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A LYAPUNOV–TYPE INEQUALITY FOR A TWO–TERM EVEN–ORDER DIFFERENTIAL EQUATION

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Abstract. In this note, a new inequality is obtained for a two-term even-order linear differential equation, which generalizes the well-known Lyapunov-type inequality for second order linear differential equations.

1. Introduction

In this paper, the well-known Lyapunov inequality for linear second-order differential equation

$$x'' + q(t)x = 0 \tag{1}$$

is generalized to a two-term even-order linear differential equation. It is well-known [5] that if $q \in C[a,b]$ and $x(t)$ is a nonzero solution of (1) such that $x(a) = x(b) = 0$, then the following Lyapunov inequality holds:

$$(b-a)\int_a^b |q(t)|dt > 4 \tag{2}$$

and the constant 4 is sharp, that is, it can not be replaced by a larger one.

This result applies to the study of various properties of solutions of differential equation (1) such as oscillation theory, disconjugacy and eigenvalue problems. There have been many proofs and generalizations of (2) such as to nonlinear second order equations, to delay differential equations, to higher order differential equations, to discrete differential equations and to Hamiltonian systems. See, for example, the references [1-14] and the references therein. But so far, few results have been achieved for the two-term even-order differential equations given below.

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2. Main results

THEOREM 1. *Consider the following* $2(n+1)$*-order linear differential equation:*

$$\left(r_n(t)\left(r_{n-1}(t)\left(\cdots\left(r_2(t)\left(r_1(t)x''\right)''\right)''\cdots\right)''\right)''\right)''+q(t)x=0, \tag{3}$$

where $r_k\in C^{2n-2k+2}([a,b],(0,+\infty))$, $\quad k=1,2,\cdots,n,$ $q\in C([a,b],\mathbb{R})$. *If* $x(t)$ *is a nonzero solution of (3) satisfying*

$$x_k(a)=x_k(b)=0, \quad k=0,1,2,\cdots,n, \tag{4}$$

where the functions $\{x_k\}$ *are defined as*

$$x_0=x,\ x_1=r_1(t)x_0'',\ x_2=r_2(t)x_1'',\cdots,x_n=r_n(t)x_{n-1}'', \tag{5}$$

then we have

$$(b-a)^{n+1}\cdot\left[\prod_{k=1}^{n}\int_a^b\frac{dt}{r_k(t)}\right]\cdot\int_a^b|q(t)|dt>4^{n+1}. \tag{6}$$

COROLLARY 1. *Assume that* $r_1(t)=\cdots=r_n(t)=r(t)\in C^{2n}([a,b],(0,+\infty))$, $q\in C([a,b],\mathbb{R})$. *If there exists a nonzero solution* $x(t)$ *of (3) satisfying* $x_k(a)=x_k(b)=0,\ k=0,1,2,\cdots,n$, *where*

$$x=x_0,\ x_1=r(t)x_0'',\ x_2=r(t)x_1'',\cdots,x_n=r(t)x_{n-1}'',$$

then

$$(b-a)^{n+1}\cdot\left[\int_a^b\frac{dt}{r(t)}\right]^n\cdot\int_a^b|q(t)|dt>4^{n+1}.$$

COROLLARY 2. *Assume that* $r_1(t)=r_2(t)=\cdots=r_n(t)\equiv 1,$ $q\in C([a,b],\mathbb{R})$. *Then (3) reduces to*

$$x^{(2n+2)}+q(t)x=0. \tag{7}$$

Assume that there exists a nonzero solution $x(t)$ *of (7) satisfying* $x^{(k)}(a)=x^{(k)}(b)=0$, $k=0,1,2,\cdots,n$. *Then*

$$(b-a)^{2n+1}\cdot\int_a^b|q(t)|dt>4^{n+1}.$$

REMARK 1. The paper of Reid [12] gives better results in this case, however, Reid's paper does not consider the more general equation like (3).

REMARK 2. If we define $r_0(t)\equiv 0$, then for $n=0$, (3) is reduced to (1) and the result of (6) becomes (2). Hence, the result of Theorem 1 is a nature generalization of the well-known Lyapunov inequality (2) for equation (1).

3. Proof of Theorem 1

We define functions $x_k, k=0,1,2,\cdots,n$ in (5) above, then (3) is equivalent to the following first order differential system:

$$\begin{aligned} x'' &= \tfrac{x_1}{r_1(t)}, \\ x_1'' &= \tfrac{x_2}{r_2(t)}, \\ &\cdots\cdots\cdots\cdots, \\ x_{n-1}'' &= \tfrac{x_n}{r_n(t)}, \\ x_n'' &= -q(t)x. \end{aligned} \tag{8}$$

Proof of Theorem 1. Since $x(a)=x(b)=0$ and $x(t)$ is a nonzero solution of (3), we see that there exists a $c \in (a, b)$ such that $x'(c)=0$ and $|x(c)| = \max_{t\in[a,b]} |x(t)| > 0$. We have therefore,

$$x(c) = \int_a^b G(c,s)x''(s)ds = \int_a^b \frac{G(c,s)x_1(s)}{r_1(s)}ds, \tag{9}$$

where

$$G(t,s) = \begin{cases} \frac{(t-a)(b-s)}{b-a}, \ a \leqslant t \leqslant s \leqslant b, \\ \frac{(s-a)(b-t)}{b-a}, \ a \leqslant s \leqslant t \leqslant b. \end{cases}$$

is the Green function for the boundary value problem $x''=0, \quad x(a)=x(b)=0$.

Since $x_1(a)=x_1(b)=0$, we get from (8),

$$x_1(s) = \int_a^b G(s,s_1)x_1''(s_1)ds_1 = \int_a^b \frac{G(s,s_1)x_2(s_1)}{r_2(s_1)}ds_1.$$

Similarly, for $k=2,3,\cdots,n-1$, we get

$$x_k(s) = \int_a^b G(s,\tau)x_k''(\tau)d\tau = \int_a^b \frac{G(s,\tau)x_{k+1}(\tau)}{r_{k+1}(\tau)}d\tau,$$

and

$$x_n(s) = \int_a^b G(s,\tau)x_n''(\tau)d\tau = -\int_a^b G(s,\tau)q(\tau)x(\tau)d\tau.$$

Combing the above discussion, we obtain

$$\begin{aligned} x(c) = &-\textstyle\int_a^b \frac{G(c,s)}{r_1(s)} \cdot \int_a^b \frac{G(s,s_1)}{r_2(s_1)} \cdots \\ &\cdot \textstyle\int_a^b \frac{G(s_{n-2},s_{n-1})}{r_n(s_{n-1})} \int_a^b G(s_{n-1},s_n)q(s_n)x(s_n)ds_n ds_{n-1}\cdots ds_1 ds, \end{aligned}$$

from which and the fact that $0 \leqslant G(t,s) \leqslant G(t,t)$, we obtain

$$\begin{aligned} |x(c)| &< |x(c)| \textstyle\int_a^b \frac{G(c,c)}{r_1(s)} \cdot \int_a^b \frac{G(c,c)}{r_2(s_1)} \cdots \cdot \int_a^b \frac{G(c,c)}{r_n(s_{n-1})} \int_a^b G(c,c)|q(s_n)|ds_n ds_{n-1}\cdots ds_1 ds \\ &= |x(c)|\,(G(c,c))^{n+1} \textstyle\int_a^b \frac{dt}{r_1(t)} \cdot \int_a^b \frac{dt}{r_2(t)} \cdots \int_a^b \frac{dt}{r_n(t)} \cdot \int_a^b |q(t)|dt. \end{aligned}$$

The strict inequality holds since the function $x(t)$ is not a constant. From $|x(c)| > 0$, we get

$$\int_a^b \frac{dt}{r_1(t)} \cdot \int_a^b \frac{dt}{r_2(t)} \cdots \int_a^b \frac{dt}{r_n(t)} \cdot \int_a^b |q(t)|dt > \frac{1}{(G(c,c))^{n+1}} \geqslant \frac{4^{n+1}}{(b-a)^{n+1}},$$

since $G(c,c) \leqslant G(\frac{a+b}{2}, \frac{a+b}{2}) = \frac{(b-a)^{n+1}}{4^{n+1}}$.

This finishes the proof of Theorem 1. □

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