Mathematical
Inequalities
& Applications
Volume 15, Number 3 (2012), 559–569 doi:10.7153/mia-15-49

UNIVERSAL INEQUALITIES FOR EIGENVALUES OF QUADRATIC POLYNOMIAL OPERATOR OF THE KOHN LAPLACIAN

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Abstract. In this paper, we investigate the Dirichlet weighted eigenvalue problem of quadratic polynomial operator of the Kohn Laplacian on a bounded domain in the Heisenberg group $\mathbb{H}^n$. We establish two inequalities for eigenvalues of this problem. One of them implies an explicit estimate for the upper bound of the $(k+1)$-th eigenvalue in terms of the first k eigenvalues. Moreover, as a special case, we give some universal inequalities and estimates for eigenvalues of the bi-Kohn Laplacian.

1. Introduction

One of important research subjects in geometric analysis is to obtain some universal inequalities and bounds for eigenvalues of differential operators on various kinds of manifolds. There have been some results (e.g., [1, 2, 3, 11, 9, 20, 21, 22, 23], etc.) for the Laplacian, the biharmonic operator and elliptic operators with variable coefficients on some Riemannian manifolds such as the Euclidean space, a unite sphere, a complex projective space and so on.

Let $\mathbb{H}^n$ be an $(2n+1)$-dimensional Heisenberg group with coordinates $(x,y,t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ and non-commutative group law given by

$$(x,y,t)(x',y',t') = \left(x+x', y+y', t+t' + \frac{1}{2}(\langle x',y\rangle_{\mathbb{R}^n} - \langle x,y'\rangle_{\mathbb{R}^n})\right),$$

where $(x',y',t') \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ and $\langle\cdot,\cdot\rangle$ denotes the inner product in $\mathbb{R}^n$. The Heisenberg group is one of classical sub-Riemannian manifolds. Its geometric structure has many differences with the Euclidean space. The Lie algebra $\mathscr{H}^n$ of $\mathbb{H}^n$ has a basis formed by $2n+1$ left-invariant vector fields $\{X_j, Y_j, T\}$ for $j = 1,\cdots,n,$ where

$$X_j = \frac{\partial}{\partial x_j} + \frac{y_j}{2}\frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} - \frac{x_j}{2}\frac{\partial}{\partial t}, \quad T = \frac{\partial}{\partial t}.$$

The Kohn Laplacian $\Delta_{\mathbb{H}^n}$ on the Heisenberg group $\mathbb{H}^n$ is named after Kohn [13] (cf. [12]). It is also called the sub-Laplacian and is defined by

$$\Delta_{\mathbb{H}^n} = \sum_{j=1}^{n}(X_j^2 + Y_j^2) = \Delta_{xy}^{\mathbb{R}^{2n}} + \frac{1}{4}\left(|x|^2+|y|^2\right)\frac{\partial^2}{\partial^2 t} + \frac{\partial}{\partial t}\sum_{j=1}^{n}\left(y_j\frac{\partial}{\partial x_j} - x_j\frac{\partial}{\partial y_j}\right). \quad (1.1)$$

Mathematics subject classification (2010): Primary 35P15, 53C17; Secondary 58C40.
Keywords and phrases: Eigenvalue, universal inequality, Heisenberg group, Kohn Laplacian, quadratic polynomial operator.
The first author was supported by the National Natural Science Foundation of China (Grant No. 11001130) and the NUST Research Funding (Grant No. 2010ZYTS064).

As an invariant differential operator on $\mathbb{H}^n$, the spectrum of $\Delta_{\mathbb{H}^n}$ has strong background in physics. Moreover, it has many connections with several complex variables and partial differential equation (see [4, 19]). Unlike the Laplacian and the biharmonic operator, $\Delta_{\mathbb{H}^n}$ is non-elliptic. Therefore, it is significant to give some estimates for eigenvalues of some problems relating to $\Delta_{\mathbb{H}^n}$.

The physical nature and intrinsic properties of $\mathbb{H}^n$ are decided by the Heisenberg relations, that is

$$[Y_j, X_i] = \delta_{ij} T, \quad \text{for} \quad i, j = 1, \cdots, n, \tag{1.2}$$

and all other commutators vanish. The non-commutativity of X_j and Y_j makes it complicated to estimate eigenvalues of $\Delta_{\mathbb{H}^n}$ and the bi-Kohn Laplacian $\Delta^2_{\mathbb{H}^n}$. Compared with the Laplacian and the biharmonic operator, there were fewer references on estimates for eigenvalues of $\Delta_{\mathbb{H}^n}$ and $\Delta^2_{\mathbb{H}^n}$. Let Ω be a bounded domain in $\mathbb{H}^n$. Denote by λ_r the r-th eigenvalue of the Dirichlet eigenvalue problem of $\Delta_{\mathbb{H}^n}$:

$$\begin{cases} -\Delta_{\mathbb{H}^n} u = \lambda u, & \text{in } \Omega, \\ u|_{\partial\Omega} = 0. \end{cases} \tag{1.3}$$

In 2003, Niu and Zhang [16] proved that eigenvalues of problem (1.3) satisfy

$$\lambda_{k+1} - \lambda_k \leqslant \frac{2}{nk} \sum_{r=1}^{k} \lambda_r. \tag{1.4}$$

In 2009, Soufi, Harrel II and Ilias [18] obtained a sharper inequality

$$\sum_{r=1}^{k} (\lambda_{k+1} - \lambda_r)^2 \leqslant \frac{2}{n} \sum_{r=1}^{k} (\lambda_{k+1} - \lambda_r)\lambda_r. \tag{1.5}$$

Compared to $\Delta_{\mathbb{H}^n}$, the difficulties caused by the non-commutativity of X_i and Y_j become more obvious to the Dirichlet eigenvalue problem of $\Delta^2_{\mathbb{H}^n}$, which is described by

$$\begin{cases} \Delta^2_{\mathbb{H}^n} u = \Gamma u, & \text{in } \Omega, \\ u|_{\partial\Omega} = \dfrac{\partial u}{\partial \nu}|_{\partial\Omega} = 0, \end{cases} \tag{1.6}$$

where ν denotes the outward unit normal vector to $\partial\Omega$. To the author's knowledge, the previous main result for problem (1.6) was also derived by Niu and Zhang [16]. They obtained an estimate for its gap of the consecutive eigenvalues:

$$\Gamma_{k+1} - \Gamma_k \leqslant \frac{4(n+1)}{n^2 k^2} \left(\sum_{r=1}^{k} \Gamma_r^{\frac{1}{2}}\right)^{\frac{1}{2}}. \tag{1.7}$$

In this paper, we investigate the weighted eigenvalue problem of quadratic polynomial operator of $\Delta_{\mathbb{H}^n}$, which is described by

$$\begin{cases} \Delta^2_{\mathbb{H}^n} u - a\Delta_{\mathbb{H}^n} u + bu = \Lambda \rho u, & \text{in } \Omega, \\ u|_{\partial\Omega} = \dfrac{\partial u}{\partial \nu}|_{\partial\Omega} = 0, \end{cases} \tag{1.8}$$

where ρ is a positive continuous function on $\overline{\Omega}$ and the constants $a, b \geqslant 0$. As we know (see [5, 7]), this problem has a real and discrete spectrum:

$$0 < \Lambda_1 \leqslant \Lambda_2 \leqslant \cdots \leqslant \Lambda_r \leqslant \cdots \nearrow \infty,$$

where each eigenvalue is repeated with its multiplicity. It characterizes the vibration problem of a clamped plate in $\mathbb{H}^n$. In 1997, Guo [8] derived some priori estimates for eigenvalues of the polynomial operator of the Kohn Laplacian, which relate to the Pólya's conjecture for the Laplacian. For more references in this direction, we refer the reader to [17, 14, 15].

The purpose of this paper is to obtain some inequalities and estimates for eigenvalues of problem (1.6) and problem (1.8). This paper is organized as follows: In Section 2, we establish some general inequalities for eigenvalues of problem (1.8). In Section 3, making use of these general inequalities, we obtain two inequalities for eigenvalues of problem (1.8) in Theorems 3 and 4. Noticing that (3.1) is a quadratic inequality with respect to Λ_{k+1}, we give an upper bound of Λ_{k+1} in terms of the first k eigenvalues in Corollary 1. A significance of the preceding estimates lies in the convenience to obtain some estimates for eigenvalues of problem (1.6). Since it is only a special case of problem (1.8), we easily derive some results for eigenvalues of problem (1.6) (see Corollaries 2-6). Even for this special case, our results is sharper than (1.7).

2. Some general inequalities

In this section, some general inequalities for eigenvalues of problem (1.8) are obtained. They will play an important role in the proofs of Theorems 3 and 4 in Section 3.

LEMMA 1. *Let Ω be a bounded domain in $\mathbb{H}^n$. Denote by u_r the weighted orthonormal eigenfunction corresponding to the r-th eigenvalue Λ_r of problem (1.8), $r = 1, 2, \cdots, k$. Namely, u_r satisfies*

$$\begin{cases} \Delta^2_{\mathbb{H}^n} u_r - a\Delta_{\mathbb{H}^n} u_r + b u_r = \Lambda_r \rho u_r, & \text{in } \Omega, \\ u_r|_{\partial\Omega} = \dfrac{\partial u_r}{\partial \nu}|_{\partial\Omega} = 0, \\ \displaystyle\int_\Omega \rho u_r u_s = \delta_{rs}. \end{cases} \tag{2.1}$$

Then we have

$$\begin{aligned} \sum_{r=1}^{k} (\Lambda_{k+1} - \Lambda_r)^2 \int_\Omega u_r^2 \leqslant & \sum_{r=1}^{k} \frac{1}{\gamma_r} (\Lambda_{k+1} - \Lambda_r) \int_\Omega \frac{1}{\rho} (X_i u_r)^2 \\ & + \sum_{r=1}^{k} \gamma_r (\Lambda_{k+1} - \Lambda_r)^2 \int_\Omega \left[4(X_i u_r)^2 + 2|\nabla_{\mathbb{H}^n} u_r|^2 + a u_r^2 \right], \end{aligned} \tag{2.2}$$

where $\nabla_{\mathbb{H}^n}$ is the gradient operator on $\mathbb{H}^n$ and the constants γ_r form a non-increasing sequence of positive numbers.

Proof. We define the trial functions

$$\varphi_{rx_i} = x_i u_r - \sum_{s=1}^{k} a_{rsx_i} u_s, \quad \text{for } i = 1, \cdots, n, \text{ and } r = 1, \cdots, k. \tag{2.3}$$

where

$$a_{rsx_i} = \int_\Omega \rho x_i u_r u_s.$$

Then it is not hard to check that for $i = 1, \cdots, n$, and $r, s = 1, \cdots, k$,

$$\int_\Omega \rho \varphi_{rx_i} u_s = 0. \tag{2.4}$$

Moreover, it yields

$$\int_\Omega \rho \varphi_{rx_i} x_i u_r = \int_\Omega \rho \varphi_{rx_i}^2. \tag{2.5}$$

Since

$$\Delta_{\mathbb{H}^n}^2 (x_i u_r) = 2\Delta_{\mathbb{H}^n} X_i u_r + 2X_i \Delta_{\mathbb{H}^n} u_r + x_i \Delta_{\mathbb{H}^n}^2 u_r, \tag{2.6}$$

we have

$$\begin{aligned} &\Delta_{\mathbb{H}^n}^2 \varphi_{rx_i} - a\Delta_{\mathbb{H}^n} \varphi_{rx_i} + b\varphi_{rx_i} \\ =&2(\Delta_{\mathbb{H}^n} X_i u_r + X_i \Delta_{\mathbb{H}^n} u_r - aX_i u_r) + \Lambda_r \rho x_i u_r - \sum_{s=1}^{k} a_{rsx_i} \Lambda_s \rho u_s. \end{aligned} \tag{2.7}$$

Substituting (2.7) into the Rayleigh-Ritz inequality

$$\Lambda_{k+1} \leqslant \frac{\int_\Omega \varphi_{rx_i} (\Delta_{\mathbb{H}^n}^2 \varphi_{rx_i} - a\Delta_{\mathbb{H}^n} \varphi_{rx_i} + b\varphi_{rx_i})}{\int_\Omega \rho \varphi_{rx_i}^2},$$

and using (2.4) and (2.5), we obtain

$$(\Lambda_{k+1} - \Lambda_r) \int_\Omega \rho \varphi_{rx_i}^2 \leqslant 2 \int_\Omega x_i u_r (\Delta_{\mathbb{H}^n} X_i u_r + X_i \Delta_{\mathbb{H}^n} u_r - aX_i u_r) + 2 \sum_{s=1}^{k} a_{rsx_i} b_{rsx_i}, \tag{2.8}$$

where

$$b_{rsx_i} = -\int_\Omega u_s (\Delta_{\mathbb{H}^n} X_i u_r + X_i \Delta_{\mathbb{H}^n} u_r - aX_i u_r) = -b_{srx_i}.$$

Using integration by parts and utilizing (2.6), we find

$$\Lambda_r a_{rsx_i} = \int_\Omega x_i u_s (\Delta_{\mathbb{H}^n}^2 u_r - a\Delta_{\mathbb{H}^n} u_r + bu_r) = -2b_{srx_i} + \Lambda_s a_{rsx_i}.$$

That is

$$2b_{rsx_i} = (\Lambda_r - \Lambda_s) a_{rsx_i}. \tag{2.9}$$

At the same time, since

$$\int_\Omega x_i u_r \Delta_{\mathbb{H}^n} X_i u_r = -2 \int_\Omega u_r X_i^2 u_r - \int_\Omega u_r \Delta_{\mathbb{H}^n} u_r - \int_\Omega u_r x_i X_i \Delta_{\mathbb{H}^n} u_r,$$

we have

$$\int_\Omega x_i u_r(\Delta_{\mathbb{H}^n} X_i u_r + X_i \Delta_{\mathbb{H}^n} u_r - aX_i u_r) = \int_\Omega \left[2(X_i u_r)^2 + |\nabla_{\mathbb{H}^n} u_r|^2 + \frac{a}{2}u_r^2\right]. \tag{2.10}$$

Substituting (2.9) and (2.10) into (2.8), we deduce

$$(\Lambda_{k+1}-\Lambda_r)\int_\Omega \rho\varphi_{rx_i}^2 \leqslant \int_\Omega \left[4(X_i u_r)^2 + 2|\nabla_{\mathbb{H}^n} u_r|^2 + a u_r^2\right] + \sum_{s=1}^k (\Lambda_r - \Lambda_s) a_{rsx_i}^2. \tag{2.11}$$

Using integration by parts again, we have

$$-2\int_\Omega \varphi_{rx_i} X_i u_r = \int_\Omega u_r^2 + 2\sum_{s=1}^k a_{rsx_i} d_{rsx_i}, \tag{2.12}$$

where

$$d_{rsx_i} = \int_\Omega u_s X_i u_r = -d_{srx_i}.$$

Multiplying (2.12) by $(\Lambda_{k+1}-\Lambda_r)^2$, using the Schwarz inequality and (2.4), we deduce

$$\begin{aligned}
&(\Lambda_{k+1}-\Lambda_r)^2\int_\Omega u_r^2 + 2\sum_{s=1}^k (\Lambda_{k+1}-\Lambda_r)^2 a_{rsx_i} d_{rsx_i} \\
=&-2(\Lambda_{k+1}-\Lambda_r)^2\int_\Omega \sqrt{\rho}\varphi_{rx_i}\Big(\frac{1}{\sqrt{\rho}}X_i u_r - \sqrt{\rho}\sum_{s=1}^k d_{rsx_i} u_s\Big) \\
\leqslant&\gamma_r(\Lambda_{k+1}-\Lambda_r)^3\int_\Omega \rho\varphi_{rx_i}^2 + \frac{\Lambda_{k+1}-\Lambda_r}{\gamma_r}\int_\Omega \Big(\frac{1}{\sqrt{\rho}}X_i u_r - \sqrt{\rho}\sum_{s=1}^k d_{rsx_i} u_s\Big)^2.
\end{aligned} \tag{2.13}$$

Substituting (2.11) into (2.13) and summing over r from 1 to k, we obtain

$$\begin{aligned}
&\sum_{r=1}^k (\Lambda_{k+1}-\Lambda_r)^2\int_\Omega u_r^2 + 2\sum_{r,s=1}^k (\Lambda_{k+1}-\Lambda_r)^2 a_{rsx_i} d_{rsx_i} \\
\leqslant&\sum_{r=1}^k \gamma_r(\Lambda_{k+1}-\Lambda_r)^2\int_\Omega \left[4(X_i u_r)^2 + 2|\nabla_{\mathbb{H}^n} u_r|^2 + a u_r^2\right] - \sum_{r,s=1}^k \frac{1}{\gamma_r}(\Lambda_{k+1}-\Lambda_r) d_{rsx_i}^2 \\
&+\sum_{r=1}^k \frac{1}{\gamma_r}(\Lambda_{k+1}-\Lambda_r)\int_\Omega \frac{1}{\rho}(X_i u_r)^2 + \sum_{r,s=1}^k \gamma_r(\Lambda_{k+1}-\Lambda_r)^2(\Lambda_r-\Lambda_s) a_{rsx_i}^2.
\end{aligned} \tag{2.14}$$

Because the sequence $\{\gamma_r\}$ is non-increasing, one can get

$$\sum_{r,s=1}^k \gamma_r(\Lambda_{k+1}-\Lambda_r)^2(\Lambda_r-\Lambda_s) a_{rsx_i}^2 \leqslant -\sum_{r,s=1}^k \gamma_r(\Lambda_{k+1}-\Lambda_r)(\Lambda_r-\Lambda_s)^2 a_{rsx_i}^2. \tag{2.15}$$

Moreover, it follows from $a_{rsx_i} = a_{srx_i}$ and $d_{rsx_i} = -d_{srx_i}$ that

$$\sum_{r,s=1}^k (\Lambda_{k+1}-\Lambda_r)^2 a_{rsx_i} d_{rsx_i} = -\sum_{r,s=1}^k (\Lambda_{k+1}-\Lambda_r)(\Lambda_r-\Lambda_s) a_{rsx_i} d_{rsx_i}. \tag{2.16}$$

Hence, using (2.15) and (2.16), we can eliminate the unwanted terms in both sides of (2.14) and obtain (2.2). □

In order to make some estimates in Theorems 3 and 4 of Section 3, we need another general inequality which contains the factor $\int_\Omega (Y_i u_r)^2$. Now we define the trial functions

$$\varphi_{ry_i} = y_i u_r - \sum_{s=1}^{k} a_{rsy_i} u_s, \quad i = 1, \cdots, n \text{ and } r = 1, \cdots, k, \tag{2.17}$$

where

$$a_{rsy_i} = \int_\Omega \rho y_i u_r u_s.$$

Similarly to the proof of Lemma 1, we can get the following lemma:

LEMMA 2. *Under the same assumptions as in Lemma 1, we have*

$$\begin{aligned}\sum_{r=1}^{k} (\Lambda_{k+1} - \Lambda_r)^2 \int_\Omega u_r^2 \leqslant \sum_{r=1}^{k} \frac{1}{\gamma_r} (\Lambda_{k+1} - \Lambda_r) \int_\Omega \frac{1}{\rho} (Y_i u_r)^2 \\ + \sum_{r=1}^{k} \gamma_r (\Lambda_{k+1} - \Lambda_r)^2 \int_\Omega \left[4(Y_i u_r)^2 + 2|\nabla_{\mathbb{H}^n} u_r|^2 + a u_r^2 \right],\end{aligned} \tag{2.18}$$

where the constants γ_r form a non-increasing sequence of positive numbers.

Combining Lemma 1 and Lemma 2, we can obtain one of two wanted general inequalities.

THEOREM 1. *Under the same assumptions as in Lemma 1, we have*

$$\begin{aligned}\sum_{r=1}^{k} (\Lambda_{k+1} - \Lambda_r)^2 \int_\Omega u_r^2 \leqslant \sum_{r=1}^{k} \frac{1}{2\gamma_r} (\Lambda_{k+1} - \Lambda_r) \int_\Omega \frac{1}{\rho} \left[(X_i u_r)^2 + (Y_i u_r)^2 \right] \\ + \sum_{r=1}^{k} \gamma_r (\Lambda_{k+1} - \Lambda_r)^2 \int_\Omega \left\{ 2\left[(X_i u_r)^2 + (Y_i u_r)^2 \right] + 2|\nabla_{\mathbb{H}^n} u_r|^2 + a u_r^2 \right\},\end{aligned} \tag{2.19}$$

where the constants γ_r form a non-increasing sequence of positive numbers.

We can derive another general inequality by making some modifications in the proofs of Lemmas 1 and 2.

THEOREM 2. *Under the same assumptions as in Lemma 1, we have*

$$\begin{aligned}\sum_{r=1}^{k} (\Lambda_{k+1} - \Lambda_r)^2 \int_\Omega u_r^2 \leqslant \frac{1}{n} \left[\sum_{r=1}^{k} (\Lambda_{k+1} - \Lambda_r) \int_\Omega \frac{1}{\rho} |\nabla_{\mathbb{H}^n} u_r|^2 \right]^{\frac{1}{2}} \\ \times \left\{ \sum_{r=1}^{k} (\Lambda_{k+1} - \Lambda_r)^2 \left[4(n+1) \int_\Omega |\nabla_{\mathbb{H}^n} u_r|^2 + 2na \int_\Omega u_r^2 \right] \right\}^{\frac{1}{2}}.\end{aligned} \tag{2.20}$$

Proof. Replacing γ_r in (2.14) by a positive constant γ, using (2.16) and the equality

$$\sum_{r,s=1}^{k}(\Lambda_{k+1}-\Lambda_r)^2(\Lambda_r-\Lambda_s)a_{rsx_i}^2=-\sum_{r,s=1}^{k}(\Lambda_{k+1}-\Lambda_r)(\Lambda_r-\Lambda_s)^2a_{rsx_i}^2, \tag{2.21}$$

we can also eliminate the unwanted terms. As a result of modifications in the proofs of Lemmas 1 and 2, we can deduce

$$\begin{aligned}\sum_{r=1}^{k}(\Lambda_{k+1}-\Lambda_r)^2\int_\Omega u_r^2 &\leqslant \frac{1}{2\gamma}\sum_{r=1}^{k}(\Lambda_{k+1}-\Lambda_r)\int_\Omega\frac{1}{\rho}\Big[(X_iu_r)^2+(Y_iu_r)^2\Big]\\ &+\gamma\sum_{r=1}^{k}(\Lambda_{k+1}-\Lambda_r)^2\Big\{2\int_\Omega\Big[(X_iu_r)^2+(Y_iu_r)^2\Big]+2\int_\Omega|\nabla_{\mathbb{H}^n}u_r|^2+a\int_\Omega u_r^2\Big\}.\end{aligned} \tag{2.22}$$

Summing over i from 1 to n in (2.22), we can get

$$\begin{aligned}&\gamma^2\sum_{r=1}^{k}(\Lambda_{k+1}-\Lambda_r)^2\Big[4(n+1)\int_\Omega|\nabla_{\mathbb{H}^n}u_r|^2+2na\int_\Omega u_r^2\Big]\\ &-2n\gamma\sum_{r=1}^{k}(\Lambda_{k+1}-\Lambda_r)^2\int_\Omega u_r^2+\sum_{r=1}^{k}(\Lambda_{k+1}-\Lambda_r)\int_\Omega\frac{1}{\rho}|\nabla_{\mathbb{H}^n}u_r|^2\geqslant 0.\end{aligned} \tag{2.23}$$

The left side of (2.23) is just a quadratic polynomial of γ. Therefore, its discriminant must be nonpositive. This yields (2.20). □

3. Some inequalities for eigenvalues of problem (1.8)

In this section, we establish some inequalities for eigenvalues of problem (1.8) by utilizing the general inequalities in Theorems 1 and 2.

THEOREM 3. *Let Λ_r be the r-th eigenvalue of problem (1.8). Set $\sigma=\big(\min_{x\in\overline{\Omega}}\rho(x)\big)^{-1}$ and $\tau=\big(\max_{x\in\overline{\Omega}}\rho(x)\big)^{-1}$. Then we have*

$$\sum_{r=1}^{k}(\Lambda_{k+1}-\Lambda_r)^2\leqslant\frac{4\sigma}{n^2\tau^2}\sum_{r=1}^{k}(\Lambda_{k+1}-\Lambda_r)\Big[(n+1)E_r+\frac{na\sigma}{2}\Big]E_r, \tag{3.1}$$

where

$$E_r=\frac{1}{2}\Big[-a\sigma+\sqrt{a^2\sigma^2+4\sigma(\Lambda_r-b\tau)}\Big].$$

Proof. According to the assumptions, it is easy to find

$$0<\tau\leqslant\int_\Omega u_r^2\leqslant\sigma. \tag{3.2}$$

Noticing that the constants $a,b \geqslant 0$ and the weight function $\rho > 0$, and utilizing

$$\int_\Omega |\nabla_{\mathbb{H}^n} u_r|^2 \leqslant \left[\int_\Omega u_r^2 \int_\Omega (\Delta_{\mathbb{H}^n} u_r)^2\right]^{\frac{1}{2}} \leqslant \left[\sigma \int_\Omega (\Delta_{\mathbb{H}^n} u_r)^2\right]^{\frac{1}{2}}, \tag{3.3}$$

we know that

$$a^2\sigma^2 + 4\sigma(\Lambda_r - b\tau) \geqslant 0.$$

Substituting (3.3) into

$$\Lambda_r = \int_\Omega u_r(\Delta_{\mathbb{H}^n}^2 u_r - a\Delta_{\mathbb{H}^n} u_r + bu_r) = \int_\Omega (\Delta_{\mathbb{H}^n} u_r)^2 + a\int_\Omega |\nabla_{\mathbb{H}^n} u_r|^2 + b\int_\Omega u_r^2,$$

we have

$$\left(\int_\Omega |\nabla_{\mathbb{H}^n} u_r|^2\right)^2 + a\sigma \int_\Omega |\nabla_{\mathbb{H}^n} u_r|^2 - \sigma(\Lambda_r - b\tau) \leqslant 0. \tag{3.4}$$

This is a quadratic inequality of $\int_\Omega |\nabla_{\mathbb{H}^n} u_i|^2$. Solving it, we conclude that

$$\int_\Omega |\nabla_{\mathbb{H}^n} u_r|^2 \leqslant E_r. \tag{3.5}$$

Summing over i from 1 to n in (2.19), we have

$$\begin{aligned} n\sum_{r=1}^{k}(\Lambda_{k+1}-\Lambda_r)^2 \int_\Omega u_r^2 \leqslant{}& \sum_{r=1}^{k} \frac{1}{2\gamma_r}(\Lambda_{k+1}-\Lambda_r)\int_\Omega \frac{1}{\rho}|\nabla_{\mathbb{H}^n} u_r|^2 \\ &+ \sum_{r=1}^{k} \gamma_r(\Lambda_{k+1}-\Lambda_r)^2\left[2(n+1)\int_\Omega |\nabla_{\mathbb{H}^n} u_r|^2 + na\int_\Omega u_r^2\right]. \end{aligned} \tag{3.6}$$

Substituting (3.2) and (3.5) into (3.6), we arrive at

$$\begin{aligned} &n\tau\sum_{r=1}^{k}(\Lambda_{k+1}-\Lambda_r)^2 \\ \leqslant{}& \sum_{r=1}^{k}\gamma_r(\Lambda_{k+1}-\Lambda_r)^2\left[2(n+1)E_r + na\sigma\right] + \frac{1}{2}\sigma\sum_{r=1}^{k}\frac{1}{\gamma_r}(\Lambda_{k+1}-\Lambda_r)E_r. \end{aligned} \tag{3.7}$$

Now we minimize the right side of (3.7). For this goal, putting

$$\gamma_r = \frac{\gamma}{2(n+1)E_r + na\sigma}$$

in (3.7), we have

$$\begin{aligned} &n\tau\sum_{r=1}^{k}(\Lambda_{k+1}-\Lambda_r)^2 \\ \leqslant{}& \gamma\sum_{r=1}^{k}(\Lambda_{k+1}-\Lambda_r)^2 + \frac{1}{\gamma}\sigma\sum_{r=1}^{k}(\Lambda_{k+1}-\Lambda_r)\left[(n+1)E_r + \frac{na\sigma}{2}\right]E_r. \end{aligned} \tag{3.8}$$

Then putting

$$\gamma = \left[\sum_{r=1}^{k}(\Lambda_{k+1}-\Lambda_r)^2\right]^{-\frac{1}{2}}\left\{\sigma\sum_{r=1}^{k}(\Lambda_{k+1}-\Lambda_r)\left[(n+1)E_r+\frac{na\sigma}{2}\right]E_r\right\}^{\frac{1}{2}}$$

in (3.8), it yields inequality (3.1). This completes the proof of Theorem 3. □

Substituting (3.2) and (3.5) into (2.20), we can derive the following result:

THEOREM 4. *Under the same assumptions as in Theorem 3, we have*

$$\begin{aligned}&\sum_{r=1}^{k}(\Lambda_{k+1}-\Lambda_r)^2\\ \leqslant&\frac{\sigma^{\frac{1}{2}}}{n\tau}\left[\sum_{r=1}^{k}(\Lambda_{k+1}-\Lambda_r)E_r\right]^{\frac{1}{2}}\left\{\sum_{r=1}^{k}(\Lambda_{k+1}-\Lambda_r)^2\left[4(n+1)E_r+2na\sigma\right]\right\}^{\frac{1}{2}}.\end{aligned} \tag{3.9}$$

We can get an explicit upper bound of Λ_{k+1} in terms of the first k eigenvalues from (3.1). In fact, (3.1) is a quadratic inequality of Λ_{k+1}. Solving it, we have the following corollary:

COROLLARY 1. *Under the same assumptions as in Theorem 3, we have*

$$\Lambda_{k+1}\leqslant A_k+\sqrt{A_k^2-B_k}, \tag{3.10}$$

where

$$A_k=\frac{1}{k}\left\{\sum_{r=1}^{k}\Lambda_r+\frac{\sigma}{n^2\tau^2}\sum_{r=1}^{k}\left[2(n+1)E_r+na\sigma\right]E_r\right\},$$

$$B_k=\frac{1}{k}\left\{\sum_{r=1}^{k}\Lambda_r^2+\frac{2\sigma}{n^2\tau^2}\sum_{r=1}^{k}\Lambda_r\left[2(n+1)E_r+na\sigma\right]E_r\right\}.$$

4. Some inequalities for eigenvalues of problem (1.6)

A significance of the estimates in Section 3 lies in the convenience to get some results for some special cases of problem (1.8). Here we only give some universal inequalities and estimates for problem (1.6).

COROLLARY 2. *Let Γ_r be the r-th eigenvalue of problem (1.6). Then we have*

$$\sum_{r=1}^{k}(\Gamma_{k+1}-\Gamma_r)^2\leqslant\frac{4(n+1)}{n^2}\sum_{r=1}^{k}(\Gamma_{k+1}-\Gamma_r)\Gamma_r. \tag{4.1}$$

COROLLARY 3. *Under the same assumptions as in Corollary 2, we have*

$$\sum_{r=1}^{k}(\Gamma_{k+1}-\Gamma_r)^2\leqslant\frac{2(n+1)^{\frac{1}{2}}}{n}\left[\sum_{r=1}^{k}(\Gamma_{k+1}-\Gamma_r)\Gamma_r^{\frac{1}{2}}\sum_{r=1}^{k}(\Gamma_{k+1}-\Gamma_r)^2\Gamma_r^{\frac{1}{2}}\right]^{\frac{1}{2}}. \tag{4.2}$$

From Corollary 1 or Corollary 2, we can obtain an estimate for the upper bound of Γ_{k+1} in terms of the first k eigenvalues.

COROLLARY 4. *Under the same assumptions as in Corollary 2, we have*

$$\begin{aligned}\Gamma_{k+1} \leqslant & \left[1+\frac{2(n+1)}{n^2}\right]\frac{1}{k}\sum_{r=1}^{k}\Gamma_r \\ & +\left\{\left[\frac{2(n+1)}{n^2}\frac{1}{k}\sum_{r=1}^{k}\Gamma_r\right]^2-\left[1+\frac{4(n+1)}{n^2}\right]\frac{1}{k}\sum_{s=1}^{k}\Big(\Gamma_s-\frac{1}{k}\sum_{r=1}^{k}\Gamma_r\Big)^2\right\}^{\frac{1}{2}}.\end{aligned} \tag{4.3}$$

Furthermore, a simpler bound can be derived by using the Cauchy-Schwarz inequality.

COROLLARY 5. *Under the same assumptions as in Corollary 2, we have*

$$\Gamma_{k+1} \leqslant \left(\frac{n+1}{n}\right)^2\frac{1}{k}\sum_{r=1}^{k}\Gamma_r. \tag{4.4}$$

At the same time, an explicit estimate on the gap of any two consecutive eigenvalues of problem (1.6) can also be obtained.

COROLLARY 6. *Under the same assumptions as in Corollary 2, we have*

$$\Gamma_{k+1}-\Gamma_k \leqslant 2\left\{\left[\frac{2(n+1)}{n^2}\frac{1}{k}\sum_{r=1}^{k}\Gamma_r\right]^2-\left(\frac{n+2}{n}\right)^2\frac{1}{k}\sum_{s=1}^{k}\Big(\Gamma_s-\frac{1}{k}\sum_{r=1}^{k}\Gamma_r\Big)^2\right\}^{\frac{1}{2}}. \tag{4.5}$$

These results are sharper than (1.7).

Acknowledgement. The authors would like to express their gratitude to the referee for valuable suggestions, which have improved the final version of this paper.

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(Received April 14, 2010)

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