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TWO GENERALIZATIONS OF LANDAU'S INEQUALITY

NICUŞOR MINCULETE

Dedicated to the memory of
the great professor, Laurenţiu Panaitopol

(*Communicated by Ivan Perić*)

Abstract. Let $\pi(x)$ be the number of prime numbers not exceeding x. In the introduction we present some inequalities related to this function; these have been presented in several articles about the *Number Theory*. In the present paper we obtain two inequalities which generalize Landau's Inequality, $\pi(2x) \leqslant 2\pi(x)$ for any integer $x \geqslant 2$. Also, we obtain the inequality $2[x\pi(x)+y\pi(y)] > (x+y)\pi(x+y)$, for all integers $x,y \geqslant 67$, and an inequality which refers to the Hardy and Littlewood conjecture. To demonstrate them, we used the Personal Computer, in order to extend the domain of the variables for which these inequalities are true.

1. Introduction

Let $\pi(x)$ be the number of prime numbers not exceeding x. In [3], Landau proposed the following conjecture:

$$\pi(2x) \leqslant 2\pi(x), \tag{1}$$

for all integers $x \geqslant 2$.

Landau's Inequality was proved by Rosser and Schoenfeld in [11]. They also established other interesting inequalities connected to $\pi(x)$; among them:

$$\pi(x) > \frac{x}{\log x}, \tag{2}$$

for all integers $x \geqslant 17$, and

$$\frac{x}{\log x - 1.5} > \pi(x) > \frac{x}{\log x - 0.5}, \tag{3}$$

for all integers $x \geqslant 67$.

Panaitopol [6], [7] gives some refinements:

$$\pi(x) > \frac{x}{\log x - 1}, \tag{4}$$

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for all integers $x \geqslant 5,393$, and

$$\pi(x) < \frac{x}{\ln x - 1.12}, \tag{5}$$

for all integers $x \geqslant 4$.

The inequalities (4) and (5) were improved (see [8]):

$$\pi(x) < \frac{x}{\log x - 1 - \frac{1}{\sqrt{\log x}}}, \tag{6}$$

for all integers $x \geqslant 6$, and

$$\pi(x) > \frac{x}{\log x - 1 + \frac{1}{\sqrt{\log x}}}, \tag{7}$$

for all integers $x \geqslant 59$.

Hardy and Littlewood [2] proposed the following conjecture:

$$\pi(x+y) < \pi(x) + \pi(y), \tag{8}$$

for all integers $x, y \geqslant 2$. Schinzel [12] proved that for all positive integers x, y with $\min(x, y) \leqslant 146$, the inequality

$$\pi(x+y) \leqslant \pi(x) + \pi(y) \tag{9}$$

holds.

Montgomery and Vaughan [5] proved that for all integers $x \geqslant 1$, $y \geqslant 2$ the inequality

$$\pi(x+y) < \pi(x) + 2\pi(y) \tag{10}$$

holds. Karanikolov (see e.g. [1], [4]) noticed that for $\varepsilon \geqslant \sqrt{e} - 1$ and for all integers $x \geqslant 347$, one has

$$\pi((1+\varepsilon)x) < (1+\varepsilon)\pi(x). \tag{11}$$

Dusart [1] established the following result:

$$\frac{1}{2}[\pi(x) + \pi(2x)] \leqslant \pi(3x) \leqslant \pi(x) + \pi(2x), \tag{12}$$

for all integers $x \geqslant 2$, and

$$\pi(kx) \leqslant k\pi(x), \tag{13}$$

for all integers $x \geqslant 3$ and k a positive integer. For all integers $x > 1$

$$\pi(2x) - \pi(x) \leqslant \frac{x}{\log x}. \tag{14}$$

Panaitopol [9] proved that for integers $x, y \geqslant 2$,

$$\pi^2(x+y) \leqslant 2[\pi^2(x) + \pi^2(y)], \tag{15}$$

and for integers $x \geqslant y \geqslant 2$ one has

$$\pi(x+y) \leqslant \pi(x)+\pi(y)+\pi(x-y) \tag{16}$$

and

$$2\frac{\pi(x+y)}{x+y} \leqslant \frac{\pi(x)}{x}+\frac{\pi(y)}{y}. \tag{17}$$

The inequality (17) holds except the cases: $x=3, y=2$ and $x=5, y=2$.

2. Other inequalities involving the function $\pi(x)$

The last three results of Panaitopol, mentioned above, are generalizations of Landau's Inequality. Next, we will find other generalizations. The aim of this paper is to prove an inequality related to the Hardy and Littlewood conjecture. We use the following technique: if we want to prove that $A<B$, where A and B are functions involving $\pi(x)$, then we first write a smooth upper bound C for A and a smooth lower bound D for B using the Panaitopol, Rosser and Schoenfeld inequalities, and then all we need to check is that $C<D$.

THEOREM 1. *For every positive integers a,b,x with $bx \geqslant 5{,}393$, $ax \geqslant 4$, $a \geqslant be^{0.12}$, the inequality*

$$a\pi(bx) > b\pi(ax) \tag{18}$$

holds.

Proof. Using the inequalities (4) and (5), for all integers $t \geqslant 5{,}393$ we have $\pi(t) > \dfrac{t}{\log t - 1}$, and for all integers $t \geqslant 4$, we have $\pi(t) < \dfrac{t}{\log 2 - 1.12}$. Now, we evaluate the expression $a\pi(bx)-b\pi(ax)$, thus:if $a \geqslant be^{0.12}$, then $\log\dfrac{a}{b} \geqslant 0.12$, and this yields

$$\begin{aligned} a\pi(bx)-b\pi(ax) &> abx\left(\frac{1}{\log bx - 1} - \frac{1}{\log ax - 1.12}\right) \\ &= abx\frac{\log\frac{a}{b} - 0.12}{(\log bx - 1)(\log ax - 1.12)} \geqslant 0. \end{aligned}$$

Therefore, we obtain the claimed inequality.

COROLLARY 1.1. *It holds:*
i) for all integers $x \geqslant 2$

$$\pi(2x) \geqslant 2\pi(x), \tag{19}$$

ii) for all integers $x \geqslant 2$

$$3\pi(2x) \geqslant 2\pi(3x) \tag{20}$$

and
iii) for all integers $x \geqslant 1$ and $x \neq 12$

$$4\pi(3x) \geqslant 3\pi(4x). \tag{21}$$

Proof. In Theorem 1, taking $a=2$ and $b=1$, we get

$$\log\frac{a}{b}=\log 2=0.69....\geqslant 0.12$$

which means that, for $x\geqslant 5,393$, we obtain Landau's Inequality, $\pi(2x)\leqslant 2\pi(x)$. But using a computer, we can descend the values for x, so that this inequality is true. Checking for $x<5,393$, we conclude that Landau's Inequality is true for all integers $x\geqslant 2$. If we take $a=3$ and $b=2$, in Theorem 1, it follows that

$$\log\frac{a}{b}=\log\frac{3}{2}=0.40...\geqslant 0.12,$$

hence, we obtain $3\pi(2x)>2\pi(3x)$, for all integers $x>2,697$. Again, helped by computer, we deduce that $2\pi(2x)>2\pi(3x)$, for all integers $x\geqslant 2$. The equality holds for $x\in\{3,6,9\}$.

To prove inequality (21), in Theorem 1 we make the substitution $a=4$ and $b=3$, so,

$$\log\frac{a}{b}=\log\frac{4}{3}=0.28...\geqslant 0.12,$$

hence, we obtain $4\pi(3x)>3\pi(4x)$, for all integers $x>1,797$. By using a PC, stated inequality is true. The equality holds for $x=\{3,5\}$. □

THEOREM 2. *For all integers $x,y\geqslant 67$, the following inequality holds:*

$$2[x\pi(x)+y\pi(y)]>(x+y)\pi(x+y). \tag{22}$$

Proof. We consider the inequality (3), $\pi(t)>\dfrac{t}{\log t-0.5}$, for all integers $\geqslant 67$. In this inequality, we take $t=x$ and $t=y$ and we deduce the inequalities

$$\pi(x)>\frac{x}{\log x-0.5}\text{ and }\pi(y)>\frac{y}{\log y-0.5}.$$

Let the function $f(x)=\dfrac{x^2}{\log x-0.5}$, defined for $x\geqslant 67$.

Since $f''(x)=\dfrac{2\log^2 x-5\log x+4}{(\log x-0.5)^3)}$, for $x\geqslant 67$, we deduce that the function f is convex, which means that we can apply Jensen's Inequality, namely, $f(x)+f(y)\geqslant 2f\left(\frac{x+y}{2}\right)$. Hence

$$\begin{aligned}x\pi(x)+y\pi(y)&>\frac{x^2}{\log x-0.5}+\frac{y^2}{\log y-0.5}\geqslant 2\frac{\left(\frac{x+y}{2}\right)^2}{\log\frac{x+y}{2}-0.5}\\&>\frac{x+y}{2}\cdot\frac{x+y}{\log(x+y)-1.12}>\frac{x+y}{2}\cdot\pi(x+y).\end{aligned}$$

We used the inequality (5) and the inequality $\frac{1}{\log t - 0.5} > \frac{1}{\log 2t - 1.12}$, for all integers $t \geqslant 2$. $\square$

REMARK. 1. If we take $x = y$, in inequality (22), then we obtain Landau's Inequality, for all integer $x \geqslant 67$. From the table of prime numbers we check the rest of the numbers. Hence, Theorem 2, is a generalization of Landau's Inequality.

THEOREM 3. *For all integers* $x, y \geqslant 1,525$*, the following inequality:*

$$xy\pi(x)\pi(y) > 4\pi^2(xy) \tag{23}$$

holds.

Proof. From inequalities (6) and (7), we have $\pi(x) < \dfrac{x}{\log x - 1 - \frac{1}{\sqrt{\log x}}}$, for all integers $x \geqslant 6$, and $\pi(x) > \dfrac{x}{\log x - 1 + \frac{1}{\sqrt{\log x}}}$, for all integers $x \geqslant 59$. Therefore, we have $xy\pi(x)\pi(y) > \dfrac{x^2y^2}{\left(\log x - 1 + \frac{1}{\sqrt{\log x}}\right)\left(\log y - 1 + \frac{1}{\sqrt{\log y}}\right)}$, for all integers $x, y \geqslant 59$, and $\dfrac{4x^2y^2}{\left(\log xy - 1 - \frac{1}{\sqrt{\log xy}}\right)^2} > 4\pi^2(xy)$, for all integers x, y with $xy \geqslant 6$. If we show that

$$\left(\log xy - 1 - \frac{1}{\sqrt{\log xy}}\right)^2 > 4\left(\log x - 1 + \frac{1}{\sqrt{\log x}}\right)\left(\log y - 1 + \frac{1}{\sqrt{\log y}}\right),$$

then the proof is completed. But, for $x, y \geqslant 59$, we have $\log x,\ \log y > 4$, so, we deduce that

$$\log x - 1 + \frac{1}{\sqrt{\log x}},\ \log y - 1 + \frac{1}{\sqrt{\log y}} > 0.$$

Therefore, using the arithmetic-geometric mean inequality, we obtain

$$\log x - 1 + \frac{1}{\sqrt{\log x}} + \log y - 1 + \frac{1}{\sqrt{\log y}} \geqslant$$
$$2\sqrt{\left(\log x - 1 + \frac{1}{\sqrt{\log x}}\right)\left(\log y - 1 + \frac{1}{\sqrt{\log y}}\right)},$$

which is equivalent to

$$\left(\log xy - 2 + \frac{1}{\sqrt{\log x}} + \frac{1}{\sqrt{\log y}}\right)^2 \geqslant$$
$$4\left(\log x - 1 + \frac{1}{\sqrt{\log x}}\right)\left(\log y - 1 + \frac{1}{\sqrt{\log y}}\right).$$

At this point, we need to show that

$$\log xy - 1 - \frac{1}{\sqrt{\log xy}} > \log xy - 2 + \frac{1}{\sqrt{\log x}} + \frac{1}{\sqrt{\log y}},$$

so

$$1 > \frac{1}{\sqrt{\log xy}} + \frac{1}{\sqrt{\log x}} + \frac{1}{\sqrt{\log y}}.$$

Since $x, y \geqslant 1,525$, we obtain $\log x, \log y > 7.329$, so, we deduce that $\sqrt{\log x}$, $\sqrt{\log y} > 2.71$ and $\sqrt{\log xy} > 3.83$, which means that the inequality $1 > \frac{1}{\sqrt{\log xy}} + \frac{1}{\sqrt{\log x}} + \frac{1}{\sqrt{\log y}}$ holds true. □

COROLLARY 3.1. *For all integers $x \geqslant 67$, the following inequality holds.*

$$x\pi(x) < 2\pi(x^2). \tag{24}$$

Proof. Taking $x = y$ in inequality (23), we have $x\pi(x) \leqslant 2\pi(x^2)$, for all integers $x, y \geqslant 1,525$. Using a PC, we check the rest of the numbers up to 67. □

REMARK. 2. In relation (24), equality holds for $x \in \{13, 38, 66\}$.

THEOREM 4. *For $x \geqslant y \geqslant 10,544,111$ and $\delta \geqslant 0.00042$ the following inequality*

$$(1-\delta)\pi(x+y) < (1+\delta)\pi(x) + \pi(y) \tag{25}$$

holds.

Proof. We consider Chebyshev's function $\theta(x) = \sum_{p \leqslant x} \log p$. Then:

$$\sum_{x < p \leqslant x+y} \log p = \theta(x+y) - \theta(x).$$

From [1], for $x \geqslant 10,544,111$, we have

$$|\theta(x) - x| \leqslant 0.006788 \frac{x}{\log x}. \tag{26}$$

We note $\alpha = 0.006788$ and inequality (26) becomes

$$-\alpha \frac{x}{\log x} < \theta(x) - x < \alpha \frac{x}{\log x}. \tag{27}$$

It is easy to see that

$$\pi(x+y) = \sum_{p \leqslant x+y} 1 = \sum_{p \leqslant x} 1 + \sum_{x < p \leqslant x+y} 1 = \pi(x) + \sum_{x < p \leqslant x+y} 1.$$

For $x < p \leqslant x+y$, we have $\log x < \log p \leqslant \log(x+y)$, which means that $\frac{\log p}{\log x} > 1 \geqslant \frac{\log p}{\log(x+y)}$, and summing up the prime numbers between x and $x+y$, we obtain

$$\sum_{x<p\leqslant x+y} \frac{\log p}{\log(x+y)} < \pi(x+y) - \pi(x) < \sum_{x<p\leqslant x+y} \frac{\log p}{\log x},$$

which is equivalent to the inequality

$$\frac{1}{\log(x+y)} \sum_{x<p\leqslant x+y} \log p < \pi(x+y) - \pi(x) < \frac{1}{\log x} \sum_{x<p\leqslant x+y} \log p.$$

Therefore, we deduce the following inequality

$$\frac{\theta(x+y)-\theta(x)}{\log(x+y)} < \pi(x+y) - \pi(x) < \frac{\theta(x+y)-\theta(x)}{\log x}. \tag{28}$$

If we have $x \geqslant y \geqslant 10,544,111$, then we apply inequality (27) and we get

$$-\alpha \frac{x+y}{\log(x+y)} < \theta(x+y) - x - y < \alpha \frac{x+y}{\log(x+y)}.$$

Combining this inequality with inequality (27), we obtain

$$-\alpha \frac{x+y}{\log(x+y)} - \alpha \frac{x}{\log x} < \theta(x+y) - \theta(x) - y < \alpha \frac{x+y}{\log(x+y)} + \alpha \frac{x}{\log x},$$

and using relation (2), $\pi(x) > \frac{x}{\log x}$, for $x \geqslant 17$, we have

$$|\theta(x+y) - \theta(x) - y| < \alpha[\pi(x+y) + \pi(x)]. \tag{29}$$

But, combining inequalities (28) and (29) we obtain the sequence of inequalities

$$\pi(x+y) - \pi(x) < \frac{\theta(x+y)-\theta(x)}{\log x} <$$

$$\frac{y}{\log x} + \frac{\alpha}{\log x}[\pi(x+y) + \pi(x)] \leqslant \frac{y}{\log y} + \frac{\alpha}{\log x}[\pi(x+y) + \pi(x)].$$

Consequently,

$$\pi(x+y) - \pi(x) < \pi(y) + \frac{\alpha}{\log x}[\pi(x+y) + \pi(x)].$$

But $\log x \geqslant \ln 10,544,111 > 16.171$, which means that

$$\frac{\alpha}{\log x} \leqslant \frac{0.006788}{16.171} \cong 0.00042 \leqslant \delta,$$

so, obtain

$$(1-\delta)\pi(x+y) < (1+\delta)\pi(x)+\pi(y),$$

for all $x \geqslant y \geqslant 10,544,111$ and $\delta \geqslant 0.00042$. $\square$

Open problem. There exists a natural number n_0, large enough, such that for all $x, y \geqslant n_0$, we have the inequality

$$\pi(x^2)\pi(y^2) \geqslant \pi^2(xy).$$

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Nicuşor Minculete
"Dimitrie Cantemir" University of Braşov
Str. Bisericii Române nr. 107
Braşov, Romania
e-mail: minculeten@yahoo.com