}V_n''(f)\|_w \leqslant M\|\varphi^{2(2-\lambda)}f''\|_w. \tag{15}$$

Proof. (1) For the case $\lambda\neq 1$ or $a\neq 0$. If $\lambda-2+b\geqslant 0$, using (5), and Lemma 2.1, we have

$$|w(x)\varphi^{2(2-\lambda)}(x)V_n''(f;x)|$$
$$= \left|w(x)\varphi^{2(2-\lambda)}(x)n(n+1)\sum_{k=0}^{\infty} v_{n+2,k}(x)\int_0^{\frac{1}{n}}\int_0^{\frac{1}{n}} f''\left(\frac{k}{n}+u+v\right)dudv\right|$$
$$\leqslant \|\varphi^{2(2-\lambda)}f''\|_w\left|w(x)\varphi^{2(2-\lambda)}(x)n(n+1)\sum_{k=0}^{\infty} v_{n+2,k}(x)\right.$$
$$\left.\cdot\int_0^{\frac{1}{n}}\int_0^{\frac{1}{n}}\left(\frac{k}{n}+u+v\right)^{\lambda-a-2}\left(1+\frac{k}{n}+u+v\right)^{\lambda+b-2}dudv\right|$$
$$\leqslant \|\varphi^{2(2-\lambda)}f''\|_w\left|w(x)\varphi^{2(2-\lambda)}(x)n(n+1)\sum_{k=1}^{\infty} v_{n+2,k}(x)\right.$$
$$\left.\cdot\int_0^{\frac{1}{n}}\int_0^{\frac{1}{n}}\left(\frac{k}{n}\right)^{\lambda-a-2}\left(1+\frac{k+2}{n}\right)^{\lambda+b-2}dudv\right|$$

$$+\|\varphi^{2(2-\lambda)}f''\|_w\Big|w(x)\varphi^{2(2-\lambda)}(x)n(n+1)v_{n+2,0}(x)$$
$$\cdot\int_0^{\frac{1}{n}}\int_0^{\frac{1}{n}}(u+v)^{\lambda-a-2}(1+u+v)^{\lambda+b-2}dudv\Big|$$
$$\leqslant \|\varphi^{2(2-\lambda)}f''\|_w w(x)\varphi^{2(2-\lambda)}(x)n(n+1)\sum_{k=1}^{\infty}v_{n+2,k}(x)n^{-2}\left(\frac{k}{n}\right)^{\lambda-a-2}\left(1+\frac{k+2}{n}\right)^{\lambda+b-2}$$
$$+2\cdot 3^{\lambda+b-2}\|\varphi^{2(2-\lambda)}f''\|_w w(x)\varphi^{2(2-\lambda)}(x)n(n+1)v_{n+2,0}(x)\int_0^{\frac{1}{n}}\frac{1}{1+a-\lambda}u^{\lambda-a-1}du$$
$$\leqslant 2\cdot 3^{\lambda+b-2}\|\varphi^{2(2-\lambda)}f''\|_w w(x)\varphi^{2(2-\lambda)}(x)\sum_{k=1}^{\infty}v_{n+2,k}(x)\left(\frac{k}{n}\right)^{\lambda-a-2}\left(1+\frac{k}{n}\right)^{\lambda+b-2}$$
$$+2\cdot 3^{\lambda+b-2}\|\varphi^{2(2-\lambda)}f''\|_w w(x)\varphi^{2(2-\lambda)}(x)n(n+1)v_{n+2,0}(x)\frac{1}{(1+a-\lambda)(\lambda-a)}\left(\frac{1}{n}\right)^{\lambda-a}$$
$$\leqslant 2\cdot 3^{\lambda+b-2}\|\varphi^{2(2-\lambda)}f''\|_w\left(1+\frac{x^{2+a-\lambda}n(n+1)}{(1+a-\lambda)(\lambda-a)(1+x)^n}\left(\frac{1}{n}\right)^{\lambda-a}\right). \tag{16}$$

(i) If $x\in E_n^c$,

$$\frac{x^{2+a-\lambda}n(n+1)}{(1+a-\lambda)(\lambda-a)(1+x)^n}\left(\frac{1}{n}\right)^{\lambda-a}\leqslant\frac{n(n+1)}{(1+a-\lambda)(\lambda-a)}\left(\frac{1}{n}\right)^2$$
$$\leqslant\frac{2}{(1+a-\lambda)(\lambda-a)}. \tag{17}$$

(ii) If $x\in E_n, n\geqslant 2$,

$$\frac{x^{2+a-\lambda}n(n+1)}{(1+a-\lambda)(\lambda-a)(1+x)^n}\left(\frac{1}{n}\right)^{\lambda-a}\leqslant\frac{n^{\lambda-a}x^2n(n+1)}{(1+a-\lambda)(\lambda-a)(1+x)^n}\left(\frac{1}{n}\right)^{\lambda-a}$$
$$\leqslant\frac{x^2n(n+1)}{(1+a-\lambda)(\lambda-a)n(n-1)x^2}$$
$$\leqslant\frac{3}{(1+a-\lambda)(\lambda-a)}. \tag{18}$$

Combining (16)-(18), we have

$$|w(x)\varphi^{2(2-\lambda)}(x)V_n''(f;x)|\leqslant M\|\varphi^{2(2-\lambda)}f''\|_w.$$

Thus,

$$\|\varphi^{2(2-\lambda)}(x)V_n''(f)\|_w\leqslant M\|\varphi^{2(2-\lambda)}f''\|_w. \tag{19}$$

If $\lambda-2+b<0$, we have

$$
\begin{aligned}
&|w(x)\varphi^{2(2-\lambda)}(x)V_n''(f;x)|\\
\leqslant\ &\|\varphi^{2(2-\lambda)}f''\|_w\left|w(x)\varphi^{2(2-\lambda)}(x)n(n+1)\sum_{k=1}^{\infty}v_{n+2,k}(x)\right.\\
&\left.\cdot\int_0^{\frac{1}{n}}\int_0^{\frac{1}{n}}\left(\frac{k}{n}\right)^{\lambda-a-2}\left(1+\frac{k}{n}\right)^{\lambda+b-2}dudv\right|\\
&+\|\varphi^{2(2-\lambda)}f''\|_w|w(x)\varphi^{2(2-\lambda)}(x)n(n+1)v_{n+2,0}(x)\int_0^{\frac{1}{n}}\int_0^{\frac{1}{n}}(u+v)^{\lambda-a-2}dudv|.
\end{aligned}
$$

By using the method similar to that of (16)-(19), it is not difficult to obtain same inequality as (19).

(2) For the case $\lambda=1, a=0$, the proof is similar to that of (1) and even simpler. Therefore the proof is complete. □

LEMMA 2.4. *For $f\in C^0_{a,b,\lambda}$, we have*

$$
\|\varphi^{2(2-\lambda)}V_n''(f)\|_w\leqslant M\left(\sum_{k=1}^{n}\|\varphi^{2(1-\lambda)}(V_k(f)-f)\|_w+\|\varphi^{2(1-\lambda)}f\|_w\right). \tag{20}
$$

Proof. Let $n\in\mathbb{N}$, $1\leqslant k\leqslant n$, by (9), (15), we have

$$
\begin{aligned}
n^{-1}\|\varphi^{2(2-\lambda)}V_n''(f)\|_w &\leqslant n^{-1}\|\varphi^{2(2-\lambda)}V_n''(V_k(f))\|_w+n^{-1}\|\varphi^{2(2-\lambda)}V_n''(V_k(f)-f)\|_w\\
&\leqslant M_2n^{-1}\|\varphi^{2(2-\lambda)}V_k''(f)\|_w+M_1\|\varphi^{2(1-\lambda)}(V_k(f)-f)\|_w\\
&\leqslant M_2\left(n^{-1}\|\varphi^{2(2-\lambda)}(V_k''-V_1'')(f)\|_w+n^{-1}\|\varphi^{2(2-\lambda)}V_1''(f)\|_w\right)\\
&\quad+M_1\|\varphi^{2(1-\lambda)}(V_k(f)-f)\|_w\\
&=M_2n^{-1}\|\varphi^{2(2-\lambda)}(V_k''-V_1'')(f)\|_w+M_1\|\varphi^{2(1-\lambda)}(V_k(f)-f)\|_w\\
&\quad+M_1M_2n^{-1}\|\varphi^{2(1-\lambda)}f\|_w.
\end{aligned}\tag{21}
$$

We write

$$
\|\varphi^{2(1-\lambda)}(V_q(f)-f)\|_w=\max_{1\leqslant k\leqslant n}\|\varphi^{2(1-\lambda)}(V_k(f)-f)\|_w.
$$

For $\|\varphi^{2(1-\lambda)}(V_q(f)-f)\|_w$, there exists $M_3>0$, such that

$$
\|\varphi^{2(1-\lambda)}(V_q(f)-f)\|_w\leqslant M_3\|\varphi^{2(1-\lambda)}(V_k(f)-f)\|_w
$$

for $1\leqslant k\leqslant n$ and $\|\varphi^{2(1-\lambda)}(V_k(f)-f))\|_w\neq 0$. Let

$$
\|\varphi^{2(2-\lambda)}V_l''(f)\|_w=\min_{1\leqslant k\leqslant n}\|\varphi^{2(2-\lambda)}V_k''(f)\|_w.
$$

We have

$$
\begin{aligned}
&M_2n^{-1}\|\varphi^{2(2-\lambda)}(V_k''-V_1'')(f)\|_w\\
\leqslant\ &M_2n^{-1}\|\varphi^{2(2-\lambda)}V_k''(f)\|_w+M_2n^{-1}\|\varphi^{2(2-\lambda)}V_1''(f)\|_w
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant M_2 n^{-1}\|\varphi^{2(2-\lambda)}V_k''(V_l(f)-f)\|_w + M_2 n^{-1}\|\varphi^{2(2-\lambda)}V_k''(V_l(f))\|_w \\
&\quad + M_1 M_2 n^{-1}\|\varphi^{2(1-\lambda)} f\|_w \\
&\leqslant M_1 M_2\|\varphi^{2(1-\lambda)}(V_l(f)-f)\|_w + M_2^2 n^{-1}\|\varphi^{2(2-\lambda)}V_l''(f)\|_w + M_1 M_2 n^{-1}\|\varphi^{2(1-\lambda)} f\|_w \\
&\leqslant M_1 M_2\|\varphi^{2(1-\lambda)}(V_q(f)-f)\|_w + M_2^2 n^{-1}\|\varphi^{2(2-\lambda)}V_1''(f)\|_w + M_1 M_2 n^{-1}\|\varphi^{2(1-\lambda)} f\|_w \\
&\leqslant M_1 M_2 M_3\|\varphi^{2(1-\lambda)}(V_k(f)-f)\|_w + M_1 M_2 (M_2+1) n^{-1}\|\varphi^{2(1-\lambda)} f\|_w . \qquad (22)
\end{aligned}
$$

By (21), (22), we have

$$
n^{-1}\|\varphi^{2(2-\lambda)}V_n''(f)\|_w \leqslant M\left(\|\varphi^{2(1-\lambda)}(V_k(f)-f)\|_w + n^{-1}\|\varphi^{2(1-\lambda)} f\|_w\right),
$$

where $M=\max\{M_1+M_1M_2M_3, M_1M_2(M_2+2)\}$. Therefore,

$$
\|\varphi^{2(2-\lambda)}V_n''(f)\|_w \leqslant M\left(\sum_{k=1}^{n}\|\varphi^{2(1-\lambda)}(V_k(f)-f)\|_w + \|\varphi^{2(1-\lambda)} f\|_w\right)
$$

The proof is complete. □

LEMMA 2.5. *If $h>0$, and either of*

$$
\text{(i)}\quad 0 < h\varphi^{\lambda}(x) < 1, \qquad \text{(ii)}\quad x \geqslant h\varphi^{\lambda}(x)
$$

is satisfied. Then

$$
\int_0^{h\varphi^{\lambda}}\int_0^{h\varphi^{\lambda}} \varphi^{-2(2-\lambda)}(x+u+v)w^{-1}(x+u+v)\,dudv \leqslant Mh^2\varphi^{-4(1-\lambda)}(x)w^{-1}(x). \quad (23)
$$

Proof. (1) For $\lambda-2+b\leqslant 0$, we have

$$
\begin{aligned}
&\int_0^{h\varphi^{\lambda}}\int_0^{h\varphi^{\lambda}} \varphi^{-2(2-\lambda)}(x+u+v)w^{-1}(x+u+v)\,dudv \\
&= \int_0^{h\varphi^{\lambda}}\int_0^{h\varphi^{\lambda}} (x+u+v)^{\lambda-2-a}(1+x+u+v)^{\lambda-2+b}\,dudv \\
&= \int_0^{h\varphi^{\lambda}}\int_0^{h\varphi^{\lambda}} x^{\lambda-2-a}(1+x)^{\lambda-2+b}\,dudv \leqslant Mh^2\varphi^{-4(1-\lambda)}(x)w^{-1}(x).
\end{aligned}
$$

(2) For $\lambda-2+b>0$, due to one has either of $x\geqslant h\varphi^{\lambda}(x)$ and $0<h\varphi^{\lambda}(x)<1$, thus

$$
\begin{aligned}
&\int_0^{h\varphi^{\lambda}}\int_0^{h\varphi^{\lambda}} \varphi^{-2(2-\lambda)}(x+u+v)w^{-1}(x+u+v)\,dudv \\
&\leqslant \int_0^{h\varphi^{\lambda}}\int_0^{h\varphi^{\lambda}} x^{\lambda-2-a}(1+x+2h\varphi^{\lambda}(x))^{\lambda-2+b}\,dudv \\
&\leqslant 3^{\lambda-2+b}h^2\varphi^{2\lambda}(x)x^{\lambda-2-a}(1+x)^{\lambda-2+b} \leqslant Mh^2\varphi^{-4(1-\lambda)}(x)w^{-1}(x).
\end{aligned}
$$

The proof is complete. □

3. Proofs of Theorems

Proof of Theorem 1.2. For $n \geqslant 2$, there exists $m \in \mathbb{N}$, such that $n/2 \leqslant m \leqslant n$, and

$$\|\varphi^{2(1-\lambda)}(V_m(f)-f)\|_w = \min_{\frac{n}{2}\leqslant k\leqslant n} \|\varphi^{2(1-\lambda)}(V_k(f)-f)\|_w,$$
$$\|\varphi^{2(1-\lambda)}(V_m(f)-f)\|_w \leqslant 2n^{-1} \sum_{\frac{n}{2}\leqslant k\leqslant n} \|\varphi^{2(1-\lambda)}(V_k(f)-f)\|_w.$$

Therefore, using the definition of $K_{\varphi^\lambda}(f;t)_{w,\lambda}$, and Lemma 2.4, we have

$$\begin{aligned}
&K_{\varphi^\lambda}(f;n^{-1})_{w,\lambda} \leqslant \|\varphi^{2(1-\lambda)}(V_m(f)-f)\|_w + n^{-1}\|\varphi^{2(2-\lambda)}V_m''(f)\|_w \\
&\leqslant 2n^{-1} \sum_{\frac{n}{2}\leqslant k\leqslant n} \|\varphi^{2(1-\lambda)}(V_k(f)-f)\|_w \\
&\quad + Mn^{-1}\left(\sum_{k=1}^{m} \|\varphi^{2(1-\lambda)}(V_k(f)-f)\|_w + \|\varphi^{2(1-\lambda)}f\|_w\right) \\
&\leqslant Mn^{-1}\left(\sum_{k=1}^{n} \|\varphi^{2(1-\lambda)}(V_k(f)-f)\|_w + \|\varphi^{2(1-\lambda)}f\|_w\right).
\end{aligned}$$

The proof is complete. □

Proof of Theorem 1.3. According to the definition of $K_{\varphi^\lambda}(f;n^{-1})_{w,\lambda}$, there exists $g \in D$, such that

$$\|\varphi^{2(1-\lambda)}(f-g)\|_w + n^{-1}\|\varphi^{2(2-\lambda)}g''\|_w \leqslant 2K_{\varphi^\lambda}(f;n^{-1})_{w,\lambda}. \tag{24}$$

On the other hand,

$$\begin{aligned}
&|\varphi^{2(1-\lambda)}(x)w(x)\Delta^2_{h\varphi^\lambda} f(x)| \\
&\leqslant |\varphi^{2(1-\lambda)}(x)w(x)\Delta^2_{h\varphi^\lambda}(f-g)(x)| + |\varphi^{2(1-\lambda)}(x)w(x)\Delta^2_{h\varphi^\lambda} g(x)|.
\end{aligned} \tag{25}$$

For the first term of (25), if one has either of $x \geqslant h\varphi^\lambda(x)$ and $0 < h\varphi^\lambda(x) < 1$, thus

$$\begin{aligned}
&|\varphi^{2(1-\lambda)}(x)w(x)\Delta^2_{h\varphi^\lambda}(f-g)(x)| \\
&\leqslant \varphi^{2(1-\lambda)}(x)w(x)\|\varphi^{2(1-\lambda)}(f-g)\|_w \Big(\varphi^{-2(1-\lambda)}(x+2h\varphi^\lambda(x))w^{-1}(x+2h\varphi^\lambda(x)) \\
&\quad +2\varphi^{-2(1-\lambda)}(x+h\varphi^\lambda(x))w^{-1}(x+h\varphi^\lambda(x)) + \varphi^{-2(1-\lambda)}(x)w^{-1}(x)\Big) \\
&\leqslant \left(\max\left\{3^{\lambda+b-1},1\right\} + 2\max\left\{2^{\lambda+b-1},1\right\} + 1\right)\|\varphi^{2(1-\lambda)}(f-g)\|_w \\
&\leqslant M\|\varphi^{2(1-\lambda)}(f-g)\|_w.
\end{aligned} \tag{26}$$

For the second term of (25), by Lemma 2.5, we have

$$
\begin{aligned}
&|\varphi^{2(1-\lambda)}(x)w(x)\Delta^2_{h\varphi^\lambda} g(x)| \\
=&\left|\varphi^{2(1-\lambda)}(x)w(x)\int_0^{h\varphi^\lambda}\int_0^{h\varphi^\lambda} g''(x+u+v)dudv\right| \\
\leqslant& M\|\varphi^{2(2-\lambda)}g''\|_w\left|\varphi^{2(1-\lambda)}(x)w(x)\int_0^{h\varphi^\lambda}\int_0^{h\varphi^\lambda} \varphi^{-2(2-\lambda)}(x+u+v)w^{-1}(x+u+v)dudv\right| \\
\leqslant& Mh^2\varphi^{-2(1-\lambda)}(x)\|\varphi^{2(2-\lambda)}g''\|_w. \qquad (27)
\end{aligned}
$$

Let $h \leqslant n^{-\frac{1}{2}}\varphi^{1-\lambda}(x)$.

(i) If $0<x<1$, then $0<h\varphi^\lambda(x)<1$ $(n\geqslant 2)$. (ii) If $x\geqslant 1$, then $x\geqslant h\varphi^\lambda(x)$ $(n\geqslant 2)$. Thus, if $h\leqslant n^{-\frac{1}{2}}\varphi^{1-\lambda}(x)$, by (25), (26) and (27), we have

$$
\begin{aligned}
&|\varphi^{2(1-\lambda)}(x)w(x)\Delta^2_{h\varphi^\lambda} f(x)| \\
\leqslant& M\left(\|\varphi^{2(1-\lambda)}(f-g)\|_w+n^{-1}\|\varphi^{2(2-\lambda)}g''\|_w\right).
\end{aligned}
$$

By the definition of $\omega^2_{\varphi^\lambda}(f;t)_{w,\lambda}$ and (24), we get

$$
\begin{aligned}
\omega^2_{\varphi^\lambda}(f;n^{-\frac{1}{2}}\varphi^{1-\lambda}(x))_{w,\lambda} &\leqslant M\left(\|\varphi^{2(1-\lambda)}(f-g)\|_w+n^{-1}\|\varphi^{2(2-\lambda)}g''\|_w\right) \\
&\leqslant MK_{\varphi^\lambda}(f;n^{-1})_{w,\lambda}. \qquad (28)
\end{aligned}
$$

Therefore, by Theorem 1.2, we have

$$
\omega^2_{\varphi^\lambda}(f;n^{-\frac{1}{2}}\varphi^{1-\lambda}(x))_{w,\lambda} \leqslant Mn^{-1}\left(\sum_{k=1}^n \|\varphi^{2(1-\lambda)}(V_k(f)-f)\|_w+\|\varphi^{2(1-\lambda)}f\|_w\right).
$$

This completes the proof of Theorem 1.3. □

Proof of Theorem 1.4. By Theorem 1.3, we have

$$
\begin{aligned}
\omega^2_{\varphi^\lambda}(f;n^{-\frac{1}{2}}\varphi^{1-\lambda}(x))_{w,\lambda} &\leqslant Mn^{-1}\left(\sum_{k=1}^n \|\varphi^{2(1-\lambda)}(V_k(f)-f)\|_w+\|\varphi^{2(1-\lambda)}f\|_w\right) \\
&\leqslant Mn^{-1}\left(\sum_{k=1}^n k^{-\alpha/2}+\|\varphi^{2(1-\lambda)}f\|_w\right)\leqslant Mn^{-\alpha/2}.
\end{aligned}
$$

Let $(n+1)^{-\frac{1}{2}}\varphi^{1-\lambda}(x)<t\leqslant n^{-\frac{1}{2}}\varphi^{1-\lambda}(x)$, we get

$$
\omega^2_{\varphi^\lambda}(f;t)_{w,\lambda}\leqslant M\omega^2_{\varphi^\lambda}(f;n^{-\frac{1}{2}}\varphi^{1-\lambda}(x))_{w,\lambda}\leqslant Mn^{-\alpha/2}\leqslant M(t\varphi^{\lambda-1}(x))^\alpha.
$$

This completes the proof of Theorem 1.4. □

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