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BOUNDS FOR LAPLACIAN GRAPH EIGENVALUES

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Abstract. Let G be a connected simple graph whose Laplacian eigenvalues are $0=\mu_n(G)\leqslant \mu_{n-1}(G)\leqslant\cdots\leqslant\mu_1(G)$. In this paper, we establish some upper and lower bounds for the algebraic connectivity and the largest Laplacian eigenvalue of G.

1. Introduction

Let $G=(V,E)$ be a simple graph with the vertex set $V=\{v_1,v_2,\cdots,v_n\}$ and edge set E. For $v_i\in V$, the degree of v_i and the average of the degrees of the vertices adjacent to v_i are denoted by d_i and m_i, respectively. If v_iv_j is an edge of G, then this can be shown by $i\sim j$. Also let N_i be the set of neighbours of v_i. The diameter of G is the maximum distance between any two vertices of G. Let $A(G)$ be the adjacency matrix of G and let $D(G)$ be the diagonal matrix of vertex degrees. The Laplacian matrix of G is $L(G)=D(G)-A(G)$. Clearly, $L(G)$ is a real symmetric matrix. Thus, its eigenvalues are nonnegative real numbers. Morever, since the sum of rows is 0, it is clear that 0 is the smallest eigenvalue of $L(G)$ with the all ones vector as an eigenvector. The eigenvalues of $L(G)$ are denoted by

$$\mu_1(G)\geqslant\mu_2(G)\geqslant\cdots\geqslant\mu_{n-1}(G)\geqslant\mu_n(G)=0.$$

It is easy to show that $\mu_{n-1}(G)=0$ if and only if G is not connected. The number $\mu_{n-1}(G)$ is known as the algebraic connectivity of the graph G which has a relation to the classical connectivity parameters of a graph G (the vertex connectivity $\upsilon(G)$ and the edge connectivity $\eta(G)$) ([5]).

For simplicity, we write $L(G)=L$ and $\mu_i(G)=\mu_i$, $i=1,\cdots,n$, when no confusion can arise. In addition, by the extremal non-trivial Laplacian eigenvalues we mean μ_{n-1} and μ_1.

In [1, Theorem 1, p. 143], it is proved that if μ_1 is an eigenvalue of L; then, $\mu_1\leqslant n$ and that the multiplicity of 0 equals the number of components of G.

The eigenvalues of the Laplacian matrix are important in the graph theory because they have a relation to numerous graph invariants, including connectivity, expanding property, isoperimetric number, maximum cut, independence number, genus, diameter,

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mean distance and bandwidth-type parameters of a graph (see for example, [3,10,11] and the references therein). In many applications one needs good lower bound and upper bound of extremal non-trivial Laplacian eigenvalues [3,10,11]. In addition to of the reference above, we may also refer to the remarkable paper [13] since the authors established the k-th largest Laplacian eigenvalue of a graph.

In this paper we always assume without loss of generality that G is a simple connected graph of order n. Firstly, we give the lower and upper bounds of the largest eigenvalue of L. Morever, we will give the upper and lower bounds for the k-th largest eigenvalue μ_k of L.

Among the known upper bounds for μ_1 are the following:

1. Anderson and Morley's bound [1] :

$$\mu_1 \leqslant \max\left\{d_i + d_j : i \sim j\right\} \tag{1}$$

2. Li and Zhang's bound [6]: If $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_n$ are the degrees of the vertices of G (here, we are not assuming that d_i is the degree of v_i), then

$$\mu_1 \leqslant 2 + \sqrt{(d_1 + d_2 - 2)(d_1 + d_3 - 2)}. \tag{2}$$

3. Another Li and Zhang's bound [6]: If r is right-hand side of (1), if $xy \in E$ is such that $d_x + d_y = r$ and if $s = \max\left\{d_i + d_j : ij \in E - \{xy\}\right\}$, then

$$\mu_1 \leqslant 2 + \sqrt{(r-2)(s-2)}. \tag{3}$$

4. Merris's bound [9]:

$$\mu_1 \leqslant \max\left\{d_i + m_i : i \in V\right\}. \tag{4}$$

5. In [7], Li and Zhang obtained the bound:

$$\mu_1 \leqslant \max\left\{\frac{d_u(d_u + m_u) + d_v(d_v + m_v)}{d_u + d_v} : uv \in E\right\}. \tag{5}$$

6. In [12], Rojo et al. obtained:

$$\mu_1 \leqslant \max\left\{d_i + d_j - \left|N_i \cap N_j\right| : 1 \leqslant i < j \leqslant n\right\}, \tag{6}$$

where d_i denotes the degree of v_i and $\left|N_i \cap N_j\right|$ is the number of common neighbors of v_i and v_j.

7. Das's bounds [4]:

$$\mu_1 \leqslant \max\left\{d_i + d_j - \left|N_i \cap N_j\right| : 1 \leqslant i < j \leqslant n, v_i v_j \in E\right\}, \tag{7}$$

where d_i denotes the degree of v_i and $\left|N_i \cap N_j\right|$ is the number of common neighbours of v_i and v_j. This upper bound for μ_1 does not exceed n.

8. Another Das's bound [4]:

$$\mu_1 \leqslant \max\left\{\sqrt{2\left(d_i^2 + d_i m_i^{'}\right)} : 1 \leqslant i \leqslant n\right\}, \tag{8}$$

where

$$m_i^{'} = \frac{\sum_j \left\{d_j - \left|N_i \cap N_j\right| : v_i v_j \in E\right\}}{d_i},$$

d_i denotes the degree of v_i and $\left|N_i \cap N_j\right|$ is the number of common neighbours of v_i and v_j.

2. Estimating the extremal non-trivial Laplacian eigenvalues

To obtain the upper and lower bounds for extremal non-trivial Laplacian eigenvalues, we need the following lemma and theorem.

LEMMA 1. [2] *Suppose that W, $\lambda = (\lambda_j) \in \mathbb{R}^n$ are nonzero column vectors, the $n \times n$ I_n is identity matrix and $e = (1, 1, \cdots, 1)^T$. Let $C = I_n - \frac{ee^T}{n}$, $m = \frac{\lambda^T e}{n}$ and $s^2 = \frac{\lambda^T C \lambda}{n}$. Then,*

1)
$$-s\sqrt{nW^TCW} \leqslant W^T\lambda - mW^Te = W^TC\lambda \leqslant s\sqrt{nW^TCW}.$$

2)
$$\sum_j (\lambda_j - \lambda_n)^2 = n\left[\, s^2 + (m - \lambda_n)^2\right] \quad and \quad \sum_j (\lambda_1 - \lambda_j)^2 = n\left[\, s^2 + (\lambda_1 - m)^2\right].$$

3)
$$\lambda_n \leqslant m - \frac{s}{\sqrt{n-1}} \leqslant m + \frac{s}{\sqrt{n-1}} \leqslant \lambda_1.$$

4) *Let $W = \frac{1}{k}\sum_{j=1}^{k} e_j - \frac{1}{n-r+1}\sum_{j=r}^{n} e_j$ with $k < r$. Then*

$$\langle W, W\rangle = \frac{1}{k} + \frac{1}{n-r+1}, \quad W^Te = 0 \quad and \quad W^TCW = W^TW,$$

where e_j is column vector whose j-th component is one and the other enries are zero.

The following result can also be found in [2].

THEOREM 2. [2] *Let A be an $n \times n$ complex matrix. A^* denotes the conjugate transpose of A. Let $B = AA^*$ with eigenvalues $\lambda_n(B) \leqslant \cdots \leqslant \lambda_1(B)$. Then*

$$m - s\sqrt{n-1} \leqslant \lambda_n(B) \leqslant m - \frac{s}{\sqrt{n-1}}$$

and

$$m+\frac{s}{\sqrt{n-1}} \leqslant \lambda_1(B) \leqslant m+s\sqrt{n-1},$$

where $m=\dfrac{trB}{n}$ *and* $s^2=\dfrac{trB^2}{n}-m^2$.

Now we are ready to show our main results.

THEOREM 3. *Let G be a simple graph. Then,*

$$\sqrt{m+\frac{s}{\sqrt{n-1}}} \leqslant \mu_1 \leqslant \sqrt{m+s\sqrt{n-1}}, \tag{9}$$

where

$$m=\frac{\sum\limits_{i=1}^{n} d_i\left(d_i+1\right)}{n}$$

and

$$s^2=\frac{\sum\limits_{i=1}^{n}\left(d_i^2+d_i\right)^2+2\sum\limits_{i<j,i\sim j}\left(d_i+d_j\right)\left(d_i+d_j-2\left|N_i\cap N_j\right|\right)+2\sum\limits_{i<j}\left|N_i\cap N_j\right|^2}{n}-m^2.$$

Proof. Clearly

$$trL^2=\sum_{i=1}^{n} d_i\left(d_i+1\right)$$

and

$$trL^4=\sum_{i=1}^{n}\left(d_i^2+d_i\right)^2+2\sum_{i<j,i\sim j}\left(d_i+d_j\right)\left(d_i+d_j-2\left|N_i\cap N_j\right|\right)+2\sum_{i<j}\left|N_i\cap N_j\right|^2.$$

Since L is a real symmetric matrix, we obtain the result from Theorem 2. □

THEOREM 4. *Let G be a simple graph order n, and let m, s^2 be as above. Suppose $\mu_{(k,r)}^2=\sum\limits_{j=k}^{r}\frac{\mu_j^2}{r-k+1}$. Then,*

$$m-s\sqrt{\frac{k-1}{n-k+1}} \leqslant \mu_{(k,r)}^2 \leqslant m+s\sqrt{\frac{n-r}{r}}. \tag{10}$$

If we take $r=k$, then we have

$$m-s\sqrt{\frac{k-1}{n-k+1}} \leqslant \mu_k^2 \leqslant m+s\sqrt{\frac{n-k}{k}}. \tag{11}$$

Proof. In Lemma 1-1), we take

$$W=\sum_{j=k}^{r}\frac{e_j}{r-k+1}. \tag{12}$$

Hence we write

$$W^Te=1. \tag{13}$$

Since $W^TCW=W^TW-\frac{e^TW}{n}=\langle W,W\rangle-\frac{1}{n}\langle e,W\rangle$, from (12) we obtain

$$W^TCW=\left\langle\sum_{j=k}^{r}\frac{e_j}{r-k+1},\sum_{j=k}^{r}\frac{e_j}{r-k+1}\right\rangle-\frac{1}{n}\left\langle e,\sum_{j=k}^{r}\frac{e_j}{r-k+1}\right\rangle \tag{14}$$

and so

$$\begin{aligned}&=\frac{1}{(r-k+1)^2}\sum_{j=k}^{r}\langle e_j,e_j\rangle-\frac{1}{n}\frac{1}{r-k+1}\sum_{j=k}^{r}\langle e,e_j\rangle\\&=\frac{1}{r-k+1}-\frac{1}{n}.\end{aligned}$$

Using the equalities (13) and (14), and by Lemma 1-1), we have

$$m-s\sqrt{\frac{n-r+k-1}{r-k+1}}\leqslant W^T\lambda\leqslant m+s\sqrt{\frac{n-r+k-1}{r-k+1}}.$$

Since $W^T\lambda=\mu^2_{(k,r)}$, we then write

$$m-s\sqrt{\frac{n-r+k-1}{r-k+1}}\leqslant \mu^2_{(k,r)}\leqslant m+s\sqrt{\frac{n-r+k-1}{r-k+1}}. \tag{15}$$

In the equality (15), if $k=1$,

$$m-s\sqrt{\frac{n-r}{r}}\leqslant \mu^2_{(1,r)}\leqslant m+s\sqrt{\frac{n-r}{r}}$$

and if $r=n$

$$m-s\sqrt{\frac{k-1}{n-k+1}}\leqslant \mu^2_{(k,n)}\leqslant m+s\sqrt{\frac{k-1}{n-k+1}}.$$

Consequently, we have

$$m-s\sqrt{\frac{k-1}{n-k+1}}\leqslant \mu^2_{(k,n)}\leqslant \mu^2_{(k,r)}\leqslant \mu^2_{(1,r)}\leqslant m+s\sqrt{\frac{n-r}{r}}. \qquad \square$$

COROLLARY 5. *Let G be a simple graph of order n. Let m and s^2 be as defined in Theorem 3. Then*

$$m-s\sqrt{\frac{n-2}{2}}\leqslant \mu^2_{n-1}\leqslant m+s\sqrt{\frac{1}{n-1}}. \tag{16}$$

REMARK 6. In [8], Lu et al. showed that

$$\mu_{n-1} \geqslant \frac{2n}{2+n(n-1)d-2md}, \tag{17}$$

where G is a connected simple graph of order n, size m and diameter d. The lower bounds of (16) and (17) are incomparable. However we can see some graphs that the lower bound (16) is better than (17) in some cases as in the following example.

EXAMPLE 7. Let $G=(V,E)$ with $V=\{1,2,3,4\}$ and edge set

$$E=\left\{\begin{array}{c}\{1,2\},\{1,3\},\{1,4\},\\ \{2,4\},\{3,4\}\end{array}\right\}.$$

For this graph, $\mu_{n-1}=2$ and, by (16), $\mu_{n-1}\geqslant 1.36$, while by (17), $\mu_{n-1}\geqslant 1.33$.

THEOREM 8. *Let G be a simple graph of order n and let s be as previously. Then, for $1\leqslant k\leqslant r\leqslant n$,*

$$\mu_k^2-\mu_r^2 \leqslant s\sqrt{n\left(\frac{1}{k}+\frac{1}{n-r+1}\right)}. \tag{18}$$

Proof. Define W as given in Lemma 1. For $k<r$, by Lemma 1-1), we write

$$-s\sqrt{nW^TCW} \leqslant W^T\lambda - mW^Te \leqslant s\sqrt{nW^TCW}. \tag{19}$$

On the other hand, we have

$$\begin{aligned} W^T\lambda &= \langle W,\lambda\rangle = \frac{1}{k}\sum_{j=1}^{k}\mu_j^2 - \frac{1}{n-r+1}\sum_{j=r}^{n}\mu_j^2 \\ &= \mu_{(1,k)}^2-\mu_{(r,n)}^2. \end{aligned}$$

Using (19), we get the result. □

As a consequence of the above theorem, we have the following corollary with the same assumptions on G and s:

COROLLARY 9.

$$\mu_1 \leqslant \sqrt{s\sqrt{2n}} \tag{20}$$

and

$$\mu_{n-1} \leqslant \sqrt{\frac{sn}{\sqrt{n-1}}}. \tag{21}$$

EXAMPLE 10. Let $G=(V,E)$ with $V=\{1,2,3,4,5,6,7,8\}$ and

$$E=\left\{\begin{array}{c}\{1,2\},\{1,3\},\{2,3\},\{2,4\},\{3,4\},\{2,5\},\\ \{2,6\},\{2,8\},\{4,5\},\{4,6\},\{4,8\},\{6,7\},\{7,8\}\end{array}\right\}.$$

EXAMPLE 11. For this graph, $\mu_1 = 7.1$. The upper bounds for μ_1 are as follows:

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(20)
G	11	9.93	9.93	9	9.05	8	8	9.38	7.86	8.36

The above table shows that in some cases, the bound (9) is the best among the above mentioned upper bounds for μ_1. But in a general sense, they are incomparable.

In the following, we will explain that why we did not choose to use Cauchy-Schwartz ineuality as a method in the proofs of our results although it gives the required bounds directly.

First, from Theorem 2, we get

$$s = \sqrt{\frac{trL^4}{n} - \left(\frac{trL^2}{n}\right)^2} = \sqrt{\frac{n\left(trL^4\right) - \left(trL^2\right)^2}{n}}$$
$$= \frac{\sqrt{\sum_{1 \leqslant i < j \leqslant n} \left(\mu_i^2 - \mu_j^2\right)^2}}{n}.$$

Therefore, inequality (9) is equivalent to

$$\mu_1 \leqslant \sqrt{\frac{\sum_{i=1}^n \mu_i^2}{n} + \frac{\sqrt{(n-1)\sum_{1 \leqslant i < j \leqslant n} \left(\mu_i^2 - \mu_j^2\right)^2}}{n}}$$

which is the same as

$$\left[(n-1)\mu_1^2 - \sum_{j=2}^n \mu_j^2\right]^2 \leqslant (n-1) \sum_{1 \leqslant i < j \leqslant n} \left(\mu_i^2 - \mu_j^2\right)^2.$$

This follows immediately from the inequality

$$\left[\sum_{j=2}^n \left(\mu_1^2 - \mu_j^2\right)\right]^2 \leqslant (n-1) \sum_{j=2}^n \left(\mu_1^2 - \mu_j^2\right)^2$$

which is true by Cauchy-Schwartz (C-S) inequality. In fact, by this approximation, it is quite easy to see that equality happens if and only if $\mu_2 = \cdots = \mu_{n-1}$.

Secondly, the similar numerical approximations can also be applied to the inequality in (19). In fact, the inequality in (19) is equivalent to

$$\frac{n}{2}\mu_1^4 \leqslant \sum_{1 \leqslant i < j \leqslant n} \left(\mu_i^2 - \mu_j^2\right)^2.$$

This inequality follows from the fact that $\left(\mu_1^2 - \mu_n^2\right)^2 = \mu_1^4$ and $\left(\mu_1^2 - \mu_j^2\right)^2 + \left(\mu_j^2 - \mu_n^2\right)^2 \geqslant \frac{\mu_1^4}{2}$ for $2 \leqslant j \leqslant n$. Similarly, one can obtains inequality (19) by C-S inequality. Here, the equality holds if and only if $\mu_j = \frac{\mu_1}{2}$ for each $2 \leqslant j \leqslant n$.

REMARK 12. Although C-S inequality seems easier than our method (used in this paper), our choice here has more advantages than C-S inequality even the bounds obtained for the Laplacian eigenvalues of G are very complicated in all our results. For instance, if we did prefer C-S inequality in this paper, then we would not have a relation between graph invariants (μ_1, degree of vertices etc.) since our bounds depend not only on the degree sequence of G, but also on the quantities $|N_i \cap N_j|$ for each $i \neq j$, where N_i denotes the set of neighbours of the vertex i of G.

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