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SHEPHARD TYPE PROBLEMS FOR THE NEW GEOMETRIC BODY $\Gamma_{-p}K$

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Abstract. Lutwak, Yang and Zhang proposed the notion of the new geometric body $\Gamma_{-p}K$. In this article, we research the Shephard-type problems for the new geometric body $\Gamma_{-p}K$.

1. Introduction and main results

Let $\mathscr{K}^n$ denote the set of convex bodies (compact, convex subsets with nonempty interiors) in Euclidean space $\mathbb{R}^n$. For the set of convex bodies containing the origin in their interiors and the set of origin-symmetric convex bodies in $\mathbb{R}^n$, we write $\mathscr{K}_o^n$ and $\mathscr{K}_c^n$, respectively. Let $\mathscr{S}_o^n$ denote the set star bodies (about the origin) in $\mathbb{R}^n$. Let S^{n-1} denote the unit sphere in $\mathbb{R}^n$, denote by $V(K)$ the n-dimensional volume of body K, for the standard unit ball B in $\mathbb{R}^n$, denote $\omega_n = V(B)$.

Lutwak, Yang and Zhang in [2] introduced the new geometric body $\Gamma_{-p}K$ as follows: Let $K \in \mathscr{K}_o^n$, real $p > 0$, the new geometric body $\Gamma_{-p}K$ is defined as the body whose radial function is given by:

$$\rho_{\Gamma_{-p}K}^{-p}(u) = \frac{1}{V(K)} \int_{S^{n-1}} |u \cdot v|^p dS_p(K,v) \tag{1.1}$$

for all $u \in S^{n-1}$. Note for $p \geqslant 1$ the body $\Gamma_{-p}K$ is a convex body. Define $\Gamma_{-\infty}K$ by

$$\Gamma_{-\infty}K = \lim_{p\to\infty} \Gamma_{-p}K.$$

Further, when K is origin-symmetric, $\Gamma_{-\infty}K = K$.

From (1.1), we easily see

$$\Gamma_{-p}(-K) = \Gamma_{-p}K. \tag{1.2}$$

The new geometric body $\Gamma_{-p}K$ has great attention from other articles (see [8, 9, 11, 12]). Especially, Wang and Leng in [12] gave the Shephard-type problem for the new geometric body $\Gamma_{-p}K$ as follows.

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THEOREM A. *If* $K \in \mathscr{Z}_p$, $L \in \mathscr{K}_o^n$, $p \geqslant 1$ *and* $\Gamma_{-p}K \subseteq \Gamma_{-p}L$, *then*

$$V(K) \leqslant V(L), \tag{1.3}$$

with equality for $p = 1$ *if and only if* K *and* L *are translation, for* $p > 1$ *if and only if* $K = L$. *Here* $\mathscr{Z}_p$ *denotes the class of* L_p*-centroid bodies, i.e.*

$$\mathscr{Z}_p = \{Z | h^p(Z, \cdot) = h^p(\Gamma_p M, \cdot) = \frac{1}{c_{n,p}V(M)} \int_M | u \cdot x |^p \, dx, for\ M \in \mathscr{S}_o^n\},$$

where $\Gamma_p M$ *denotes the* L_p*-centroid body of* M *which was introduced by Lutwak and Zhang (see [6]).*

In this article, we shall continuously study the new geometric body $\Gamma_{-p}K$. Firstly, associated with Theorem A, we give the negative form of the Shephard-type problem for the new geometric body $\Gamma_{-p}K$ as follows:

THEOREM 1.1. *If* $1 \leqslant p \leqslant n$, K *is not an origin-symmetric body, then there exists an origin-symmetric star body* L, *such that*

$$\Gamma_{-p}K \subset \Gamma_{-p}L,$$

but

$$V(K) > V(L). \tag{1.4}$$

Secondly, we also get L_p-affine surface area form of Shephard-type problem for the new geometric body $\Gamma_{-p}K$.

THEOREM 1.2. *For* $K \in \mathscr{W}_n^p$, $L \in \mathscr{F}_o^n$, $p \geqslant 1$, *if* $\Gamma_{-p}K \subseteq \Gamma_{-p}L$, *then*

$$\frac{\Omega_p^{\frac{n+p}{n}}(K)}{V(K)} \geqslant \frac{\Omega_p^{\frac{n+p}{n}}(L)}{V(L)}, \tag{1.5}$$

with equality for $p = 1$ *if and only if* K *and* L *are translations, for* $p > 1$ *if and only if* $K = L$.

Further, we characterize the equality of two new geometric bodies by L_p-mixed volume as follows:

THEOREM 1.3. *If* $K, L \in \mathscr{K}_o^n$, $p \geqslant 1$, *then* $\Gamma_{-p}K = \Gamma_{-p}L$ *if and only if for any* $Q \in \mathscr{K}_c^n$

$$\frac{V_p(K, Q)}{V(K)} = \frac{V_p(L, Q)}{V(L)}. \tag{1.6}$$

From Theorem 1.3, we get an improve version of Theorem A:

THEOREM 1.4. *For* $K \in \mathscr{K}_c^n$, $L \in \mathscr{K}_o^n$, $p \geqslant 1$, *if*

$$\Gamma_{-p}K = \Gamma_{-p}L,$$

then

$$V(K) \leqslant V(L). \tag{1.7}$$

with equality for $p = 1$ *if and only if* K *and* L *are translations, for* $p > 1$ *if and only if* $K = L$.

2. Preliminaries

2.1. Support function and radial function

If $K \in \mathscr{K}^n$, then its support function, $h_K = h(K,\cdot)\colon \mathbb{R}^n \longrightarrow (-\infty,\infty)$, is defined by (see [1, 7])

$$h(K,\cdot) = \max\{u\cdot x : x \in K\}, \quad u \in S^{n-1},$$

where $u\cdot x$ denotes the standard inner product of u and x.

If K is a compact star-shaped (about the origin) in $\mathbb{R}^n$, then its radial function, $\rho_K = \rho(K,\cdot)\colon \mathbb{R}^n \setminus \{0\} \longrightarrow [0,\infty)$, is defined by (see [1, 7])

$$\rho(K,u) = \max\{\lambda \geqslant 0 : \lambda\cdot u \in K\}, \quad u \in S^{n-1}.$$

If ρ_K is continuous and positive, then K will be called a star body. Two star bodies K,L are said to be dilates (of one another) if $\rho_K(u)/\rho_L(u)$ is independent of $u \in S^{n-1}$.

2.2. L_p-mixed volume

For $K,L \in \mathscr{K}_o^n$, $p \geqslant 1$ and $\varepsilon > 0$, the Firey L_p-combination $K +_p \varepsilon\cdot L \in \mathscr{K}_o^n$ is defined by (see [4])

$$h(K +_p \varepsilon\cdot L,\cdot)^p = h(K,\cdot)^p + \varepsilon h(L,\cdot)^p,$$

where "$\cdot$" in $\varepsilon\cdot L$ denotes the Firey scalar multiplication.

If $K,L \in \mathscr{K}_o^n$ in $\mathbb{R}^n$, then for $p \geqslant 1$, the L_p-mixed volume, $V_p(K,L)$, of K and L is defined by (see [4])

$$\frac{n}{p}V_p(K,L) = \lim_{\varepsilon\longrightarrow 0^+}\frac{V(K +_p \varepsilon\cdot L) - V(K)}{\varepsilon}.$$

Corresponding to each $K,L \in \mathscr{K}_o^n$, there is a positive Borel measure, $S_p(K,\cdot)$, on S^{n-1} such that (see [4])

$$V_p(K,L) = \frac{1}{n}\int_{S^{n-1}} h_L^p(u)\,dS_p(K,u). \tag{2.1}$$

From (2.1), we have

$$V_p(K,K) = V(K). \tag{2.2}$$

The Minkowski inequality for the L_p-mixed volume is called L_p-Minkowski inequality. The L_p-Minkowski inequality was given by Lutwak (see [4, 5]):

THEOREM B. *If $K,L \in \mathscr{K}_o^n$ and $p \geqslant 1$ then*

$$V_p(K,L) \geqslant V(K)^{\frac{n-p}{n}}V(L)^{\frac{p}{n}}, \tag{2.3}$$

with equality for $p = 1$ if and only if K and L are homothetic, for $p > 1$ if and only if K and L are dilates.

2.3. L_p-dual mixed volume

For $K,L \in \mathscr{S}_o^n$, and $\varepsilon > 0$, the L_p-harmonic radial combination, $K \tilde{+}_{-p} \varepsilon \star L \in \mathscr{S}_o^n$, of K and L is defined by (see [4, 5])

$$\rho(K \tilde{+}_{-p} \varepsilon \star L, \cdot)^{-p} = \rho(K,\cdot)^{-p} + \varepsilon \rho(L,\cdot)^{-p}.$$

If $K,L \in \mathscr{S}_o^n$, for $P \geqslant 1$, the L_p-dual mixed volume, $V_{-p}(K,L)$, of the K and L is defined by (see [4, 5])

$$-\frac{n}{p} V_{-p}(K,L) = \lim_{\varepsilon \longrightarrow 0^+} \frac{V(K \tilde{+}_{-p} \varepsilon \star L) - V(K)}{\varepsilon}.$$

The definition above and the polar coordinate formula for volume give the following integral representation of the L_p-dual mixed volume, $V_{-p}(K,L)$ of $K,L \in \mathscr{S}_o^n$:

$$V_{-p}(K,L) = \frac{1}{n} \int_{S^{n-1}} \rho_K^{n+p}(u) \rho_L^{-p}(u) dS(u), \tag{2.4}$$

where the integration is with respect to spherical Lebesgue measure S on $\mathscr{S}_o^n$.

From the formula (2.4), we get

$$V_{-p}(K,K) = V(K).$$

2.4. L_p-mixed affine surface area

For $K,L \in \mathscr{F}_o^n$, $p \geqslant 1$ and $i \in \mathbb{R}$, the L_p-mixed affine surface area, $\Omega_{p,i}(K,L)$, of K and L is defined by (see [10])

$$\Omega_{p,i}(K,L) = \int_{S^{n-1}} f_p(K,u)^{\frac{n-i}{n+p}} f_p(L,u)^{\frac{i}{n+p}} dS(u).$$

Specially, for the case $i = -p$, we have that

$$\Omega_{p,-p}(K,L) = \int_{S^{n-1}} f_p(K,u) f_p(L,u)^{\frac{-p}{n+p}} dS(u). \tag{2.5}$$

Obviously, from (2.5),we have that

$$\Omega_{p,i}(K,K) = \Omega_p(K). \tag{2.6}$$

The Minkowski inequality for the L_p-mixed affine surface area was given by Wang and Leng (see [10]):

THEOREM C. *If $K,L \in \mathscr{F}_o^n$ and $p \geqslant 1$, $i \in \mathbb{R}$, then for $i < 0$ or $i > n$,*

$$\Omega_{p,i}(K,L)^n \geqslant \Omega_p(K)^{n-i} \Omega_p(L)^i, \tag{2.7}$$

with equality for $p = 1$ if and only if K and L are homothetic, for $n \neq p > 1$ if and only if K and L are dilates; for $0 < i < n$, (2.7) is reverse; for $i = 0$ or $i = n$, (2.7) is identical.

2.5. L_p-surface area combination

A convex body $K \in \mathscr{K}_o^n$ is said to have a L_p-curvature function (see [5]) $f_p(K,\cdot)$: $S^{n-1} \longrightarrow \mathbb{R}$, if its L_p-surface area measure $S_p(K,\cdot)$ is absolutely continuous with respect to spherical Lebesgue measure S, and

$$\frac{dS_p(K,\cdot)}{dS} = f_p(K,\cdot). \tag{2.8}$$

Let $\mathscr{F}_o^n$, $\mathscr{F}_c^n$ denote the set of all bodies in $\mathscr{K}_o^n$, $\mathscr{K}_c^n$, respectively, and both of them have a positive continuous curvature function.

Associate with the volume normalized L_p-Minkowski problem given by Lutwak, Yang and Zhang, they obtained such result in [3]: Suppose $p \geqslant 1$. If μ is an even Borel measure on S^{n-1} whose support is not contained in a great subsphere of S^{n-1}, then there exists a convex body K, symmetric about the origin, such that

$$\frac{h(K)^{1-p}}{V(K)} dS(K,\cdot) = d\mu. \tag{2.9}$$

Now according to (2.9), we give the notion of L_p-surface area combination as follows:

For $K, L \in \mathscr{K}_c^n$, λ, $n \neq p \geqslant 1$, the L_p-surface area combination, $K \hat{+}_p L \in \mathscr{K}_c^n$ of K and L is defined by

$$\frac{dS_p(K \hat{+}_p L,\cdot)}{V(K \hat{+}_p L)} = \frac{dS_p(K,\cdot)}{V(K)} + \frac{dS_p(L,\cdot)}{V(L)}. \tag{2.10}$$

According to (2.9), we also should define the L_p-surface area body, $\hat{\nabla}_p K \in \mathscr{K}_c^n$, of $K \in \mathscr{K}_o^n$ by

$$\frac{dS_p(\hat{\nabla}_p K,\cdot)}{V(\hat{\nabla}_p K)} = \frac{1}{2} \cdot \frac{dS_p(K,\cdot)}{V(K)} + \frac{1}{2} \cdot \frac{dS_p(-K,\cdot)}{V(-K)}. \tag{2.11}$$

Obviously, L_p-surface area body $\hat{\nabla}_p K$ is origin-symmetric.

3. The Shephard type problems

In this section, we will complete the proof of Theorems. Here we need the following several lemmas.

LEMMA 3.1. *If $K \in \mathscr{K}_o^n$, $p \geqslant 1$, then*

$$V(\hat{\nabla}_p K) \leqslant V(K), \tag{3.1}$$

with equality if and only if K is an origin-symmetric body.

Proof. From the definition (2.1) and using (2.10), (2.11), we have that for any $Q \in \mathscr{K}_c^n$

$$\begin{aligned}
\frac{V_p(\hat{\nabla}_p K, Q)}{V(\hat{\nabla}_p K)} &= \frac{1}{n}\frac{1}{V(\hat{\nabla}_p K)}\int_{S^{n-1}} h_Q^p(v)dS_p(\hat{\nabla}_p K, v) \\
&= \frac{1}{n}\int_{S^{n-1}} h_Q^p(v)\frac{dS_p(\hat{\nabla}_p K, v)}{V(\hat{\nabla}_p K)} \\
&= \frac{1}{nV(K)}\int_{S^{n-1}} h_Q^p(v)\left[\frac{1}{2}dS_p(K,v)+\frac{1}{2}dS_p(-K,v)\right] \\
&= \frac{1}{nV(K)}\int_{S^{n-1}} \left[\frac{1}{2}h_Q^p(v)dS_p(K,v)\right]+\left[\frac{1}{2}h_Q^p(v)dS_p(K,-v)\right]
\end{aligned}$$

Since $Q \in \mathscr{K}_c^n$, then $h_Q(v) = h_Q(-v)$ for all $v \in S^{n-1}$. Therefore

$$\begin{aligned}
\frac{V_p(\hat{\nabla}_p K, Q)}{V(\hat{\nabla}_p K)} &= \frac{1}{nV(K)}\int_{S^{n-1}} \left[\frac{1}{2}h_Q^p(v)dS_p(K,v)+\frac{1}{2}h_Q^p(-v)dS_p(K,-v)\right] \\
&= \frac{V_p(K,Q)}{2V(K)}+\frac{V_p(K,Q)}{2V(K)} \\
&= \frac{V_p(K,Q)}{V(K)}.
\end{aligned}$$

Taking $Q = \hat{\nabla}_p K$, from (2.2), we know

$$V_p(K, \hat{\nabla}_p K) = V(K).$$

By using (2.3), we can get

$$V(K) = V_p(K, \hat{\nabla}_p K) \geqslant V(K)^{\frac{n-p}{n}} V(\hat{\nabla}_p K)^{\frac{p}{n}}.$$

So we have

$$V(\hat{\nabla}_p K) \leqslant V(K).$$

According to the equality condition of (2.3), we see that equality holds in (3.1) if and only if K is an origin-symmetric body. □

LEMMA 3.2. *If $K \in \mathscr{K}_o^n$, $p \geqslant 1$, then*

$$\Gamma_{-p}\hat{\nabla}_p K = \Gamma_{-p} K. \tag{3.2}$$

Proof. From the definitions (1.1), (2.11) and (1.2), we know for all $u \in S^{n-1}$,

$$\begin{aligned}
\rho^{-p}_{\Gamma_{-p}\hat{\nabla}_p K}(u) &= \frac{1}{V(\hat{\nabla}_p K)} \int_{S^{n-1}} |u\cdot v|^p dS_p(\hat{\nabla}_p K, v) \\
&= \int_{S^{n-1}} |u\cdot v|^p \frac{dS_p(\hat{\nabla}_p K, v)}{V(\hat{\nabla}_p K)} \\
&= \frac{1}{V(K)} \int_{S^{n-1}} |u\cdot v|^p \left[\frac{1}{2} dS_p(K,v) + \frac{1}{2} dS_p(-K,v)\right] \\
&= \frac{1}{2}\rho^{-p}_{\Gamma_{-p}K}(u) + \frac{1}{2}\rho^{-p}_{\Gamma_{-p}(-K)}(u) = \rho^{-p}_{\Gamma_{-p}K}(u).
\end{aligned}$$

So we get for all $u \in S^{n-1}$,

$$\rho_{\Gamma_{-p}\hat{\nabla}_p K}(u) = \rho_{\Gamma_{-p}K}(u).$$

From this, (3.2) is obtained. □

Proof of Theorem 1.1. From (3.1), when K is not origin-symmetric, we can get

$$V(\hat{\nabla}_p K) < V(K).$$

Since $L \in \mathscr{K}_c^n$, taking $L = (1+\varepsilon)\hat{\nabla}_p K$ $(0 < \varepsilon < 1)$ such that

$$V(L) < V(K).$$

By using (3.2), we know for $n \geqslant p$,

$$\begin{aligned}
\Gamma_{-p}L &= \Gamma_{-p}(1+\varepsilon)\hat{\nabla}_p K \\
&= (1+\varepsilon)^{n-p}\Gamma_{-p}\hat{\nabla}_p K \\
&= (1+\varepsilon)^{n-p}\Gamma_{-p}K \supset \Gamma_{-p}K.
\end{aligned}$$

So (1.4) is obtained. □

LEMMA 3.3. [12] *If $K \in \mathscr{K}_o^n$, $L \in \mathscr{S}_o^n$, $p \geqslant 1$, then*

$$V_p(K, \Gamma_p L) = \frac{V(K)}{nc_{n-2,p}V(L)} V_{-p}(L, \Gamma_{-p}K). \tag{3.3}$$

LEMMA 3.4. *If $K, L \in \mathscr{K}_o^n$, $p \geqslant 1$, and $\Gamma_{-p}K \subseteq \Gamma_{-p}L$, then for all $Z \in \mathscr{Z}_p$,*

$$\frac{V_p(K,Z)}{V(K)} \geqslant \frac{V_p(L,Z)}{V(L)}. \tag{3.4}$$

Here $\mathscr{Z}_p$ denotes the class of L_p-centroid bodies.

Proof. Since $Z \in \mathscr{Z}_p$, then exists a $M \in \mathscr{K}_o^n$, such that $Z = \Gamma_p M$. Hence from (3.3), we have

$$\frac{V_p(K,Z)}{V(K)} = \frac{V_p(K,\Gamma_p M)}{V(K)} = \frac{1}{nc_{n-2,p}V(M)} V_{-p}(M, \Gamma_{-p}K)$$

Since $\Gamma_{-p}K \subseteq \Gamma_{-p}L$, according to (2.4), we have that

$$V_{-p}(M,\Gamma_{-p}K) \geqslant V_{-p}(M,\Gamma_{-p}L)$$

Therefore,

$$\frac{V_p(K,\Gamma_p M)}{V(K)} \geqslant \frac{V_p(L,\Gamma_p M)}{V(L)}.$$

The (3.4) is obtained. □

LEMMA 3.5. *If* $K,L \in \mathscr{F}_o^n$, $Q \in \mathscr{W}_n^p$, $p \geqslant 1$, *then*

$$\frac{\Omega_{p,-p}(K,Q)}{\Omega_{p,-p}(L,Q)} = \frac{V_p(K,Z)}{V_p(L,Z)}. \tag{3.5}$$

Here $\mathscr{W}_n^p = \{Q \in \mathscr{F}_o^n :$ *there exists a* $Z \in \mathscr{Z}_p$ *with* $f_p(Q,u) = h_Z^{-(n+p)}(u)\}$.

Proof. From (2.1) and (2.5), we have for all $Z \in \mathscr{Z}_p$,

$$\begin{aligned} V_p(K,Z) &= \frac{1}{n}\int_{S^{n-1}} h_Z^p(u)dS_p(K,u) \\ &= \frac{1}{n}\int_{S^{n-1}} f_p(Q,u)^{\frac{-p}{n+p}} dS_p(K,u) \\ &= \frac{1}{n}\int_{S^{n-1}} f_p(L,u) f_p(Q,u)^{\frac{-p}{n+p}} dS(u) \\ &= \frac{1}{n}\Omega_{p,-p}(K,Q). \end{aligned}$$

Similarly, we may get

$$V_p(L,Z) = \frac{1}{n}\Omega_{p,-p}(L,Q).$$

Thus

$$\frac{\Omega_{p,-p}(K,Q)}{\Omega_{p,-p}(L,Q)} = \frac{V_p(K,Z)}{V_p(L,Z)},$$

This give (3.5). □

Proof of Theorem 1.2. Since $\Gamma_{-p}K \subseteq \Gamma_{-p}L$, thus by (3.4) and (3.5) we know

$$\frac{\Omega_{p,-p}(K,Q)}{V(K)} \Big/ \frac{\Omega_{p,-p}(L,Q)}{V(L)} = \frac{V_p(K,Z)}{V(K)} \Big/ \frac{V_p(L,Z)}{V(L)} \geqslant 1.$$

Then we have

$$\frac{\Omega_{p,-p}(K,Q)}{V(K)} \geqslant \frac{\Omega_{p,-p}(L,Q)}{V(L)}.$$

But $Q \in \mathscr{W}_n^p$, taking $Q = K$ and using (2.6), (2.7), we get

$$\frac{\Omega_p(K)}{V(K)} \geqslant \frac{\Omega_{p,-p}(L,Q)}{V(L)} \geqslant \frac{\Omega_p^{\frac{n+p}{n}}(L)\Omega_p^{\frac{-p}{n}}(K)}{V(L)}.$$

So

$$\frac{\Omega_p^{\frac{n+p}{n}}(K)}{V(K)} \geqslant \frac{\Omega_p^{\frac{n+p}{n}}(L)}{V(L)}.$$

Equality holds in equality (2.7) for $p = 1$ if and only if K and L are homothetic, for $p > 1$ if and only if K and L are dilates. This together with the condition of $\Gamma_{-p}K = \Gamma_{-p}L$, we know that equality holds in inequality (1.5) for $p = 1$ if and only if K and L are translations, for $p > 1$ if and only if $K = L$. $\square$

Proof of Theorem 1.3. First, we prove that if for any $Q \in \mathscr{K}_c^n$,

$$\frac{V_p(K,Q)}{V(K)} = \frac{V_p(L,Q)}{V(L)},$$

then $\Gamma_{-p}K = \Gamma_{-p}L$.

From (2.1), we have

$$\frac{V_p(K,Q)}{V(K)} = \frac{1}{nV(K)} \int_{S^{n-1}} h_Q^p(v) dS_p(K,v),$$

$$\frac{V_p(L,Q)}{V(L)} = \frac{1}{nV(L)} \int_{S^{n-1}} h_Q^p(v) dS_p(L,v). \tag{3.6}$$

Since $Q \in \mathscr{K}_c^n$, taking $Q = [-u,u]$, then we know for every $v \in \mathscr{S}^{n-1}$

$$h_Q(v) = |u \cdot v|. \tag{3.7}$$

Combing with (3.6), (3.7) and (1.1), we have

$$\frac{V_p(K,Q)}{V(K)} = \frac{1}{nV(K)} \int_{S^{n-1}} |u \cdot v|^p dS_p(K,v) = \frac{1}{n} \rho_{\Gamma_{-p}K}^{-p}(u),$$

$$\frac{V_p(L,Q)}{V(L)} = \frac{1}{nV(L)} \int_{S^{n-1}} |u \cdot v|^p dS_p(L,v) = \frac{1}{n} \rho_{\Gamma_{-p}L}^{-p}(u).$$

Therefore, if for $K, L \in \mathscr{K}_o^n$ and any $Q \in \mathscr{K}_c^n$,

$$\frac{V_p(K,Q)}{V(K)} = \frac{V_p(L,Q)}{V(L)},$$

then we have

$$\Gamma_{-p}K = \Gamma_{-p}L.$$

Second, we prove that if $\Gamma_{-p}K = \Gamma_{-p}L$, then for any $Q \in \mathscr{K}_c^n$

$$\frac{V_p(K,Q)}{V(K)} = \frac{V_p(L,Q)}{V(L)},$$

From definition (1.1) and (1.2), we know that

$$\begin{aligned}\rho_{\Gamma_{-p}K}^{-p}(u) &= \frac{1}{2}\rho_{\Gamma_{-p}K}^{-p}(u)+\frac{1}{2}\rho_{\Gamma_{-p}(-K)}^{-p}(u)\\ &= \frac{1}{2V(K)}\int_{S^{n-1}}|u\cdot v|^p d[S_p(K,v)+S_p(-K,v)]\\ &= \frac{1}{2V(K)}\int_{S^{n-1}}|u\cdot v|^p d[S_p(K,v)+S_p(K,-v)].\end{aligned}$$

Similarly,

$$\rho_{\Gamma_{-p}L}^{-p}(u) = \frac{1}{2V(L)}\int_{S^{n-1}}|u\cdot v|^p d[S_p(L,v)+S_p(L,-v)].$$

Thus if $\Gamma_{-p}K=\Gamma_{-p}L$, then we get

$$\begin{aligned}&\int_{S^{n-1}}|u\cdot v|^p d\left[\frac{1}{V(K)}(S_p(K,v)+S_p(K,-v))\right]\\ =&\int_{S^{n-1}}|u\cdot v|^p d\left[\frac{1}{V(L)}(S_p(L,v)+S_p(L,-v))\right],\end{aligned}$$

i.e.

$$\int_{S^{n-1}}|u\cdot v|^p d\left[\frac{1}{V(K)}(S_p(K,v)+S_p(K,-v))-\frac{1}{V(L)}(S_p(L,v)+S_p(L,-v))\right]=0. \tag{3.8}$$

Let

$$\mu(v)=\frac{1}{V(K)}[S_p(K,v)+S_p(K,-v)]-\frac{1}{V(L)}[S_p(L,v)+S_p(L,-v)],$$

then (3.8) can be written as

$$\int_{S^{n-1}}|u\cdot v|^p d\mu(v)=0. \tag{3.9}$$

Note that $\mu(v)$ is a continuous, even and non-negative function on the sphere. Therefore, together with (3.9), we obtain $\mu(v)=0$, thus

$$\frac{1}{V(K)}[S_p(K,v)+S_p(K,-v)]-\frac{1}{V(L)}[S_p(L,v)+S_p(L,-v)]=0.$$

So

$$d\left[\frac{1}{V(K)}(S_p(K,v)+S_p(K,-v))\right]=d\left[\frac{1}{V(L)}[S_p(L,v)+S_p(L,-v)\right]. \tag{3.10}$$

Notice that $h_Q(v)=h_Q(-v)$ for $Q\in\mathscr{K}_c^n$, this combing with (2.1), we have for any $Q\in\mathscr{K}_c^n$,

$$\frac{V_p(K,Q)}{V(K)}=\frac{1}{2nV(K)}\int_{S^{n-1}}h_Q(v)^p d[S_p(K,v)+S_p(K,-v)],$$

$$\frac{V_p(L,Q)}{V(L)} = \frac{1}{2nV(L)} \int_{S^{n-1}} h_Q(v)^p d[S_p(L,v) + S_p(L,-v)].$$

Hence, by (3.10) we see

$$\frac{V_p(K,Q)}{V(K)} = \frac{V_p(L,Q)}{V(L)}.$$

Then (1.6) is obtained. □

Proof of Theorem 1.4. Since $K \in \mathscr{K}_c^n$, thus let $Q = K$ in (1.6), then by (2.2) and inequality (2.3) we get

$$V(L) = V_p(L,K) \geqslant V(L)^{\frac{n-p}{n}} V(K)^{\frac{p}{n}},$$

i.e.

$$V(K) \leqslant V(L).$$

According to equality conditions of inequality and combine with $\Gamma_{-p}K = \Gamma_{-p}L$, we know that equality hold for $p = 1$ if and only if K and L are translation, for $p > 1$ if and only if $K = L$. □

As an application of Theorem 1.3, we also obtain an interesting result as follows:

COROLLARY 3.1. *If $K, L \in \mathscr{K}_c^n$, $n \neq p > 1$, and $\Gamma_{-p}K = \Gamma_{-p}L$, then $K = L$.*

Proof. From (1.6) and $K \in \mathscr{K}_c^n$, taking $K = Q$, and using (2.2) and (2.3), we have

$$V(L) = V_p(L,K) \geqslant V(L)^{\frac{n-p}{n}} V(K)^{\frac{p}{n}}.$$

Thus

$$V(K) \leqslant V(L), \tag{3.10}$$

and equality holds for $p > 1$ if and only if K and L are dilates.

Similarly, taking $L = Q$ in (1.6), we get

$$V(K) \geqslant V(L), \tag{3.11}$$

and equality holds for $p > 1$ if and only if K and L are dilates.

Combing with the (3.10) and (3.11), we have $V(K) = V(L)$, and K and L are dilates. From this, let $K = cL$ $(c > 0)$ in $V(K) = V(L)$, then

$$V(cL) = c^n V(L) = V(L),$$

thus $c = 1$. This yields $K = L$. □

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