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INTERPOLATION OF POSITIVE OPERATORS ON VARIABLE LEBESGUE SPACES

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Abstract. We prove an interpolation theorem for integral operators with positive kernel on the variable Lebesgue spaces. As an application we show that the set of exponents for spaces on which the Hardy-Littlewood maximal operator is bounded is convex.

The variable Lebesgue spaces are Banach function spaces that generalize the classical L^p spaces. Given a set $\Omega \subset \mathbb{R}^n$, a measurable function $p(\cdot) : \Omega \to [1,\infty]$ is called an exponent function, and we denote this by $p(\cdot) \in \mathscr{P}(\Omega)$. For $p(\cdot) \in \mathscr{P}(\Omega)$, let $\Omega_\infty^{p(\cdot)} = \{x \in \Omega : p(x) = \infty\}$, and let $p_+ = \operatorname{ess\,sup}\{p(x) : x \in \Omega\}$.

For any measurable function f defined on Ω let

$$\rho_{p(\cdot)}(f) = \int_{\Omega\setminus\Omega_\infty^{p(\cdot)}} |f(x)|^{p(x)}\,dx + \|f\|_{L^\infty(\Omega_\infty^{p(\cdot)})}.$$

Define $L^{p(\cdot)}(\Omega)$ to be the set of all functions f such that $\rho_{p(\cdot)}(f/\lambda) < \infty$ for some $\lambda > 0$. Then $L^{p(\cdot)}(\Omega)$ is a Banach space with the norm

$$\|f\|_{L^{p(\cdot)}(\Omega)} = \|f\|_{p(\cdot)} = \inf\{\lambda > 0 : \rho_{p(\cdot)}(f/\lambda) \leqslant 1\}.$$

These spaces have been studied extensively in the past twenty years, and we refer the reader to [5, 6, 7, 9, 11] for their basic properties and further references.

In the study of harmonic analysis on the classical L^p spaces an important tool is the interpolation of operators, either via the Marcinkiewicz interpolation theorem (i.e., the real interpolation method) or the Riesz-Thorin theorem (i.e., the complex interpolation method). It remains an important open question whether the Marcinkiewicz interpolation theorem can be generalized to variable Lebesgue spaces: see [6]. The Riesz-Thorin theorem was generalized to variable Lebesgue spaces by Musielak [10] as part of a more general result on interpolation in modular spaces. A simpler proof of his result was given by Diening, Hästö and Nekvinda [6]; also see [5].

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The purpose of this note is to prove an interpolation theorem for integral operators with positive kernels. Hereafter, $K:\Omega\times\Omega\to[0,\infty)$ will be a non-negative, measurable function, and we will define the integral operator T by

$$Tf(x)=\int_\Omega K(x,y)f(y)\,dy.$$

Given $p_i(\cdot)\in\mathscr{P}(\Omega)$, $i=1,2$, define the interpolation exponents

$$p_\theta(\cdot),q_\theta(\cdot):\Omega\to[1,\infty]$$

by

$$\frac{1}{p_\theta(x)}=\frac{\theta}{p_1(x)}+\frac{1-\theta}{p_2(x)},\qquad \frac{1}{q_\theta(x)}=\frac{\theta}{q_1(x)}+\frac{1-\theta}{q_2(x)},$$

with the convention that $1/\infty=0$.

THEOREM 1. *Given a set $\Omega\subseteq\mathbb{R}^n$, and $p_i(\cdot),q_i(\cdot)\in\mathscr{P}(\Omega)$, $i=1,2$, suppose*

$$\|Tf\|_{q_i(\cdot)}\leqslant B_i\|f\|_{p_i(\cdot)},\qquad i=1,2.$$

Then for each θ, $0<\theta<1$,

$$\|Tf\|_{q_\theta(\cdot)}\leqslant 48B_1^\theta B_2^{1-\theta}\|f\|_{p_\theta(\cdot)}. \tag{1}$$

REMARK 2. The constant 48 in (1) is a universal constant; however, implicit in our proof is a sharper constant that depends on the exponent functions. Details are left to the reader.

In Theorem 1 we have stronger hypotheses than those for the general complex interpolation theorem (see [6, 10]) since our result only holds for positive integral operators. In these results it was assumed that $p_+<\infty$; more recently, however, in [5] complex interpolation was extended to the case of unbounded exponents. Theorem 1 also has the advantage of a much simpler proof.

When $p_1(\cdot)=q_1(\cdot)=p(\cdot)$, $p_2(\cdot)=q_2(\cdot)=p'(\cdot)$, where $p'(\cdot)$ is the dual exponent defined by

$$1=\frac{1}{p(x)}+\frac{1}{p'(x)},$$

then for $\theta=1/2$ we have $p_\theta(\cdot)=2$. In this particular case the conclusion of Theorem 1 also follows from an interpolation result due to Karlovich and Lerner [8].

As a corollary to Theorem 1 we can prove an extension of a convexity result for the Hardy-Littlewood maximal operator. The Hardy-Littlewood maximal operator is defined by

$$Mf(x)=\sup_{Q\ni x}\frac{1}{|Q|}\int_Q|f(y)|\,dy=\sup_{Q\ni x}\fint_Q|f(y)|\,dy,$$

where the supremum is taken over all cubes Q with sides parallel the coordinate axes. (Equivalently, the averages may be taken over balls that contain x.) The boundedness of the maximal operator on the variable Lebesgue spaces has been studied extensively, and we refer the reader to [2, 4, 5] for details and further references.

In [6] it was shown that set of exponents $p(\cdot) \in \mathscr{P}(\Omega)$ such that $p_+ < \infty$ and M is bounded on $L^{p(\cdot)}(\Omega)$ is convex: i.e., if M is bounded on $L^{p_1(\cdot)}(\Omega)$ and $L^{p_2(\cdot)}(\Omega)$, then it is bounded on $L^{p_\theta(\cdot)}(\Omega)$. The proof relied on the characterization of the boundedness of the maximal operator in terms of certain linear operators. However, there exist unbounded exponents $p(\cdot)$ such that M is bounded on $L^{p(\cdot)}(\Omega)$, and this proof does not work in this setting, even using the full complex interpolation theorem. We can generalize this result by removing the restriction that p_+ is finite. Since the maximal operator is not an integral operator, we cannot apply Theorem 1 to it; rather we proceed indirectly. Given any measurable function f there exists a collection of disjoint open sets E_k and cubes Q_k with $E_k \subset Q_k$, such that if we define

$$K_0(x,y) = \sum_k \chi_{E_k}(x)|Q_k|^{-1}\chi_{Q_k}(y), \tag{2}$$

then the operator T with this kernel satisfies

$$|Tf(x)| \leqslant T(|f|)(x) \leqslant Mf(x) \leqslant 2T(|f|)(x).$$

(Note that the first two inequalities always hold for any such operator T; the key point is that it is possible to find a partition such that the last inequality holds. This is the well-known technique of “linearizing” the maximal operator: see, for instance, de la Torre [3].) It follows at once from this inequality that M is bounded on a variable Lebesgue space if and only if the family of linear operators with kernels of the form (2) are bounded with uniform constant. Therefore, we can apply Theorem 1 to these operators to get the following corollary.

COROLLARY 3. *Given Ω and $p_i(\cdot) \in \mathscr{P}(\Omega)$, $i = 1,2$, suppose the Hardy-Littlewood maximal operator satisfies $\|Mf\|_{p_i(\cdot)} \leqslant B_i\|f\|_{p_i(\cdot)}$. Then for all θ, $0 < \theta < 1$,*

$$\|Mf\|_{p_\theta(\cdot)} \leqslant 48B_1^\theta B_2^{1-\theta}\|f\|_{p_\theta(\cdot)}.$$

Proof of Theorem 1. Our proof is modelled on the proof of this result for classical Lebesgue spaces given by Bennett and Sharpley [1]. Fix θ and fix $f \in L^{p_\theta(\cdot)}(\Omega)$; since $|T(af)(x)| \leqslant |a|T(|f|)(x)$, we may assume without loss of generality that f is non-negative and $\|f\|_{p_\theta(\cdot)} = 1$. Given any exponent $p(\cdot)$,

$$\|f\|_{p(\cdot)} \leqslant 3 \sup_g \int_\Omega |f(x)g(x)|\,dx,$$

where the supremum is taken over all $g \in L^{p'(\cdot)}(\Omega)$ with $\|g\|_{p'(\cdot)} \leqslant 1$. (See [9].) Therefore, to prove (1) it will suffice to prove that for all non-negative functions $g \in L^{q'_\theta(\cdot)}(\Omega)$, $\|g\|_{q'_\theta(\cdot)} \leqslant 1$,

$$\int_\Omega Tf(x)g(x)\,dx \leqslant 16B_1^\theta B_2^\theta.$$

To prove this we first define the functions $f_i(x) = f(x)^{\frac{p_\theta(x)}{p_i(x)}}$. To make sense of this when the exponent functions are infinite, note that $\Omega_\infty^{p_\theta(\cdot)} \subset \Omega_\infty^{p_1(\cdot)} \cap \Omega_\infty^{p_2(\cdot)}$. Then for $x \in \Omega_\infty^{p_i(\cdot)}$ we define

$$\frac{p_\theta(x)}{p_i(x)} = \begin{cases} 1 & x \in \Omega_\infty^{p_\theta(\cdot)} \\ 0 & x \in \Omega_\infty^{p_i(\cdot)} \setminus \Omega_\infty^{p_\theta(\cdot)}. \end{cases}$$

We define the functions $g_i(x) = g(x)^{\frac{q'_\theta(x)}{q'_i(x)}}$ in the same way, using the fact that $\Omega_\infty^{q'_\theta(\cdot)} \subset \Omega_\infty^{q'_1(\cdot)} \cap \Omega_\infty^{q'_2(\cdot)}$ and observing that for exponent functions, "duality" and "interpolation" commute. More precisely, given exponents $q_1(\cdot), q_2(\cdot)$, then the interpolation exponent between the dual exponents $q'_1(\cdot)$ and $q'_2(\cdot)$ is the same as $q'_\theta(\cdot)$, the dual of the interpolation exponent between $q_1(\cdot)$ and $q_2(\cdot)$.

We now claim that $\|f_i\|_{p_i(\cdot)}, \|g_i\|_{q'_i(\cdot)} \leqslant 2$, $i = 1,2$. We will show this for f_1; the proofs for the other three functions are identical. Since $\|f\|_{p_\theta(\cdot)} = 1$, $\rho_{p_\theta(\cdot)}(f) \leqslant 1$ (see [9]). In particular,

$$\int_{\Omega\setminus\Omega_\infty^{p_\theta(\cdot)}} |f(x)|^{p_\theta(x)}\,dx \leqslant 1, \qquad \|f\|_{L^\infty(\Omega_\infty^{p_\theta(\cdot)})} \leqslant 1. \tag{3}$$

For almost every $x \in \Omega_\infty^{p_1(\cdot)}$, $f_1(x) \leqslant 1$. If $x \in \Omega_\infty^{p_\theta(\cdot)}$, this follows from the second inequality in (3) and the fact that $p_\theta(x)/p_1(x) = 1$; for $x \in \Omega_\infty^{p_1(\cdot)} \setminus \Omega_\infty^{p_\theta(\cdot)}$ this follows since $p_\theta(x)/p_1(x) = 0$. Hence,

$$\begin{aligned}
&\|f_1\|_{p_1(\cdot)} \\
&\quad = \inf\left\{\lambda > 0 : \int_{\Omega\setminus\Omega_\infty^{p_1(\cdot)}} \left(\frac{|f_1(x)|}{\lambda}\right)^{p_1(x)} dx + \lambda^{-1}\|f_1\|_{L^\infty(\Omega_\infty^{p_1(\cdot)})} \leqslant 1\right\} \\
&\quad \leqslant \inf\left\{\lambda > 0 : \int_{\Omega\setminus\Omega_\infty^{p_1(\cdot)}} \left(\frac{|f_1(x)|}{\lambda}\right)^{p_1(x)} dx + \lambda^{-1} \leqslant 1\right\} \\
&\quad = \inf\left\{\lambda > 1 : \int_{\Omega\setminus\Omega_\infty^{p_1(\cdot)}} \left(\frac{|f_1(x)|}{\lambda}\right)^{p_1(x)} dx + \lambda^{-1} \leqslant 1\right\} \\
&\quad \leqslant \inf\left\{\lambda > 1 : \lambda^{-1}\int_{\Omega\setminus\Omega_\infty^{p_\theta(\cdot)}} |f(x)|^{p(x)}\,dx + \lambda^{-1} \leqslant 1\right\} \\
&\quad \leqslant \inf\{\lambda > 1 : 2\lambda^{-1} \leqslant 1\} \\
&\quad = 2.
\end{aligned}$$

Again by our definition of the exponents we have the identity $f(x) = f_1(x)^\theta f_2(x)^{1-\theta}$ and similarly for g. Therefore, since K, f and g are non-negative, by Hölder's inequality with exponent θ^{-1}, the generalized Hölder's inequality in variable Lebesgue spaces (see [9]) and our hypothesis, we have that

$$\begin{aligned}
&\int_\Omega Tf(x)g(x)\,dx \\
&\quad = \int_\Omega \int_\Omega K(x,y)f(y)g(x)\,dy\,dx \\
&\quad = \int_\Omega \int_\Omega \big(K(x,y)f_1(y)g_1(x)\big)^\theta \big(K(x,y)f_2(y)g_2(x)\big)^{1-\theta}\,dy\,dx \\
&\quad \leqslant \left(\int_\Omega \int_\Omega K(x,y)f_1(y)g_1(x)\,dy\,dx\right)^\theta \\
&\qquad \times \left(\int_\Omega \int_\Omega K(x,y)f_2(y)g_2(x)\,dy\,dx\right)^{1-\theta} \\
&\quad = \left(\int_\Omega Tf_1(x)g_1(x)\,dx\right)^\theta \left(\int_\Omega Tf_2(x)g_2(x)\,dx\right)^{1-\theta} \\
&\quad \leqslant \big(4\|Tf_1\|_{q_1(\cdot)}\|g_1\|_{q_1'(\cdot)}\big)^\theta \big(4\|Tf_2\|_{q_2(\cdot)}\|g_2\|_{q_2'(\cdot)}\big)^{1-\theta} \\
&\quad \leqslant 8B_1^\theta B_2^{1-\theta}\|f_1\|_{p_1(\cdot)}^\theta \|f_2\|_{p_2(\cdot)}^{1-\theta} \\
&\quad \leqslant 16B_1^\theta B_2^{1-\theta}.
\end{aligned}$$

This completes the proof. □

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