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SOLVING THE MATRIX INEQUALITY $AXB + (AXB)^* \geqslant C$

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Abstract. A pair of complex Hermitian matrices A and B of the same size are said to satisfy an inequality $A \geqslant B$ in the Löwner partial ordering if $A - B$ is nonnegative definite. In this note, we first derive the general solutions in closed-form for the linear matrix equation $AXB + (AXB)^* = C$ by using generalized inverses of matrices, and then derive general solutions of the linear matrix inequality $AXB + (AXB)^* \geqslant C$ when C is a Hermitian nonnegative definite matrix.

1. Introduction

Throughout this note, $\mathbb{C}^{m\times n}$ and $\mathbb{C}_{\mathrm{H}}^{m}$ stand for the sets of all $m \times n$ complex matrices and all $m \times m$ complex Hermitian matrices, respectively. The symbols A^*, $r(A)$ and $\mathscr{R}(A)$ stand for the conjugate transpose, rank and range (column space) of a matrix $A \in \mathbb{C}^{m\times n}$, respectively; I_m denotes the identity matrix of order m; $[A, B]$ denotes a row block matrix consisting of A and B. The inertia of a Hermitian matrix A is defined to be the triplet $\mathrm{In}(A) = \{ i_+(A), i_-(A), i_0(A) \}$, where $i_+(A)$, $i_-(A)$ and $i_0(A)$ are the numbers of positive, negative and zero eigenvalues of A counted with multiplicities, respectively. We write $A \geqslant 0$ $(A > 0)$ if A is Hermitian nonnegative definite (positive definite). Two Hermitian matrices A and B of the same size are said to satisfy the inequality $A \geqslant B$ $(A > B)$ in the Löwner partial ordering if $A - B$ is nonnegative definite (positive definite). For a matrix $A \in \mathbb{C}^{m\times m}$, the matrix $\mathscr{H}(A) = (A + A^*)/2$ is called the Hermitian part of A. The matrix A is said to be Re-nonnegative definite if $\mathscr{H}(A) \geqslant 0$. The Moore–Penrose inverse of $A \in \mathbb{C}^{m\times n}$, denoted by $A^\dagger$, is defined to be the unique solution X satisfying the four matrix equations

$$\text{(i) } AXA = A, \quad \text{(ii) } XAX = X, \quad \text{(iii) } (AX)^* = AX, \quad \text{(iv) } (XA)^* = XA.$$

Further, the symbols E_A and F_A stand for the two orthogonal projectors $E_A = I_m - AA^\dagger$ and $F_A = I_n - A^\dagger A$. Their ranks are given by $r(E_A) = m - r(A)$ and $r(F_A) = n - r(A)$. Results on the Moore–Penrose inverse can be found, e.g., in [1, 2, 7].

The Löwner partial ordering is one of the most basic concepts for characterizing relations between two complex Hermitian (real symmetric) matrices. A challenging research topic on Hermitian matrices is to solve matrix inequalities induced from the Löwner partial ordering, which can generally be stated as:

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PROBLEM. For a given matrix function $f(X)$ that satisfies $f(X) = f^*(X)$, establish necessary and sufficient conditions for the matrix inequalities

$$f(X) \geqslant 0, \ \ f(X) > 0, \ \ f(X) \leqslant 0, \ \ f(X) < 0 \tag{1.1}$$

to be feasible, respectively, and find solutions X of the matrix inequalities.

When the $f(X)$ in (1.1) is a linear matrix function, for instance, $f(X) = A - BXB^*$ and $f(X) = A - BX - (BX)^*$, it is usually called linear matrix inequalities (LMI) in the literature. Recall that for a Hermitian matrix A of order m, $A > 0$ $(A < 0)$ if and only if $i_+(A) = m$ $(i_-(A) = m)$; $A \geqslant 0$ $(A \leqslant 0)$ if and only if $i_-(A) = 0$ $(i_+(A) = 0)$. Hence, it is possible to characterize the solvability of a matrix inequality in the Löwner partial ordering by using the inertia of Hermitian matrix. In recent papers [17, 18], Tian established some closed-form formulas for calculating the global maximum and minimum ranks and inertias of the two linear matrix functions $A - BXB^*$ and $A - BX - (BX)^*$ with respect to a variable matrix X, and used these formulas to characterize the existence of solutions of the following LMIs:

$$BXB^* > A \ (\geqslant A, \ < A, \ \leqslant A), \quad BX + (BX)^* > A \ (\geqslant A, \ < A, \ \leqslant A).$$

As an extension, we solve in this note the following LMI:

$$AXB + (AXB)^* \geqslant C, \tag{1.2}$$

where $A \in \mathbb{C}^{m \times p}$, $B \in \mathbb{C}^{q \times m}$ and $0 \leqslant C \in \mathbb{C}_{\mathrm{H}}^{m}$ are given, and $X \in \mathbb{C}^{p \times q}$ is a variable matrix. This inequality may occur in the investigation of Hermitian parts and Re-nonnegative definiteness of triple matrix products. For example, $\mathscr{H}(AXB) \geqslant C$ is equivalent to $AXB + (AXB)^* \geqslant 2C$. Some previous work on the solvability of the LMI in (1.2) and its special cases can be found in [3, 6, 8, 15], while the work on Re-nonnegative definiteness of a complex matrix and its applications can be found in [4, 9, 19, 20].

Matrix equations and matrix inequalities have been main objects of study in matrix theory and applications. Generalized inverses, ranks, inertias and ranges of matrices were successfully used to represent solvability conditions and general solutions of matrix equations and inequalities. The following are some known results on matrix equations, which will be used in the latter part of this note.

LEMMA 1.1. ([10]) *Let $A, B \in \mathbb{C}^{m \times n}$ be given. Then,*

(a) *There exists an $X \in \mathbb{C}_{\mathrm{H}}^{n}$ such that*

$$AX = B \tag{1.3}$$

if and only if $\mathscr{R}(B) \subseteq \mathscr{R}(A)$ and $AB^* = BA^*$. In this case, the general Hermitian solution of (1.3) can be written in the following parametric form

$$X = A^{\dagger}B + (A^{\dagger}B)^* - A^{\dagger}BA^{\dagger}A + F_AWF_A, \tag{1.4}$$

where $W \in \mathbb{C}_{\mathrm{H}}^{n}$ is arbitrary.

(b) *There exists an* $X \in \mathbb{C}^{n\times n}$ *such that*

$$AXX^* = B \tag{1.5}$$

if and only if $\mathscr{R}(B) \subseteq \mathscr{R}(A)$, $AB^* = BA^* \geqslant 0$ *and* $r(AB^*) = r(B)$. *In this case, the general solution of* (1.5) *can be written in the following parametric form*

$$XX^* = B^*(AB^*)^{\dagger}B + F_A WW^* F_A, \tag{1.6}$$

where $W \in \mathbb{C}^{n\times n}$ *is arbitrary.*

LEMMA 1.2. ([14]) *Let* $A \in \mathbb{C}^{m\times p}$, $B \in \mathbb{C}^{q\times n}$ *and* $C \in \mathbb{C}^{m\times n}$ *be given. Then, there exists an* $X \in \mathbb{C}^{p\times q}$ *such that*

$$AXB = C \tag{1.7}$$

if and only if $\mathscr{R}(C) \subseteq \mathscr{R}(A)$ *and* $\mathscr{R}(C^*) \subseteq \mathscr{R}(B^*)$. *In this case, the general solution of* (1.7) *can be written as*

$$X = A^{\dagger}CB^{\dagger} + F_A W_1 + W_2 E_B, \tag{1.8}$$

where $W_1, W_2 \in \mathbb{C}^{p\times q}$ *are arbitrary.*

LEMMA 1.3. *Let* $A_1 \in \mathbb{C}^{m\times p}$, $B_1 \in \mathbb{C}^{q\times n}$, $A_2 \in \mathbb{C}^{m\times r}$, $B_2 \in \mathbb{C}^{s\times n}$ *and* $C \in \mathbb{C}^{m\times n}$ *be given. Then,*

(a) [13] *There exist* $X \in \mathbb{C}^{p\times q}$ *and* $Y \in \mathbb{C}^{r\times s}$ *such that*

$$A_1XB_1 + A_2YB_2 = C \tag{1.9}$$

if and only if the following four rank equalities

$$r[C, A_1, A_2] = r[A_1, A_2], \quad r\begin{bmatrix} C \\ B_1 \\ B_2 \end{bmatrix} = r\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \tag{1.10}$$

$$r\begin{bmatrix} C & A_1 \\ B_2 & 0 \end{bmatrix} = r(A_1) + r(B_2), \quad r\begin{bmatrix} C & A_2 \\ B_1 & 0 \end{bmatrix} = r(A_2) + r(B_1) \tag{1.11}$$

hold, or equivalently,

$$[A_1, A_2][A_1, A_2]^{\dagger}C = C, \quad C\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}^{\dagger}\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = C, \quad E_{A_1}CF_{B_2} = 0, \quad E_{A_2}CF_{B_1} = 0. \tag{1.12}$$

(b) [16] *Under* (1.10) *and* (1.11), *the general solutions of* (1.9) *can be decomposed as*

$$X = X_0 + X_1X_2 + X_3 \quad \text{and} \quad Y = Y_0 - Y_1Y_2 + Y_3, \tag{1.13}$$

where X_0 *and* Y_0 *are a pair of special solutions of* (1.9), X_1, X_2, X_3 *and* Y_1, Y_2, Y_3 *are the general solutions of the following four homogeneous matrix equations*

$$A_1X_1 + A_2Y_1 = 0, \quad X_2B_1 + Y_2B_2 = 0, \quad A_1X_3B_1 = 0, \quad A_2Y_3B_2 = 0. \tag{1.14}$$

By using generalized inverses of matrices, (1.13) can be written in the following parametric forms

$$X = X_0 + [I_p, 0]F_G W E_H \begin{bmatrix} I_q \\ 0 \end{bmatrix} + F_{A_1} W_1 + W_2 E_{B_1}, \tag{1.15}$$

$$Y = Y_0 - [0, I_r]F_G W E_H \begin{bmatrix} 0 \\ I_s \end{bmatrix} + F_{A_2} W_3 + W_4 E_{B_2}, \tag{1.16}$$

where $G = [A_1, A_2]$, $H = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$, *the five matrices* W, W_1, W_2, W_3 *and* W_4 *are arbitrary.*

LEMMA 1.4. ([12]) *Let* $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$ *and* $C \in \mathbb{C}^{l\times n}$. *Then, the following rank expansion formulas hold*

$$r[A, B] = r(A) + r(E_A B) = r(B) + r(E_B A), \tag{1.17}$$

$$r\begin{bmatrix} A \\ C \end{bmatrix} = r(A) + r(CF_A) = r(C) + r(AF_C), \tag{1.18}$$

$$r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = r(B) + r(C) + r(E_B A F_C), \tag{1.19}$$

$$r\begin{bmatrix} AA^* & B \\ B^* & 0 \end{bmatrix} = r[A,\ B] + r(B). \tag{1.20}$$

2. General solution of $AXB + (AXB)^* \geqslant C$

We first solve the matrix equation

$$AXB + (AXB)^* = C, \tag{2.1}$$

where $A \in \mathbb{C}^{m\times p}$, $B \in \mathbb{C}^{q\times m}$ and $C \in \mathbb{C}_{\mathrm{H}}^{m}$ are given. Using the notation for Hermitian part of a matrix, (2.1) can be rewritten as

$$\mathscr{H}(AXB) = C/2. \tag{2.2}$$

Eq. (2.1) and its applications in control theory were studied by some authors; see, e.g., [5, 21].

THEOREM 2.1.

(a) [21] *There exists an* $X \in \mathbb{C}^{p\times q}$ *such that* (2.1) *holds if and only if*

$$\mathscr{R}(C) \subseteq \mathscr{R}[A,\, B^{*}], \;\; r\begin{bmatrix} C & A \\ A^{*} & 0 \end{bmatrix} = 2r(A), \;\; r\begin{bmatrix} C & B^{*} \\ B & 0 \end{bmatrix} = 2r(B), \tag{2.3}$$

or equivalently,

$$[A,\, B^{*}][A,\, B^{*}]^{\dagger}C = C, \;\; E_ACE_A = 0, \;\; F_BCF_B = 0. \tag{2.4}$$

(b) *Under* (2.3)*, the general solution of* (2.1) *can be written as*

$$X = \frac{1}{2}(U+V^{*}), \tag{2.5}$$

where U *and* V *are general solutions of the equation* $AUB+B^{*}VA^{*}=C$*, or can explicitly be written as*

$$X = X_0 + [I_p,\, 0]F_GWE_H\begin{bmatrix} I_q \\ 0 \end{bmatrix} - [0,\, I_p]E_HW^{*}F_G\begin{bmatrix} 0 \\ I_q \end{bmatrix} + F_AW_1 + W_2E_B, \tag{2.6}$$

where X_0 *is a special solution of* (2.1)*,* $G=[A,\, B^{*}]$*,* $H=\begin{bmatrix} B \\ A^{*} \end{bmatrix}$*, and the three matrices* $W \in \mathbb{C}^{(p+q)\times(p+q)}$ *and* $W_1,\, W_2 \in \mathbb{C}^{p\times q}$ *are arbitrary.*

(c) *The solution* X *of* (2.1) *is unique if and only if* $r(A)=p$*,* $r(B)=q$ *and* $\mathscr{R}(A)\cap\mathscr{R}(B^{*})=\{0\}$*.*

(d) *The matrix* AXB *satisfying* (2.1) *is unique if and only if* $\mathscr{R}(A)\cap\mathscr{R}(B^{*})=\{0\}$*.*

Proof. If (2.1) is consistent, then $AXB+B^{*}YA^{*}=C$ is consistent as well. Hence, the following three rank equalities

$$r[C,\, A,\, B^{*}] = r[A,\, B^{*}], \;\; r\begin{bmatrix} C & A \\ A^{*} & 0 \end{bmatrix} = r(A)+r(A^{*}), \;\; r\begin{bmatrix} C & B^{*} \\ B & 0 \end{bmatrix} = r(B)+r(B^{*})$$

hold by Lemma 1.3(a), which are further equivalent to (2.3) and (2.4) by Lemma 1.4. Conversely, if (2.3) holds, there exist U and V such that $AUB+B^{*}VA^{*}=C$ by Lemma 1.3(a). Taking the conjugate transpose of this equality gives $B^{*}U^{*}A^{*}+AV^{*}B=C$. Adding these two equalities and dividing by 2 yield

$$A\left(\frac{U+V^{*}}{2}\right)B + B^{*}\left(\frac{U+V^{*}}{2}\right)^{*}A^{*} = C.$$

This equality implies that for any pair of solutions of $AUB+B^{*}VA^{*}=C$, (2.5) is a solution of (2.1). Moreover, assume that X_0 is any solution of (2.1). Then, $AUB+B^{*}VA^{*}=C$ has a pair of solutions $U=V^{*}=X_0$. Thus, X_0 can be rewritten as

$$X_0 = \frac{1}{2}(X_0+X_0) = \frac{1}{2}(U+V^{*}).$$

This expression implies that (2.5) is the general solution of (2.1). From Lemma 1.3(b), the general solution of $AUB + B^*VA^* = C$ can be written as

$$U = U_0 + [I_p, 0]F_G W E_H \begin{bmatrix} I_q \\ 0 \end{bmatrix} + F_A W_1 + W_2 E_B, \tag{2.7}$$

$$V = V_0 - [0, I_q]F_G W E_H \begin{bmatrix} 0 \\ I_p \end{bmatrix} + F_{B^*} W_3 + W_4 E_{A^*}, \tag{2.8}$$

where U_0 and V_0 are a pair of special solutions of $AUB + B^*VA^* = C$, $G = [A, B^*]$, $H = \begin{bmatrix} B \\ A^* \end{bmatrix}$, and the five matrices W, W_1, W_2, W_3 and W_4 are arbitrary. It is easy to verify that

$$E_{A^*} = I_p - A^*(A^*)^\dagger = I_p - A^\dagger A = F_A, \;\; F_{B^*} = I_q - (B^*)^\dagger B^* = I_q - BB^\dagger = E_B.$$

Substituting (2.7) and (2.8) into (2.5) yields

$$X = \frac{1}{2}(U + V^*) = \frac{1}{2}(U_0 + V_0^*) + \frac{1}{2}[I_p, 0]F_G W E_H \begin{bmatrix} I_q \\ 0 \end{bmatrix}$$
$$- \frac{1}{2}[0, I_p]E_H W^* F_G \begin{bmatrix} 0 \\ I_q \end{bmatrix} + \frac{1}{2}F_A(W_1 + W_4^*) + \frac{1}{2}(W_2 + W_3^*)E_B,$$

which can equivalently be represented as (2.6) due to the arbitrariness of W, W_1, W_2, W_3 and W_4. Results (c) and (d) follow from applying (1.17) and (1.18) to the coefficient matrices of W, W_1 and W_2 in (2.6). □

A special case of Theorem 2.1 for $C \geqslant 0$ in (2.1) is given below.

COROLLARY 2.2. *Let $A \in \mathbb{C}^{m\times p}$, $B \in \mathbb{C}^{q\times m}$ and $C \in \mathbb{C}^{m\times m}$ be given, and define $G = [A, B^*]$ and $H = [B^*, A]^*$. Then, the following three statements are equivalent:*

(a) *There exists an $X \in \mathbb{C}^{p\times q}$ such that*

$$AXB + (AXB)^* = CC^*. \tag{2.9}$$

(b) *There exists a $Y \in \mathbb{C}^{p\times q}$ such that*

$$AYB = CC^*. \tag{2.10}$$

(c) *$\mathscr{R}(C) \subseteq \mathscr{R}(A)$ and $\mathscr{R}(C) \subseteq \mathscr{R}(B^*)$ or equivalently, $E_A CC^* = 0$ and $F_B CC^* = 0$.*

In this case, the general solution of (2.9) can be written as

$$X = \frac{1}{2}A^\dagger CC^* B^\dagger + [I_p, 0]F_G W E_H \begin{bmatrix} I_q \\ 0 \end{bmatrix} - [0, I_p]E_H W^* F_G \begin{bmatrix} 0 \\ I_q \end{bmatrix} + F_A W_1 + W_2 E_B, \tag{2.11}$$

where $W \in \mathbb{C}^{(p+q)\times(p+q)}$ and $W_1, W_2 \in \mathbb{C}^{p\times q}$ are arbitrary, or equivalently,

$$X = \frac{1}{2}Y + Z, \tag{2.12}$$

where Y is the general solution of (2.10), and Z is the general solution of $AZB + (AZB)^* = 0$.

Proof. Replacing C with CC^* in the latter two rank equalities in (2.3) and applying (1.20), we obtain the two reduced rank equalities $r[A, C] = r(A)$ and $r[B^*, C] = r(B)$, which are obviously equivalent to $\mathscr{R}(C) \subseteq \mathscr{R}(A)$ and $\mathscr{R}(C) \subseteq \mathscr{R}(B^*)$. Thus, we obtain the equivalence of (a), (b) and (c) from Lemma 1.2 and Theorem 2.1. In this case, $AA^{\dagger}CC^*B^{\dagger}B = CC^*$ holds, which means that $\frac{1}{2}A^{\dagger}CC^*B^{\dagger}$ is a special solution of (2.9), so that (2.6) can be written as (2.11). Also by Lemma 1.2, the general solution of (2.10) can be written as $Y = A^{\dagger}CC^*B^{\dagger} + F_AW_1 + W_2E_B$. Comparing this formula with (2.11) leads to (2.12). □

Under $C \geqslant 0$, we write (1.2) equivalently as

$$AXB + (AXB)^* \geqslant CC^*. \tag{2.13}$$

It was recently shown in [11] that

$$\max_{X\in\mathbb{C}^{p\times q}} i_+[CC^* - AXB - (AXB)^*] = \min\{ r[A, C], \ r[B^*, C] \}, \tag{2.14}$$

$$\max_{X\in\mathbb{C}^{p\times q}} i_-[CC^* - AXB - (AXB)^*] = \min\{ r(A), \ r(B) \}, \tag{2.15}$$

$$\min_{X\in\mathbb{C}^{p\times q}} i_+[CC^* - AXB - (AXB)^*] = \max\{ r[A, C] - r(A), \ r[B^*, C] - r(B) \}, \tag{2.16}$$

$$\min_{X\in\mathbb{C}^{p\times q}} i_-[CC^* - AXB - (AXB)^*] = 0. \tag{2.17}$$

These formulas enable us to derive necessary and sufficient conditions for (2.13) to have a solution.

THEOREM 2.3. *Let $A \in \mathbb{C}^{m\times p}, B \in \mathbb{C}^{q\times m}$ and $C \in \mathbb{C}^{m\times m}$ be given, and define $M = [E_A, F_B], \ G = [A, B^*]$ and $H = [B^*, A]^*$. Then,*

(a) (i) *There exists an $X \in \mathbb{C}^{p\times q}$ that satisfies (2.13) if and only if*

$$\mathscr{R}(C) \subseteq \mathscr{R}(A) \ \text{and} \ \mathscr{R}(C) \subseteq \mathscr{R}(B^*). \tag{2.18}$$

(ii) *Under (2.18), the general solution of (2.13) and the corresponding $AXB + (AXB)^*$ can be written in the following parametric from*

$$X = \frac{1}{2}A^{\dagger}CC^*B^{\dagger} + A^{\dagger}E_MUU^*E_MB^{\dagger} + [I_p, 0]F_GWE_H\begin{bmatrix} I_q \\ 0 \end{bmatrix} - [0, I_p]E_HW^*F_G\begin{bmatrix} 0 \\ I_q \end{bmatrix} + F_AW_1 + W_2E_B, \tag{2.19}$$

$$AXB + (AXB)^* = CC^* + 2E_MUU^*E_M, \tag{2.20}$$

where $U \in \mathbb{C}^{m\times m}$, $W \in \mathbb{C}^{(p+q)\times(p+q)}$ and $W_1, W_2 \in \mathbb{C}^{p\times q}$ are arbitrary. Further, (2.13) can equivalently be written as

$$X = A^{\dagger}E_MUU^*E_MB^{\dagger} + \frac{1}{2}Y + Z, \tag{2.21}$$

where Y is the general solution of $AYB = CC^*$, and Z is the general solution of $AZB + (AZB)^* = 0$.

(iii) *Under* (2.18), *let*

$$\mathscr{S} = \{ X \in \mathbb{C}^{p\times q} \mid AXB + (AXB)^* \geqslant CC^* \}. \tag{2.22}$$

Then,

$$\max_{X\in\mathscr{S}} r[AXB + (AXB)^*] = r\begin{bmatrix} A & 0 & C \\ 0 & B^* & C \end{bmatrix} - r[A, B^*], \tag{2.23}$$

$$\min_{X\in\mathscr{S}} r[AXB + (AXB)^*] = r(C), \tag{2.24}$$

$$\max_{X\in\mathscr{S}} r[AXB + (AXB)^* - CC^*] = r(A) + r(B) - r[A, B^*]. \tag{2.25}$$

(b) *There exists an* $X \in \mathbb{C}^{p\times q}$ *such that*

$$AXB + (AXB)^* > CC^* \tag{2.26}$$

if and only if $r(A) = r(B) = m$. *In this case, the general solution of* (2.26) *and the corresponding* $AXB + (AXB)^*$ *can be written in the following parametric from*

$$X = \frac{1}{2}A^{\dagger}CC^*B^{\dagger} + A^{\dagger}UU^*B^{\dagger} + [I_p, 0]F_G W E_H \begin{bmatrix} I_q \\ 0 \end{bmatrix} - [0, I_p]E_H W^* F_G \begin{bmatrix} 0 \\ I_q \end{bmatrix} + F_A W_1 + W_2 E_B, \tag{2.27}$$

$$AXB + (AXB)^* = CC^* + 2UU^*, \tag{2.28}$$

where $U \in \mathbb{C}^{m\times m}$ *is any matrix with* $r(U) = m$, *and* $W \in \mathbb{C}^{(p+q)\times(p+q)}$ *and* $W_1, W_2 \in \mathbb{C}^{p\times q}$ *are arbitrary. Further,* (2.27) *can equivalently be written as*

$$X = A^{\dagger}UU^*B^{\dagger} + \frac{1}{2}Y + Z, \tag{2.29}$$

where Y *is the general solution of* $AYB = CC^*$, *and* Z *is the general solution of* $AZB + (AZB)^* = 0$.

(c) *There exists an* $X \in \mathbb{C}^{p\times q}$ *such that*

$$AXB + (AXB)^* \leqslant -CC^* \tag{2.30}$$

if and only if (2.18) *holds. In this case, the general solution of* (2.30) *and the corresponding* $AXB + (AXB)^*$ *can be written as*

$$X = -\frac{1}{2}A^{\dagger}CC^*B^{\dagger} - A^{\dagger}E_M UU^* E_M B^{\dagger} + [I_p, 0]F_G W E_H \begin{bmatrix} I_q \\ 0 \end{bmatrix} - [0, I_p]E_H W^* F_G \begin{bmatrix} 0 \\ I_q \end{bmatrix} + F_A W_1 + W_2 E_B, \tag{2.31}$$

$$AXB + (AXB)^* = CC^* - 2E_M UU^* E_M, \tag{2.32}$$

where $U \in \mathbb{C}^{m\times m}$, $W \in \mathbb{C}^{(p+q)\times(p+q)}$ *and* $W_1, W_2 \in \mathbb{C}^{p\times q}$ *are arbitrary.*

(d) *There exists an* $X \in \mathbb{C}^{p\times q}$ *such that*

$$AXB + (AXB)^* < -CC^* \tag{2.33}$$

if and only if $r(A) = r(B) = m$. *In this case, the general solution of* (2.33) *can be written as*

$$\begin{aligned} X = &-\frac{1}{2}A^{\dagger}CC^*B^{\dagger} - A^{\dagger}UU^*B^{\dagger} + [I_p, 0]F_GWE_H\begin{bmatrix} I_q \\ 0 \end{bmatrix} \\ &- [0, I_p]E_HW^*F_G\begin{bmatrix} 0 \\ I_q \end{bmatrix} + F_AW_1 + W_2E_B, \end{aligned} \tag{2.34}$$

$$AXB + (AXB)^* = CC^* - 2UU^*, \tag{2.35}$$

where $U \in \mathbb{C}^{m\times m}$ *is any matrix with* $r(U) = m$, *and* $W \in \mathbb{C}^{(p+q)\times(p+q)}$ *and* $W_1, W_2 \in \mathbb{C}^{p\times q}$ *are arbitrary.*

Proof. Note that (2.13) can be rewritten as $CC^* - AXB - (AXB)^* \leqslant 0$, which is obviously equivalent to

$$\min_{X\in\mathbb{C}^{p\times q}} i_+[CC^* - AXB - (AXB)^*] = 0. \tag{2.36}$$

Setting the right-hand side of (2.16) to zero, we see that (2.36) holds if and only if $r[A, C] = r(A)$ and $r[B^*, C] = r(B)$, which are equivalent to (2.18). On the other hand, (2.13) is equivalent to the following quadratic matrix equation

$$AXB + (AXB)^* = CC^* + VV^*. \tag{2.37}$$

From Corollary 2.2(a) and (c), this equation is solvable for X if and only if

$$E_A(VV^* + CC^*) = 0 \text{ and } F_B(VV^* + CC^*) = 0,$$

that is,

$$\begin{bmatrix} E_A \\ F_B \end{bmatrix} VV^* = -\begin{bmatrix} E_A \\ F_B \end{bmatrix} CC^*. \tag{2.38}$$

From Lemma 1.1(b), (2.38) is solvable for VV^* if and only if

$$\begin{bmatrix} E_A \\ F_B \end{bmatrix} CC^*[E_A, F_B] \leqslant 0 \text{ and } r\left(\begin{bmatrix} E_A \\ F_B \end{bmatrix} CC^*[E_A, F_B]\right) = r\left(\begin{bmatrix} E_A \\ F_B \end{bmatrix} CC^*\right),$$

both of which are obviously equivalent to $E_ACC^* = F_BCC^* = 0$, i.e., (2.18) holds. In this case, the general solution of (2.38), by Lemma 1.1(b), can be written as

$$VV^* = 2(I_m - [E_A, F_B][E_A, F_B]^{\dagger})UU^*(I_m - [E_A, F_B][E_A, F_B]^{\dagger}) = 2E_MUU^*E_M,$$

where $U \in \mathbb{C}^{m\times m}$ is arbitrary. Substituting this VV^* into (2.37) gives

$$AXB + (AXB)^* = CC^* + 2E_MUU^*E_M. \tag{2.39}$$

From Corollary 2.2, the general solution of (2.39) is given by

$$X = \frac{1}{2}A^{\dagger}CC^{*}B^{\dagger} + A^{\dagger}E_{M}UU^{*}E_{M}B^{\dagger} + [I_p, 0]F_{G}WE_{H}\begin{bmatrix} I_q \\ 0 \end{bmatrix} - [0, I_p]E_{H}W^{*}F_{G}\begin{bmatrix} 0 \\ I_q \end{bmatrix} + F_{A}W_{1} + W_{2}E_{B},$$

establishing (2.19).

It can be seen from (2.39) that

$$r[AXB + (AXB)^{*}] = r[C^{*}, E_{M}U], \quad r[AXB + (AXB)^{*} - CC^{*}] = r(E_{M}U). \tag{2.40}$$

It can be derived by (1.18) that

$$r(CM) = r(C[E_A, F_B]) = r\begin{bmatrix} C & C \\ A^{*} & 0 \\ 0 & B \end{bmatrix} - r(A) - r(B) = r\begin{bmatrix} A & 0 & C \\ 0 & B^{*} & C \end{bmatrix} - r(A) - r(B), \tag{2.41}$$

$$r(M) = r[E_A, F_B] = r\begin{bmatrix} A & 0 & I_m \\ 0 & B^{*} & I_m \end{bmatrix} - r(A) - r(B) = m + r[A, B^{*}] - r(A) - r(B). \tag{2.42}$$

Thus, we have

$$\max_{X\in\mathscr{S}} r[AXB + (AXB)^{*}] = r[C^{*}, E_M] = r(CM) + m - r(M) = r\begin{bmatrix} A & 0 & C \\ 0 & B^{*} & C \end{bmatrix} - r[A, B^{*}],$$

$$\min_{X\in\mathscr{S}} r[AXB + (AXB)^{*}] = (C),$$

$$\max_{X\in\mathscr{S}} r[AXB + (AXB)^{*} - CC^{*}] = r(E_M) = r(A) + r(B) - r[A, B^{*}],$$

establishing (2.23)–(2.25).

It is obvious that (1.19) holds if and only if

$$\max_{X\in\mathbb{C}^{p\times q}} i_{-}[CC^{*} - AXB - (AXB)^{*}] = m,$$

which is equivalent to $r(A) = r(B) = m$ by (2.15). In this case, $E_M = I_m$, and therefore (b) follows from (a). Replacing X with $-X$ in (a) and (b) leads to (c) and (d). □

Two simple consequences of Corollary 2.2 and Theorem 2.3 are given below.

COROLLARY 2.4. (a) *The general solution of* $X + X^{*} = I_n$ *can be written as*

$$X = \frac{1}{2}I_n + W - W^{*},$$

where $W \in \mathbb{C}^{n\times n}$ *is arbitrary.*

(b) *The general solution of* $X + X^{*} \geqslant I_n$ *can be written as*

$$X = \frac{1}{2}I_n + UU^{*} + W - W^{*},$$

where $U, W \in \mathbb{C}^{n\times n}$ *are arbitrary.*

Setting $C=0$ in (2.13), we obtain from Theorem 2.3 the analytical solution of $\mathscr{H}(AXB) \geqslant 0$ as follows.

COROLLARY 2.5. *Let $A \in \mathbb{C}^{m\times p}$ and $B \in \mathbb{C}^{q\times m}$ be given, and define $M=[E_A, F_B]$, $G=[A, B^*]$ and $H=[B^*, A]^*$. Then,*

(a) *The general solution of*

$$AXB+(AXB)^* \geqslant 0 \tag{2.43}$$

can be written in the parametric from

$$X = A^\dagger E_M UU^* E_M B^\dagger + [I_p, 0]F_G W E_H \begin{bmatrix} I_q \\ 0 \end{bmatrix} - [0, I_p]E_H W^* F_G \begin{bmatrix} 0 \\ I_q \end{bmatrix} + F_A W_1 + W_2 E_B, \tag{2.44}$$

where $U \in \mathbb{C}^{m\times m}$, $W \in \mathbb{C}^{(p+q)\times(p+q)}$ and $W_1, W_2 \in \mathbb{C}^{p\times q}$ are arbitrary.

(b) *There exists an $X \in \mathbb{C}^{p\times q}$ such that*

$$AXB+(AXB)^* > 0 \tag{2.45}$$

if and only if $r(A)=r(B)=m$. In this case, the general solution of (2.45) can be written as

$$X = A^\dagger UU^* B^\dagger + [I_p, 0]F_G W E_H \begin{bmatrix} I_q \\ 0 \end{bmatrix} - [0, I_p]E_H W^* F_G \begin{bmatrix} 0 \\ I_q \end{bmatrix} + F_A W_1 + W_2 E_B, \tag{2.46}$$

where $U \in \mathbb{C}^{m\times m}$ is any matrix with $r(U)=m$, and $W \in \mathbb{C}^{(p+q)\times(p+q)}$ and $W_1, W_2 \in \mathbb{C}^{p\times q}$ are arbitrary.

If the C in (1.2) is a general Hermitian matrix, then (1.2) can equivalently be written as

$$AXB+(AXB)^* = C + VV^* \tag{2.47}$$

for some matrix V. From Theorem 2.1(a), this equation is solvable for X if and only if VV^* satisfies

$$E_G VV^* = -E_G C, \quad E_A VV^* E_A = -E_A C E_A, \quad F_B VV^* F_B = -F_B C F_B, \tag{2.48}$$

where $G=[A, B^*]$. However, we do not know how to solve for VV^* analytically from the triple matrix equations, and therefore, we are unable to generally give an analytical solution of the LMI in (1.2).

The Moore–Penrose inverses and Löwner partial ordering for linear operators on a Hilbert space or elements in a ring with involution were defined and their algebraic properties were extensively studied in the literature. In most cases, the conclusions on the complex matrices and their counterparts in general algebraic settings are analogous. Also, note that the results in this note are derived from ordinary algebraic operations of the given matrices and their Moore–Penrose inverses. Hence, it is no doubt that most of the conclusions in this note can trivially be extended to the corresponding equations and inequalities for linear operators on a Hilbert space or elements in a ring with involution.

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