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QUADRATIC INEQUALITIES AND A CHARACTERIZATION OF INNER PRODUCT

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Abstract. Let X be a real linear space and let $a \in \mathbb{R}$, $a \neq 0$, be fixed. Assuming that the functions $g,h : X \to \mathbb{R}$ satisfy the inequalities $g(ax+y)+h(x-ay) \leqslant a^2g(x)+g(y)+h(x)+a^2h(y)$ for all $x,y \in X$, and some subhomogeneity type conditions, we prove that $h=g$, the function g is a quadratic functional, and there exists a unique symmetric biadditive function $S : X^2 \to \mathbb{R}$ such that $g(x)=S(x,x)$ for all $x \in X$.

A motivation in the theory of orthogonal additive functions is presented.

1. Introduction

According to the well known Jordan and von Neumann Theorem [4], a linear normed space $(X,\|\cdot\|)$ is an inner product space iff the function $g := \|\cdot\|^2$ satisfies the *parallelogram equality*

$$g(x+y)+g(x-y)=2g(x)+2g(y), \qquad x,y \in X.$$

This equation is also referred to as a *quadratic functional equation* (cf. for instance J. Dhombres, J. Aczél [1]). M.M. Day [3] proved that this result remains true if the parallelogram equality is replaced by the inequality in either direction (cf. also D. Amir [2], p. 47).

In this paper we consider the functional inequalities of the form

$$g(ax+y)+h(x-ay) \leqslant a^2g(x)+g(y)+h(x)+a^2h(y)$$

where $g,h : X \to \mathbb{R}$ are unknown functions and a is a fixed real parameter. In a very special case, when $h=g$ and $a=1$, this inequality reduces to the inequality

$$g(x+y)+g(x-y) \leqslant 2g(x)+2g(y).$$

Applying Theorem 1 with $a=1$ we obtain the following result. Let X be a real linear space. If the functions $g,h : X \to \mathbb{R}$ satisfy the inequality

$$g(x+y)+h(x-y) \leqslant g(x)+g(y)+h(x)+h(y), \qquad x,y \in X,$$

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g is even, and $4g(x) \leqslant g(2x)$, $4h(x) \leqslant h(2x)$ for all $x \in X$, then $h = g$ and g is a quadratic functional; moreover there exists a unique symmetric biadditive function $S : X \to \mathbb{R}$ such that $g(x) = S(x,x)$ for all $x \in X$. If g satisfies some weak regularity conditions then S is bilinear, and in the case when $g \geqslant 0$, the function $\|\cdot\| : X \to \mathbb{R}$ defined by $\|x\| := \sqrt{g(x)}$ is a seminorm in X (Theorem 2). Moreover, if $g(x) \neq 0$ for $x \neq \mathbf{0}$, then S is an inner product and $\|\cdot\|$ is a norm generated by S.

To give a motivation denote by $(\cdot|\cdot)$ the usual inner product in $\mathbb{R}^2$. Suppose that $F : \mathbb{R}^2 \to \mathbb{R}$ is orthogonally additive, i.e. such that

$$u \perp v \Rightarrow F(u+v) = F(u) + F(v), \qquad u, v \in \mathbb{R}^n,$$

where $u \perp v$ denotes that $(u|v) = 0$. Put $e_1 := (1,0)$, $e_2 := (0,1)$ and define $g, h : \mathbb{R} \to \mathbb{R}$ by

$$g(t) := F(t,0), \qquad h(t) := F(0,t), \qquad t \in \mathbb{R}.$$

Since for any $x, y \in \mathbb{R}$, the vectors xe_1 and ye_2 are orthogonal, we have

$$F((x,y)) = F(xe_1 + ye_2) = F(xe_1) + F(ye_2) = g(x) + h(y)$$

for all x, y. Let us fix $a \in \mathbb{R}$, $a \neq 0$. Since for any $x, y \in \mathbb{R}$ the vectors $(ax, -ay)$ and (y, x) are orthogonal, we have

$$\begin{aligned} g(ax+y) + h(x-ay) &= F(ax+y, x-ay) = F((ax,-ay) + (y,x)) = \\ &= F((ax,-ay)) + F(y,x)) = g(ax) + h(-ay) + g(y) + h(x). \end{aligned}$$

Hence, if h is even and $g(ax) \leqslant a^2 g(x)$, $h(ax) \leqslant a^2 h(x)$, we obtain the above inequality. In the last section we show that the power 2 occurring in these inequalities cannot be replaced by a different one.

2. Results

We begin with the following

PROPOSITION 1. *Let X be a real linear space and let $a \in \mathbb{R}$, $a \neq 0$, be fixed. If the functions $g, h : X \to \mathbb{R}$ satisfy the inequalities*

$$g(ax+y) + h(x-ay) \leqslant a^2 g(x) + g(y) + h(x) + a^2 h(y), \quad x, y \in X, \tag{1}$$

$$h(ax+y) + g(x-ay) \leqslant a^2 h(x) + h(y) + g(x) + a^2 g(y), \quad x, y \in X, \tag{2}$$

$$(1+a^2)^2 g(x) \leqslant g\left((1+a^2)x\right), \quad (1+a^2)^2 h(x) \leqslant h\left((1+a^2)x\right), \qquad x \in X, \tag{3}$$

then

$$h = g,$$

and g satisfies the functional equation

$$g(ax+y) + g(x-ay) = (1+a^2)[g(x) + g(y)], \qquad x, y \in X. \tag{4}$$

Proof. Replacing x by $ax+y$ and y by $x-ay$ in (1), we obtain

$$g\left((1+a^2)x\right)+h\left((1+a^2)y\right)$$
$$\leqslant a^2[g(ax+y)+h(x-ay)]+h(ax+y)+g(x-ay),$$

for all $x,y \in X$. Hence, by (3), (1) and (2),

$$(1+a^2)^2[g(x)+h(y)]$$
$$\leqslant a^2[a^2g(x)+g(y)+h(x)+a^2h(y)]+a^2h(x)+h(y)+g(x)+a^2g(y),$$

for all $x,y \in X$, which reduces to the inequality

$$h(y)-g(y) \leqslant h(x)-g(x), \qquad x,y \in X.$$

It follows that $h-g=c$ for some $c \in \mathbb{R}$. Setting $x=y=\mathbf{0}$ in (1) and (3) gives, respectively, $g(\mathbf{0})+h(\mathbf{0}) \geqslant 0$, and $g(\mathbf{0}) \leqslant 0$, $h(\mathbf{0}) \leqslant 0$, whence $c=0$. Consequently $h=g$ and, by (1),

$$g(ax+y)+g(x-ay) \leqslant (1+a^2)[g(x)+g(y)], \qquad x,y \in X. \tag{5}$$

Replacing in this inequality x by $ax+y$ and y by $x-ay$ we get

$$g((1+a^2)x)+g((1+a^2)y) \leqslant (1+a^2)[g(ax+y)+g(x-ay)], \quad x,y \in X,$$

whence, by (3),

$$(1+a^2)^2[g(x)+g(y)] \leqslant (1+a^2)[g(ax+y)+g(x-ay)], \quad x,y \in X,$$

and, consequently,

$$(1+a^2)[g(x)+g(y)] \leqslant g(ax+y)+g(x-ay), \qquad x,y \in X. \tag{6}$$

Now (5) and (6) imply that

$$g(ax+y)+g(x-ay)=(1+a^2)[g(x)+g(y)], \qquad x,y \in X,$$

which completes the proof. □

Applying Proposition 1 with $h=g$ we obtain

COROLLARY 1. *Let X be a real linear space and let $a \in \mathbb{R}$, $a \neq 0$, be fixed. If a function $g: X \to \mathbb{R}$ satisfies the inequalities*

$$g(ax+y)+g(x-ay) \leqslant (a^2+1)[g(x)+g(y)], \quad x,y \in X,$$
$$(1+a^2)^2 g(x) \leqslant g\left((1+a^2)x\right), \qquad x \in X,$$

then

$$g(ax+y)+g(x-ay)=(1+a^2)[g(x)+g(y)], \qquad x,y \in X.$$

PROPOSITION 2. *Let X be a real linear space and let $a \in \mathbb{R}$, $a \neq 0$, be fixed. If the functions $g, h : X \to \mathbb{R}$ are even and satisfy the inequalities*

$$g(ax+y)+h(x-ay) \leqslant a^2 g(x)+g(y)+h(x)+a^2 h(y), \quad x,y \in X,$$

$$(1+a^2)^2 g(x) \leqslant g\left((1+a^2)x\right), \quad (1+a^2)^2 h(x) \leqslant h\left((1+a^2)x\right), \qquad x \in X,$$

then

$$h = g,$$

and g satisfies the functional equation

$$g(ax+y)+g(x-ay) = (1+a^2)[g(x)+g(y)], \qquad x,y \in X.$$

Proof. Replacing x by y and y by $-x$ in the first of the assumed inequalities we obtain

$$g(ay-x)+h(y+ax) \leqslant a^2 g(y)+g(-x)+h(y)+a^2 h(-x), \quad x,y \in X$$

As g and h are even, we hence get

$$h(ax+y)+g(x-ay) \leqslant a^2 h(x)+h(y)+g(x)+a^2 g(y), \quad x,y \in X.$$

Thus g and h satisfy all the assumptions of Proposition 1. □

PROPOSITION 3. *Let X be a real linear space and let $a \in \mathbb{R}$, $a \neq 0$, be fixed. If the functions $g, h : X \to \mathbb{R}$ satisfy the inequalities*

$$g(ax+y)+h(x-ay) \leqslant a^2 g(x)+g(y)+h(x)+a^2 h(y), \quad x,y \in X,$$

$$(1+a^2)^2 g(x) \leqslant g\left((1+a^2)x\right), \quad (1+a^2)^2 h(x) \leqslant h\left((1+a^2)x\right), \qquad x \in X,$$

$$a^2 h(x) \leqslant h(ax), \qquad x \in X,$$

and g is even, then

$$h = g,$$

and g satisfies the functional equation

$$g(ax+y)+g(x-ay) = (1+a^2)[g(x)+g(y)], \qquad x,y \in X.$$

Proof. In the same way as in the proof of Proposition 1 we can show that $g(\mathbf{0}) = h(\mathbf{0}) = 0$. Setting $x = \mathbf{0}$ in the first of assumed inequalities we get

$$h(-ay) \leqslant a^2 h(y), \quad y \in X.$$

Since $a^2 h(x) \leqslant h(ax)$ for all $x \in X$, we hence get $a^2 h(-y) \leqslant a^2 h(y)$ for all $y \in X$, whence

$$h(-y) \leqslant h(y), \quad y \in X.$$

Replacing y by $-y$ gives the opposite inequality and proves that h is even. Now the result follows from Proposition 2. □

REMARK 1. If $g = \|\cdot\|^2$ where $\|\cdot\|$ is a seminorm in X, then inequalities (3) are satisfied.

PROPOSITION 4. *Let X be a real linear space. If the functions $g,h : X \to \mathbb{R}$ satisfy the inequality*

$$g(x+y)+h(x-y) \leqslant g(x)+g(y)+h(x)+h(y), \quad x,y \in X,$$

$$4g(x) \leqslant g(2x), \qquad 4h(x) \leqslant h(2x), \qquad x \in X,$$

and g is even, then

$$h = g,$$

and g is a quadratic functional, that is

$$g(x+y)+g(x-y) = 2g(x)+2g(y), \qquad x,y \in X.$$

Proof. It is easy to check that $g(\mathbf{0}) = h(\mathbf{0}) = 0$. Setting $x = \mathbf{0}$ in the first of the assumed inequalities we get

$$h(-y) \leqslant h(y), \quad y \in X,$$

which implies that h is even. Now it enough to apply Proposition 2 with $a = 1$. □

As a special case of a more general result (cf. for instance [5], Theorem 9.5) we have the following

LEMMA 1. *Assume that X is a real linear space, $n \in \mathbb{N}$ a fixed positive integer and $a_k, b_k \in \mathbb{R}$, $k = 0,\dots,n$, such that $b_1 \cdot \ldots \cdot b_n \neq 0$. If the functions $f, f_0, f_1, \dots, f_n : X \to \mathbb{R}$ satisfy the functional equation*

$$f(x) = \sum_{k=0}^{n} f_k(a_k x + b_k y), \qquad x,y \in X,$$

then f is a generalized polynomial of degree at most n, i.e., there exist a unique constant $A_0 \in \mathbb{R}$ and unique symmetric k-additive functions $A_k : X^k \to \mathbb{R}$ for $k = 1,\dots,n$ such that

$$f(x) = \sum_{k=0}^{n} A_k(x,\dots,x), \qquad x \in X.$$

Now we prove the following

THEOREM 1. *Let X be a real linear space, $a \in \mathbb{R}$, $a \neq 0$, be fixed, and $g,h : X \to \mathbb{R}$.*

Suppose that
(i) the triple (g,h,a) satisfies the assumptions of one of Propositions 1-3;
Then

$$h = g;$$

the function g is a quadratic functional, i.e.

$$g(x+y)+g(x-y)=2g(x)+2g(y), \qquad x,y\in X;$$

and there exists a unique symmetric biadditive function $S:X^2\to\mathbb{R}$ such that

$$g(x)=S(x,x), \qquad x\in X,$$

and

$$S(ax,y)=S(x,ay), \qquad S(ax,ax)=a^2S(x,x), \qquad x,y\in X. \tag{7}$$

Proof. By (i) the functions g,h and the number a satisfy the assumptions of Proposition j for some $j\in\{1,2,3\}$. Applying Proposition j, we infer that $h=g$ and g satisfies equation (4), that is

$$g(ax+y)+g(x-ay)=(1+a^2)[g(x)+g(y)], \qquad x,y\in X,$$

whence

$$g(x)=\frac{1}{a^2+1}g(ax+y)+\frac{1}{a^2+1}g(x-ay)-g(y), \qquad x,y\in X.$$

Since $a\neq 0$, by Lemma 1, the function g must be a generalized polynomial of degree at most 2, i.e., there exist a unique constant $c\in\mathbb{R}$, a unique additive function $A:X\to\mathbb{R}$, and a unique symmetric biadditive function $S:X^2\to\mathbb{R}$ such that

$$g(x)=c+A(x)+S(x,x), \qquad x\in X. \tag{8}$$

Setting this function into equation (4) gives

$$\begin{aligned}&[c+A(ax+y)+S(ax+y,ax+y)]+[c+A(x-ay)+S(x-ay,x-ay)]\\ &=\left(1+a^2\right)\{[c+A(x)+S(x,x)]+[c+A(y)+S(y,y)]\}\end{aligned} \tag{9}$$

for all $x,y\in X$.

Setting here $x=y=\mathbf{0}$ and taking into account that $A(\mathbf{0})=S(\mathbf{0},\mathbf{0})=0$, we hence get $2c=2c\left(1+a^2\right)$, whence, as $a\neq 0$, we obtain

$$c=0. \tag{10}$$

From (9) and (10), as S is biadditive, we get

$$\begin{aligned}&A(ax+y)+A(x-ay)-\left(1+a^2\right)[A(x)+A(y)]\\ &=a^2[S(x,x)+S(y,y)]-S(ax,ax)-S(ay,ay)\\ &\quad -S(ax,y)-S(y,ax)+S(x,ay)+S(ay,x)\end{aligned} \tag{11}$$

for all $x,y\in X$.

On replacing x by $-x$ and y by $-y$, the right-hand side remains unchanged, while, by the oddness of A, the left-hand side changes its sign. It follows that

$$A(ax+y)+A(x-ay)-\left(1+a^2\right)[A(x)+A(y)]=0, \qquad x,y\in X. \tag{12}$$

Since $A(\mathbf{0})=0$, setting here $y=\mathbf{0}$, we get

$$A(ax)=a^2A(x), \qquad x\in X,$$

whence, by the additivity of A, from (12) we obtain $2a^2A(y)=0$ for all $y\in X$, i.e.

$$A=0. \tag{13}$$

From (8), (10) and (13) we obtain

$$g(x)=S(x,x), \qquad x\in X,$$

which proves that g is a quadratic function, that is that

$$g(x+y)+g(x-y)=2g(x)+2g(y)=0, \qquad x,y\in X.$$

From (11) and (13), by the symmetry of S, we get

$$a^2[S(x,x)+S(y,y)]-S(ax,ax)-S(ay,ay)=2[S(x,ay)-S(ax,y)] \tag{14}$$

for all $x,y\in X$.

Setting here $y=x$ we get (by the symmetry of S),

$$S(ax,ax)=a^2S(x,x), \qquad x\in X,$$

whence, applying (14),

$$S(x,ay)=S(ax,y), \qquad x,y\in X.$$

This completes the proof. □

Taking $a=1$ in Theorem 1 and applying Proposition 4 we obtain

COROLLARY 2. *Let X be a real linear space. If the functions $g,h:X\to\mathbb{R}$ satisfy the inequalities*

$$g(x+y)+h(x-y)\leqslant g(x)+g(y)+h(x)+h(y), \qquad x,y\in X, \tag{15}$$

$$4g(x)\leqslant g(2x), \qquad 4h(x)\leqslant h(2x), \qquad x\in X,$$

and g is even, then

$$h=g,$$

g is a quadratic functional, i.e.

$$g(x+y)+g(x-y)=2g(x)+2g(y), \qquad x,y\in X,$$

and there exists a unique symmetric biadditive function $S:X^2\to\mathbb{R}$ such that

$$g(x)=S(x,x), \qquad x\in X.$$

Proof. The assumed inequalities imply that $g(\mathbf{0}) = h(\mathbf{0}) = 0$. Setting $x = \mathbf{0}$ in (15) gives $h(-y) \leqslant h(y)$ for all $y \in X$, which obviously implies that h is even. Since, by assumption, g is even, replacing in (15) y by $-y$ we get

$$g(x-y) + h(x+y) \leqslant g(x) + g(-y) + h(x) + h(-y), \qquad x, y \in X,$$

and, consequently,

$$g(x-y) + h(x+y) \leqslant g(x) + g(y) + h(x) + h(y), \qquad x, y \in X.$$

Now the result follows from Proposition 4 and Theorem 1. □

REMARK 2. Theorem 1 as well as all other results remain true if the assumed inequalities are reversed.

THEOREM 2. *Let X be a real linear space and let $a \in \mathbb{R}$, $a \neq 0$, be fixed. Suppose that the functions $g, h : X \to \mathbb{R}$ satisfy all the conditions of Theorem 1.*

If for any $x \in X \backslash \{\mathbf{0}\}$ and for any $y \in X$, there exists a subset I_x of a positive Lebesgue measure in $\mathbb{R}$ and a positive constant $M_{x,y}$ such that

$$|g(rx + y)| \leqslant M_{x,y}, \qquad r \in I_x,$$

then $h = g$, where g is a quadratic functional, and there exists a unique symmetric bilinear function $S : X^2 \to \mathbb{R}$ such that

$$g(x) = S(x,x), \qquad x \in X.$$

If, moreover, $g \geqslant 0$ then $\|\cdot\| : X \to \mathbb{R}$ defined by

$$\|x\| := \sqrt{S(x,x)} \qquad x \in X,$$

is a seminorm in X.

Proof. By Theorem 1 we have $h = g$ and there is a biadditive function $S : X^2 \to \mathbb{R}$ such that $g(x) = S(x,x)$ for all $x \in X$. For $x, y \in X$, $x \neq \mathbf{0}$, define $f : \mathbb{R} \to \mathbb{R}$ by $f(\lambda) := S(\lambda x, y)$, $\lambda \in \mathbb{R}$. The biadditivity of S implies that f is additive. From the definition of S and the assumption we get

$$|f(\lambda)| \leqslant \frac{1}{4}\left(|g(\lambda x + y)| + |g(\lambda x - y)|\right) \leqslant \frac{1}{4}\left(M_{x,y} + M_{x,-y}\right), \qquad \lambda \in I_x,$$

that is f is bounded on a set of positive Lebesgue measure. Thus (cf. for instance [1], p. 15) $f(t) = \lambda f(1)$ for all $\lambda \in \mathbb{R}$, i.e. $S(\lambda x, y) = \lambda S(x,y)$ for all $\lambda \in \mathbb{R}$, which proves that S is bilinear.

Now the "moreover" statement is obvious. □

3. Final remarks

PROPOSITION 5. *Let X be a real linear space and let $p,q \in \mathbb{R}$ be fixed. Suppose that the functions $g,h : X \to \mathbb{R}$ satisfy the inequalities*

$$g(ax+y)+h(x-ay) \leqslant a^p g(x)+g(y)+h(x)+a^p h(y), \quad x,y \in X, \tag{16}$$

$$h(ax+y)+g(x-ay) \leqslant a^p h(x)+h(y)+g(x)+a^p g(y), \quad x,y \in X, \tag{17}$$

$$(1+a^2)^q g(x) \leqslant g\left((1+a^2)x\right), \quad (1+a^2)^q h(x) \leqslant h\left((1+a^2)x\right), \qquad x \in X, \tag{18}$$

for $a=1$ and for some $a>0$, $a \neq 1$. If the function $g+h$ takes both positive and negative values, then $p=q=2$.

Proof. Replacing x by $ax+y$ and y by $x-ay$ in (16) and then applying (17), (18) and again (17), we obtain

$$\begin{aligned} &g\left((1+a^2)x\right)+h\left((1+a^2)y\right) \\ &\leqslant a^2[g(ax+y)+h(x-ay)]+h(ax+y)+g(x-ay), \end{aligned}$$

for all $x,y \in X$. Hence, by (3), (1) and (2),

$$(1+a^2)^q[g(x)+h(y)] \leqslant (a^{2p}+1)[g(x)+h(y)]+2a^p[h(x)+g(y)],$$

for all $x,y \in X$. Interchanging x and y we get

$$(1+a^2)^q[g(y)+h(x)] \leqslant (a^{2p}+1)[g(y)+h(x)]+2a^p[h(y)+g(x)],$$

for all $x,y \in X$. Adding these inequalities by sides gives

$$[(1+a^2)^q-(a^p+1)^2][g(x)+h(x)+g(y)+h(y)] \leqslant 0,$$

for all $x,y \in X$. Since $g+h$ takes both positive and negative values, we infer that $(1+a^2)^q-(a^p+1)^2=0$. Taking $a=1$ we get $q=2$. Now, taking $a>0$, $a \neq 1$, we get $(1+a^2)^2=(a^p+1)$ whence $p=2$ follows. $\square$

REMARK 3. Proposition 5 explains why in the inequalities (1), (2), (3), as well as in some other inequalities, the numbers a^2 and $(1+a^2)^2$ appear.

REMARK 4. If $r=p$ in (16)-(18) then the assumption that the range of the functions contains positive as well as negative numbers can be omitted.

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