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WEIGHTED HARDY–TYPE INEQUALITIES IN ORLICZ SPACES

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Abstract. For a given N-function M, and inner and outer weight functions $\omega, \mathrm{e}^{-\varphi}$, we obtain Hardy-type inequalities:

$$\int_a^b M(\omega(r)|u(r)|)\mathrm{e}^{-\varphi(r)}dr \leqslant C\left(\int_a^b M(|u(r)|)\mathrm{e}^{-\varphi(r)}dr + \int_a^b M(|u'(r)|)\mathrm{e}^{-\varphi(r)}dx\right),$$

holding for every $u \in \mathscr{R}$, where $\mathscr{R}$ is a suitable dilation invariant subset of $W^{1,1}_{loc}(a,b)$, containing $C_0^\infty(a,b)$. The constant C above is independent of u. In many cases considered, the set $\mathscr{R}$ is proven to be maximal possible.

1. Introduction

One of most significant tools in analysis is the following inequality, established by Hardy in the early 1920's [16, 17]:

$$\int_{\mathbb{R}_+} |u(r)|^p r^{\alpha-p} dr \leqslant \left(\frac{p}{|\alpha-p+1|}\right)^p \int_{\mathbb{R}_+} |u'(r)|^p r^\alpha dr. \tag{1.1}$$

Here $\alpha \neq p-1$, $u \in W^{1,1}_{loc}(\mathbb{R}_+)$, and u is such that $\lim_{r\to\infty} u(r) = 0$ when $\alpha > p-1$, or $\lim_{r\to 0} u(r) = 0$ when $\alpha < p-1$.

Over the years, its generalizations were subject of intensive research. For the immense literature of the subject, we refer to the monographs [24, 26, 27, 28, 33, 34] and their references. Among other ones, one finds the generalization of (1.1) of the form:

$$\int_{\mathbb{R}_+} |\omega(r)u(r)|^p d\mu(r) \leqslant C \int_{\mathbb{R}_+} |u'(r)|^p d\mu(r), \tag{1.2}$$

where μ is a Radon measure on $\mathbb{R}_+$, and $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ is a measurable function.

Inequalities of the form (1.2) has been studied in Orlicz spaces as well, see e.g. Bloom and Kerman [3, 4], Lai [29, 30, 31, 32], Heinig-Maligranda [19], Heinig-Lai [18] and their references.

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Paper MIA-15-65

The authors have contributed in this direction as well. Namely, in [21], we have studied the inequality:

$$\int_{\mathbb{R}_+} M(|\omega(r)u(r)|)\,\mathrm{e}^{-\varphi(r)}dr \leqslant C\int_{\mathbb{R}_+} M(|u'(r)|)\,\mathrm{e}^{-\varphi(r)}dr, \tag{1.3}$$

with an N-function M satisfying the Δ_2- condition, and two weight functions $\omega, \mathrm{e}^{-\varphi}$. When certain criteria concerning (M,ω,φ) were satisfied, this inequality was proven to hold for functions which do not necessarily vanish when approaching 0 or ∞. We indicated a set $\mathscr{R} \subseteq W^{1,1}_{loc}(\mathbb{R}_+)$, depending on (M,ω,φ) such that (1.3) holds for every $u \in \mathscr{R}$. This set contains $C_0^\infty(\mathbb{R}_+)$, possibly as a proper subset. In the classical case, which corresponds to $M(t) = t^p$, $\omega(r) = \frac{1}{r}$, $\varphi(r) = -\alpha \ln r$, the approach of [21] retrieved the best constant.

In some cases inequalities (1.2) and their Orlicz variants are insufficient for applications, in particular in the regularity theory in PDEs. Therefore it is plausible to consider inequalities with an additional term depending on u added to the right-hand side, which can hold even when (1.3) does not.

In the current paper we are concerned with inequalities

$$\begin{aligned}&\int_a^b M(\omega(r)|u(r)|)\mathrm{e}^{-\varphi(r)}dr\\ &\leqslant C_1\int_a^b M(|u'(r)|)\mathrm{e}^{-\varphi(r)}dr + C_2\int_a^b M(|u(r)|)\mathrm{e}^{-\varphi(r)}dr,\end{aligned} \tag{1.4}$$

with a given N-function M, and inner and outer weight functions $\omega, \mathrm{e}^{-\varphi}$. We would like (1.4) to hold for u from a reasonably large subset $\mathscr{R}$ of $W^{1,1}_{loc}(a,b)$, depending on (M,ω,φ) and containing $C_0^\infty(a,b)$. Once we have (1.4), we can also derive the Orlicz-norm inequality (see Theorem 3.3):

$$\|\omega u\|_{L^M((a,b),\mu)} \leqslant D_1\|u\|_{L^M((a,b),\mu)} + D_2\|u'\|_{L^M((a,b),\mu)}. \tag{1.5}$$

Generalizing methods from [21], we prove (1.4) (and consequently (1.5)) for N-functions M satisfying the Δ_2-condition, and locally finite weights $\omega, \mathrm{e}^{-\varphi}$, being sufficiently regular. They are supposed to satisfy certain conditions depending on the Simonenko indices of M and involving the behaviour of $\omega, \omega', \varphi', \varphi''$ near the endpoints of the interval (a,b). Those conditions are somewhat technical, but they are practical to analyze, as one has to verify only assumptions concerning some limits and suprema/infima.

The definition of the set $\mathscr{R}$ of functions u for which (1.4) holds (see e.g. Theorem 3.2) is expressed in terms of the behaviour of the function $W^u(r) = \frac{M(\omega(r)|u(r)|)}{\varphi'(r)}\mathrm{e}^{-\varphi(r)}$ near the endpoints. This set is dilation invariant and contains $C_0^\infty(a,b)$, possibly as a proper subset.

Proposition 3.2 gives a condition on φ under which the set $\mathscr{R}$ is optimal. The sets $\mathscr{R}$ were first introduced by the authors in [21], Section 5, and we refer to that paper for their more detailed analysis.

Our methods can be applied e.g. to measures $\mu(dr) = r^\beta dr$, $\mu(dr) = r^\beta(\ln(2+r))^\gamma dr$ and $\mu(dr) = r^\beta e^{-Cr^\gamma}dr$ where $C > 0$ and inner weight $\omega(r) = r^\alpha$. A thorough treatment of inequalities for those measures can be found in our subsequent paper [23]. It turns out that in all the cases considered in that paper our theorems precisely describe the range of parameters α, β, γ for which the inequality holds true on supersets of $C_0^\infty(0,\infty)$. In particular our sufficient conditions from Theorem 3.2 turn out to be necessary as well in that class of weights. Moreover, in all these cases the sets $\mathscr{R}$ are shown to be maximal possible. Let us remark that in the general case, the question of optimality of sets $\mathscr{R}$, as well as of finding necessary and sufficient conditions for the validity of (1.4), remain open.

Our motivation to consider inequalities (1.4) is twofold.

First, they can serve as a tool towards constructing inequalities on $\mathbb{R}^n$ and in general on domains. For example, Cianchi in [9] applies weighted Hardy inequalities on $\mathbb{R}_+$ to obtain Orlicz-space Sobolev inequalities on sufficiently regular domains in $\mathbb{R}^n$. Guided by that paper, one can use Hardy-type inequality (1.5) to get other Orlicz-space generalizations of Sobolev inequalities on domains in $\mathbb{R}^n$.

The theory of constructing Hardy type inequalities for supersets of $C_0^\infty(a,b)$ is far from being complete. Sharp conditions for the validity of some Poincaré-type inequalities were found by Mazy'a, see Section 2.3.2 of [33]. His abstract isoperimetric-type conditions use capacitary estimates involving compact sets. Only a few cases can be determined using those estimates and it is still a challenge to construct Hardy-Sobolev-Poincaré inequalities in a given case. Some other conditions used to derive weighted Poincaré-type inequalities use the concept of 'measure of noncompactness', see e.g. [11] and later related contributions.

Our second motivation comes from the fact that Hardy-type inequalities:

$$\int_\Omega P(|\nabla\varphi||u|)d\mu \leqslant K_1\int_\Omega P(A|\nabla u|)d\mu + K_2\int_\Omega M(|u|)d\mu, \tag{1.6}$$

holding for any $u\in C_0^\infty(\Omega)$, imply the Gagliardo-Nirenberg-type inequalities for modulars and norms:

$$\int_\Omega M(|\nabla u|)d\mu \leqslant L\int_\Omega P(\theta|\nabla^{(2)}u|)d\mu + \int_\Omega Q(\frac{B}{\theta}|u|)d\mu, \text{ where } \theta>0$$

$$\|\nabla u\|_{L^M(\Omega,\mu)} \leqslant L_1\sqrt{\|\nabla^{(2)}u\|_{L^P(\Omega,\mu)}\|u\|_{L^Q(\Omega,\mu)}} + L_2\|u\|_{L^Q(\Omega,\mu)},$$

which were obtained in [22]. Here P,Q are two other N-functions tied to M by certain Young-type inequality, and M satisfies the Δ_2-condition. It is well known (see e.g. [35]) that inequalities of Gagliardo-Nirenberg type serve as a crucial tool in the regularity theory, when one investigates elliptic and parabolic equations in nondivergent form involving ∇u, where u is unknown.

Therefore we believe that estimates of the form (1.4) and (1.5) may be useful in the regularity theory for nonlinear PDE's in Orlicz spaces. Such PDE's are motivated by many models in mathematical physics, see e.g. [1, 2, 9, 13, 14].

The idea of our proof is close in spirit to the original Hardy's proof of (1.1) from [16]. It relies on the behaviour of functions near the endpoints. Therefore in this approach it is natural that admissible functions u should satisfy certain decay conditions near the endpoints. Taking into account the fact that in many cases the sets $\mathscr{R}$ are optimal, we are convinced that the decay conditions near the endpoints are intrinsic here.

To link our inequalities with those existing in the literature, let us start with the following inequality investigated by Oinarov [36],

$$\left(\int_a^b |\omega u|^q dr\right)^{\frac{1}{q}} \leqslant C\left(\int_a^b |vu|^p dr + \int_a^b |\rho u'|^p dr\right)^{\frac{1}{p}},$$

which involves general weight functions ω, v, ρ.

Inequalities

$$\int_0^\infty M(|r^\gamma u(r)|) r^\alpha \mathrm{e}^{-r^\beta} dr \leqslant C_1 \int_0^\infty M(|u(r)|) r^\alpha \mathrm{e}^{-r^\beta} dr + C_2 \int_0^\infty M(|u'(r)|) r^\alpha \mathrm{e}^{-r^\beta} dr,$$

which we derive in paper [23] as an application of present methods, generalize the basic inequality for the Gaussian measure $\gamma(dr) = \mathrm{e}^{-\frac{r^2}{2}} dr$ (cf. [5])

$$\int_0^\infty (ru(r))^2 \gamma(dr) \leqslant 2\int_0^\infty (u(r))^2 \gamma(dr) + 4\int_0^\infty (u'(r))^2 \gamma(dr).$$

Inequalities in Orlicz norms:

$$\|\frac{u}{d^{1+\alpha}}\|_{L^B(G)} \leqslant C\left(\|\frac{u}{d^\alpha}\|_{L^A(G)} + \|\frac{Du}{d^\alpha}\|_{L^A(G)}\right), \tag{1.7}$$

where $d(x) = \mathrm{dist}(x, \partial G)$, G is a (sufficiently regular) bounded domain in $\mathbb{R}^n$, were analyzed by Cianchi in [10].

Our inequalities are close in spirit to the celebrated Cafarelli-Kohn-Nirenberg inequalities on $\mathbb{R}^n$ [7], which have the form:

$$\||x|^\gamma |u|\|_{L^r} \leqslant C(\||x|^\alpha |\nabla u|\|_{L^p})^a \cdot \left(\||x|^\beta |u|\|_{L^q}\right)^{1-a}, \tag{1.8}$$

holding true under suitable conditions on parameters $\alpha, \beta, \gamma > 0$, $p, q, r \geqslant 1$, $a \in (0,1)$. Their variants on $(0, \infty)$ were obtained by Brown and Hinton in [6].

Finally, let us mention that 1-dimensional Poincaré inequalities

$$\left(\int_a^b |f(x) - \frac{1}{v[a,b]}\int_a^b f dv|^q \rho(x) dx\right)^{\frac{1}{q}} \leqslant C\left(\int_a^b |f'(x)|^p w(x) dx\right)^{\frac{1}{p}}, \tag{1.9}$$

with weights v, w, ρ, imply Hardy-type inequalities. Sharp conditions for the validity of (1.9) can be found in [8].

2. Notation and preliminaries

2.1. Notation

Let $-\infty \leqslant a < b \leqslant +\infty$ be two given numbers (possibly infinite). We use the convention $\sup\emptyset = -\infty$, $\inf\emptyset = +\infty$, $c/\infty = 0$, $f\chi_A$ is the function f (defined on A) extended by 0 outside A, $\chi_\emptyset \equiv 0$. We write $D' = D/(D-1)$ for the Hölder conjugate to the number $D > 1$.

If $-\infty \leqslant a < b \leqslant +\infty$ and v, ρ are two locally integrable nonnegative functions on (a,b), by $W^{1,p}((a,b),v,\rho)$ we denote the completion of the set

$$\{u \in C^\infty((a,b)) : \int_a^b |u(r)|^p v(r) dr < \infty, \int_a^b |u'(r)|^p \rho(r) dr < \infty\}$$

in the norm $\left(\int_a^b |u(r)|^p v(r) dr + \int_a^b |u'(r)|^p \rho(r) dr\right)^{\frac{1}{p}}$. When $v \equiv \rho \equiv 1$, we omit it from the notation, while when $v = \rho$, we write $W^{1,p}((a,b),v)$ instead of $W^{1,p}((a,b),v,v)$. If $\mathscr{R} \subseteq W^{1,1}_{loc}((a,b))$ is an arbitrary subset, then we define the space $W^{1,p}_{\mathscr{R}}((a,b),v,\rho)$ to be the completion of the set $\mathscr{R} \cap W^{1,p}((a,b),v,\rho)$ in the space $W^{1,p}((a,b),v,\rho)$.

2.2. N-functions and Orlicz spaces

We start by recalling some known facts about Orlicz spaces (we refer to [25, 37] for details).

DEFINITION 2.1. A function $M : [0,\infty) \to [0,\infty)$ is called an N-function if it is continuous, convex, $M(0) = 0$, $\lim_{\lambda\to 0} \frac{M(\lambda)}{\lambda} = 0$ and $\lim_{\lambda\to\infty} \frac{M(\lambda)}{\lambda} = \infty$.

The function *conjugate* to an N-function (its Legendre transform) is given by the formula

$$M^*(y) = \sup_{\lambda \geqslant 0}[\lambda y - M(\lambda)].$$

DEFINITION 2.2. (Δ_2-condition) A continuous function $M : [0,\infty) \to [0,\infty)$ such that $M(0) = 0$ satisfies the Δ_2-condition if and only if

$$M(2\lambda) \leqslant CM(\lambda), \tag{2.1}$$

with a positive constant C not depending on $\lambda > 0$.

For convex, differentiable functions such that $M(0) = 0$, condition (2.1) is equivalent to the condition

$$\lambda M'(\lambda) \leqslant CM(\lambda),$$

where $C > 0$ is independent of $\lambda > 0$.

Throughout the paper, we will be assuming the following condition.

(M) $M:[0,\infty)\to[0,\infty)$ is a differentiable N-function, and

$$d_M\frac{M(\lambda)}{\lambda}\leqslant M'(\lambda)\leqslant D_M\frac{M(\lambda)}{\lambda}\quad\text{for every } \lambda>0, \tag{2.2}$$

where $1\leqslant d_M\leqslant D_M$.

Inequalities (2.2) imply the Δ_2-condition for M, and also for M^* when $d_M>1$ (see e.g. [25], Theorem 4.3 or [20], Proposition 4.1). Best possible constants d_M and D_M in (2.2) are called the Simonenko lower and upper indices of M respectively (see [12, 15, 38]).

Inequalities (2.2) result in the estimates

$$\begin{aligned} M(a\lambda)&\leqslant \max(a^{d_M},a^{D_M})M(\lambda)=:\bar{c}(a)M(\lambda),\\ M(a\lambda)&\geqslant \min(a^{d_M},a^{D_M})M(\lambda)=:\underline{c}(a)M(\lambda), \end{aligned} \tag{2.3}$$

valid for every $\lambda>0, a>0$ (see e.g. [21], Lemma 4.1, part iii)), and also in the following lemma.

LEMMA 2.1. ([21], Lemma 4.2) *Suppose that M is a differentiable N-function, and let $1\leqslant d_M\leqslant D_M$ be two constants such that (2.2) is satisfied.*

Then for every $s_1,s_2>0$ the following estimate holds true:

$$\frac{M(s_1)}{s_1}s_2\leqslant\frac{D_M-1}{d_M}M(s_1)+\frac{1}{d_M}M(s_2). \tag{2.4}$$

Let $-\infty\leqslant a<b\leqslant+\infty$. The weighted Orlicz space $L^M_\mu((a,b))$ is by definition the space

$$L^M((a,b),\mu)\stackrel{def}{=}\left\{f:(a,b)\to\mathbb{R}\text{ measurable}:\int_{(a,b)}M\left(\frac{|f(r)|}{K}\right)\mu(dr)\leqslant 1\right.$$
$$\left.\text{for some } K>0\right\},$$

equipped with the Luxemburg norm

$$\|f\|_{L^M((a,b),\mu)}:=\inf\left\{K>0:\int_{(a,b)}M\left(\frac{|f(r)|}{K}\right)\mu(dr)\leqslant 1\right\}.$$

This norm makes the space complete and turns $L^M_\mu((a,b))$ into a Banach space. For $M(\lambda)=\lambda^p$ with $p>1$, the space $L^M_\mu((a,b))$ coincides with the usual $L^p((a,b),\mu)$ space. It is known that we always have

$$\int_{(a,b)}M\left(\frac{|f(r)|}{\|f\|_{L^M((a,b),\mu)}}\right)\mu(dr)\leqslant 1, \tag{2.5}$$

and for functions M satisfying the Δ_2-condition we have an equality here. To conform with previously introduced notation, when $\mu(dx)=\rho(x)dx$, we will also write $L^M((a,b),\mu)=L^M((a,b),\rho)$.

2.3. Orlicz-Sobolev spaces

The weighted Orlicz-Sobolev space $W^{1,M}((a,b),\rho)$ is by definition the completion of the set

$$\{u \in C^\infty((a,b)) : \|u\|_{L^M((a,b),\rho)} + \|u'\|_{L^M((a,b),\rho)} < \infty\}$$

in the norm

$$\|u\|_{W^{1,M}((a,b),\rho)} := \|u\|_{L^M((a,b),\rho)} + \|u'\|_{L^M((a,b),\rho)}.$$

All the weight functions $\rho = \mathrm{e}^{-\varphi}$ considered in this paper are strictly positive and separated from zero on each compact subset of (a,b), so we have an embedding $W^{1,M}((a,b),\rho) \hookrightarrow W^{1,1}_{loc}((a,b))$. Consequently every $u \in W^{1,M}((a,b),\rho)$ is locally absolutely continuous on (a,b) – its derivative u' is well defined almost everywhere and is a locally integrable function. When additionally M satisfies the Δ_2-condition, then the quantity

$$\mathscr{A}(u) := \int_{(a,b)} M(|u(r)|)\rho(r)dr + \int_{(a,b)} M(|u'(r)|)\rho(r)dr$$

is finite. It is easy to see that for M satisfying Δ_2 one has $u_n \to u$ in $W^{1,M}((a,b),\rho)$, as $n \to \infty$, if and only if

$$\mathscr{A}(u_n - u) \overset{n\to\infty}{\to} 0.$$

We also have then $\mathscr{A}(u_n) \to \mathscr{A}(u)$.

DEFINITION 2.3. Let $\mathscr{R} \subseteq W^{1,1}_{loc}((a,b))$ be an arbitrary subset. By $W^{1,M}_{\mathscr{R}}((a,b),\rho)$ we denote the completion of $\mathscr{R} \cap W^{1,M}((a,b),\rho)$ in the space $W^{1,M}((a,b),\rho)$. In particular when $\mathscr{R} = W^{1,1}_{loc}((a,b))$, we have $W^{1,M}_{\mathscr{R}}((a,b),\rho) = W^{1,M}((a,b),\rho)$. Also write $W^{1,M}_0((a,b),\rho)$ when $\mathscr{R} = C^\infty_0((a,b))$. Analogous notation is used for Sobolev spaces.

3. Descriptions of results

Suppose that $-\infty \leqslant a < b \leqslant +\infty$. We will be concerned with the inequality

$$\int_a^b M(\omega|u|)\,d\mu \leqslant C_1 \int_a^b M(|u'|)\,d\mu + C_2 \int_a^b M(|u|)\,d\mu \tag{3.1}$$

where M is a differentiable N-function satisfying (2.2), $\mu(dx) = \mathrm{e}^{-\varphi(x)}dx$ is a Radon measure on $(0,\infty)$, φ and ω are measurable functions.

Proofs of the results presented here are postponed to Section 4.

3.1. First approach. Global assumptions

We start with the description of our first results.

In this subsection we assume that φ and ω satisfy the following conditions:

(φ) $\varphi \in C^2((a,b))$, φ' does not vanish on (a,b),

(ω) $\omega : (a,b) \to [0,\infty)$ is a measurable function of class C^1 on (a,b).

3.1.1. Auxiliary notation

To facilitate the presentation of our results, we need to to introduce some sets and functions. The reader should bear in mind the important choice $\omega(x) = |\varphi'(r)|$, when the quantities appearing below become substantially simpler.

Let $\lambda \in [0,\infty]$. We consider the following sets:

$$\begin{aligned} F &:= \{r \in (a,b) : \omega(r) \neq 0,\ \omega'(r)\varphi'(r) > 0\}, \\ G &:= \{r \in (a,b) : \omega(r) \neq 0,\ \omega'(r)\varphi'(r) < 0\}, \\ F^\lambda &:= \left\{ r \in F : \frac{\omega'(r)}{\varphi'(r)} > \lambda \right\}, \\ G^\lambda &:= \left\{ r \in G : |\frac{\omega'(r)}{\varphi'(r)}| > \lambda \right\}. \end{aligned} \tag{3.2}$$

Here F^∞, G^∞ are understood to be empty sets. Note that $F^0 = F$ and $G^0 = G$.

We will also need the following functions:

$$\begin{aligned} \Psi^{1,\lambda}(r) &= 1 + \frac{\varphi''(r)}{(\varphi'(r))^2} - \frac{\omega'(r)}{\omega(r)\varphi'(r)} \left(d_M \chi_G(r) + D_M \chi_{F^\lambda}(r) \right), \\ \Psi^{2,\lambda}(r) &= -1 - \frac{\varphi''(r)}{(\varphi'(r))^2} + \frac{\omega'(r)}{\omega(r)\varphi'(r)} \left(d_M \chi_F(r) + D_M \chi_{G^\lambda}(r) \right). \end{aligned} \tag{3.3}$$

They are well defined for all $\lambda \in [0,\infty]$.

Further, let

$$B^{i,\lambda}(a,b) := \inf\{\Psi^{i,\lambda}(\tau), \tau \in (a,b)\},\ \ i = 1,2, \tag{3.4}$$

$$L(a,b) := \sup \left\{ \frac{|\omega(r)|}{|\varphi'(r)|} : r \in (a,b), \varphi'(r) \neq 0 \right\}. \tag{3.5}$$

3.1.2. The results

We now state our first theorem, covering the case when the weights ω and φ satisfy assumptions **(**φ**)** and **(**ω**)**. It constitutes the main technical tool towards proving our main result, Theorem 3.1, and it permits to control the behaviour of the resulting constants. Also it yields the classical Hardy inequality (with best constants) as a special case.

In our approach we will consider certain classes $\mathscr{R}^1(a,b)$ or $\mathscr{R}^2(a,b)$, depending on M, ω, φ defined as follows.

$$\mathscr{R}^1(a,b) := \left\{ u \in W^{1,1}_{loc}(a,b) : \exists \lim_{n\to\infty} (W^u(b_n) - W^u(a_n)) \in [0,\infty] \text{ for some } a_n \downarrow a, b_n \uparrow b \right\},$$

$$\mathscr{R}^2(a,b) := \left\{ u \in W^{1,1}_{loc}(a,b) : \exists \lim_{n\to\infty} (W^u(b_n) - W^u(a_n)) \in [-\infty,0] \text{ for some } a_n \downarrow a, b_n \uparrow b \right\},$$

where

$$W^u(r) := \frac{M(\omega(r)|u(r)|)}{\varphi'(r)} \, \mathrm{e}^{-\varphi(r)}. \tag{3.6}$$

For discussion of sets $\mathscr{R}$ we refer to Section 5 in the paper [21], where those sets have appeared for the first time.

THEOREM 3.1. *Suppose that* $-\infty \leqslant a < b \leqslant \infty$ *are given,* M, φ, ω *satisfy* **(M)**, **(φ)**, **(ω)**, $\mathscr{R}^i(a,b)$ *be defined by (3.6). Let us consider the following assumptions:*

(A1) $L(a,b) < \infty$ *and* $B^{1,\lambda_0}(a,b) > 0$ *for some* $\lambda_0 \in [0,\infty)$,

(A2) $L(a,b) < \infty$ *and* $B^{2,\lambda_0}(a,b) > 0$ *for some* $\lambda_0 \in [0,\infty)$.

If **(Ai)** *is satisfied where* $i = 1$ *or* $i = 2$, *then the inequality*

$$\int_a^b M(\omega(r)|u(r)|)\,\mu(dr) \leqslant C_1^{(i)} \int_a^b M(|u(r)|)\,\mu(dr) + C_2^{(i)} \int_a^b M(|u^{'}(r)|)\mu(dr) \quad (3.7)$$

holds for all $u \in W^{1,M}_{\mathscr{R}^i(a,b)}((a,b),\mathrm{e}^{-\varphi})$, *with constants independent of* u.

Moreover, if **(Ai)** *holds with* $\lambda_0 = 0$, *then (3.7) is satisfied with* $C_1^{(i)} = 0$, $C_2^{(i)} = \bar{c}\left(\frac{D_M^2 L(a,b)}{d_M B^{i,0}(a,b)}\right)$.

Recall that the sets $W^{1,M}_{\mathscr{R}^i(a,b)}((a,b),\mathrm{e}^{-\varphi})$ were introduced in Definition 2.3.

REMARK 3.1. In the case when we can have **(Ai)** satisfied with $\lambda_0 = 0$, the constants obtained are identical with those from our former paper [21], where we have got the inequality:

$$\int_0^\infty M(\omega(r)|u(r)|)\,\mu(dr) \leqslant \bar{c}\left(\frac{D_M^2 L(0,\infty)}{d_M B^{i,0}(0,\infty)}\right)\int_0^\infty M(|u'(r)|)\,\mu(dr),$$

holding for all $u \in \mathscr{R}^i(0,\infty)$. For $M(\lambda) = \lambda^p$ this approach led to classical Hardy inequality with optimal constants.

REMARK 3.2. In general, the constants we obtain are:

$$C_1^{(i)} = \frac{\left(\frac{\lambda_0}{L(a,b)+\lambda_0}\right)^{\frac{1}{D_M'}} \cdot \bar{c}\left(\frac{D_M^2}{d_M B^{i,\lambda_0}(a,b)}(L(a,b)+\lambda_0)^{\frac{1}{D_M'}}\lambda_0^{\frac{1}{D_M}}\right)}{D_M\left(1-\frac{1}{D_M'}\left\{\left(\frac{\lambda_0}{L(a,b)+\lambda_0}\right)^{\frac{1}{D_M'}}+\left(\frac{L(a,b)}{L(a,b)+\lambda_0}\right)^{\frac{1}{D_M'}}\right\}\right)},$$

$$C_2^{(i)} = \frac{\left(\frac{L(a,b)}{L(a,b)+\lambda_0}\right)^{\frac{1}{D_M'}} \cdot \bar{c}\left(\frac{D_M^2}{d_M B^{i,\lambda_0}(a,b)}(L(a,b)+\lambda_0)^{\frac{1}{D_M'}}L(a,b)^{\frac{1}{D_M}}\right)}{D_M\left(1-\frac{1}{D_M'}\left\{\left(\frac{\lambda_0}{L(a,b)+\lambda_0}\right)^{\frac{1}{D_M'}}+\left(\frac{L(a,b)}{L(a,b)+\lambda_0}\right)^{\frac{1}{D_M'}}\right\}\right)}.$$

See Remark 4.1 in Section 4.

In the simplest case $M(\lambda) = \lambda^p$, we obtain a more precise statement, illustrating Theorem 3.1.

PROPOSITION 3.1. *Suppose that* $-\infty \leqslant a < b \leqslant \infty$, $p > 1$ *are given, and that* φ, ω *satisfy* (μ), (ω). *Let the quantities* $L(a,b)$ *and* $B^{i,\lambda}(a,b)$ $(i = 1,2)$ *be given by* (3.4) *and* (3.5), *and* $\mathscr{R}^i(a,b)$ *be the same as in* (3.6) *with* $W^u(r) = \frac{|u(r)|^p \omega(r)^p}{\varphi'(r)} \mathrm{e}^{-\varphi(r)}$. *Moreover, let the assumptions* **(A1)** *and* **(A2)** *be the same as in Theorem* 3.1. *If* **(Ai)** *is satisfied, where* $i = 1$ *or* $i = 2$, *then the inequality*

$$\left(\int_a^b |u(r)|^p \omega(r)^p \mu(dr) \right)^{\frac{1}{p}} \tag{3.8}$$
$$\leqslant \frac{p}{B^{i,\lambda_0}(a,b)} \left\{ \lambda_0 \left(\int_a^b |u(r)|^p \mu(dr) \right)^{\frac{1}{p}} + L(a,b) \left(\int_a^b |u'(r)|^p \mu(dr) \right)^{\frac{1}{p}} \right\}$$

holds for all $u \in W^{1,p}_{\mathscr{R}^i(a,b)}((a,b), \mathrm{e}^{-\varphi})$.

3.2. Generalization. Wider class of inequalities

It turns out that the assumptions used to obtain inequality (3.1) can be weakened. We only need to impose those assumptions near the endpoints of (a,b). However, this time we will deal with new sets $\mathscr{R}$.

For purpose of this subsection we assume that $\mu(dr) = \mathrm{e}^{-\varphi(r)} dr$ is a Radon measure on (a,b), $\omega : (a,b) \to [0,\infty)$, $\varphi : (a,b) \to \mathbb{R}$ are measurable. We will consider local versions of (φ), (ω), near right and left endpoints separately:

(φ_-) φ is of class C^2 in some neighborhood of a, φ' does not vanish in some neighborhood of a,

(ω_-) ω is locally bounded on (a,b) and of class C^1 in some neighborhood of a;

(φ_+) φ is of class C^2 in some neighborhood of b, φ' does not vanish in some neighborhood of b,

(ω_+) ω is locally bounded on (a,b) and of class C^1 in some neighborhood of b.

Here by a neighborhood of ∞ or $-\infty$ we understand any halfline (R,∞) or $(-\infty,R)$.

3.2.1. Auxiliary notation

Let us denote

$$L_- = \begin{cases} \limsup_{r \to a^+} \frac{\omega(r)}{|\varphi'(r)|} & \text{if } \varphi \in C^1((a,a+\varepsilon)) \text{ and } \varphi' \neq 0 \text{ on } (a,a+\varepsilon) \text{ for some } \varepsilon > 0, \\ +\infty & \text{else,} \end{cases}$$

$$L_+ = \begin{cases} \limsup_{r \to b^-} \frac{\omega(r)}{|\varphi'(r)|}; & \text{if } \varphi \in C^1((b-\varepsilon,b)) \text{ and } \varphi' \neq 0 \text{ on } (b-\varepsilon,b) \text{ for some } \varepsilon > 0, \\ +\infty & \text{else.} \end{cases}$$

We slightly modify the definitions of sets $F, G, F^{\lambda}, G^{\lambda}$ and also of functions $\Psi^{i,\lambda}$. To begin with, let

$$\mathscr{E} := \{r \in (a,b) : \omega \in C^1((r-\varepsilon_r, r+\varepsilon_r)) \text{ and } \varphi \in C^2((r-\varepsilon_r, r+\varepsilon_r)) \\ \text{for some } \varepsilon_r > 0,\ \omega(r)\varphi'(r) \neq 0\},$$

then we define the sets

$$\begin{aligned} F &:= \{r \in \mathscr{E} :\ \omega'(r)\varphi'(r) > 0\}, \\ G &:= \{r \in \mathscr{E} :\ \omega'(r)\varphi'(r) < 0\}, \\ F^{\lambda} &:= \{r \in F : \frac{\omega'(r)}{\varphi'(r)} > \lambda\}, \\ G^{\lambda} &:= \{r \in G : |\frac{\omega'(r)}{\varphi'(r)}| > \lambda\} \end{aligned} \tag{3.9}$$

and the subordinate functions

$$\Psi^{1,\lambda}(r) = \left\{1 + \frac{\varphi''(r)}{(\varphi'(r))^2} - \frac{\omega'(r)}{\omega(r)\varphi'(r)}\left(d_M\chi_G(r) + D_M\chi_{F^{\lambda}}(r)\right)\right\}\chi_{\mathscr{E}}(r), \tag{3.10}$$

$$\Psi^{2,\lambda}(r) = \left\{-1 - \frac{\varphi''(r)}{(\varphi'(r))^2} + \frac{\omega'(r)}{\omega(r)\varphi'(r)}\left(d_M\chi_F(r) + D_M\chi_{G^{\lambda}}(r)\right)\right\}\chi_{\mathscr{E}}(r).$$

Observe that when (φ), (ω) are satisfied, then $\mathscr{E} = (a,b) \cap \{r : \omega(r) \neq 0\}$ and the definitions (3.9) and (3.10) coincide with (3.2) and (3.3), respectively.

Then we define

$$B_-^{i,\lambda} = \liminf_{r \to a^+} \Psi^{i,\lambda}(r),\ \ B_+^{i,\lambda} = \liminf_{r \to b^-} \Psi^{i,\lambda}(r),\ \ i = 1,2.$$

Note that if $B_-^{i,\lambda} > 0$ (resp. $B_+^{i,\lambda} > 0$), then the conditions (φ_-) and (ω_-) (resp. (φ_+) and (ω_+)) are automatically fulfilled.

3.2.2. The results

We consider sets $\mathscr{R}_-^i(a,b)$ and $\mathscr{R}_+^i(a,b)$ where $i \in \{1,2\}$ defined as follows (W^u is the same as in (3.6)):

$$\begin{aligned} \mathscr{R}_-^i(a,b) &= \{u \in W^{1,1}_{loc}((a,b)) : \limsup_{r \to a^+}(-1)^i W^u(r) \geqslant 0\}, \\ \mathscr{R}_+^i(a,b) &= \{u \in W^{1,1}_{loc}((a,b)) : \limsup_{r \to b^-}(-1)^{i+1} W^u(r) \geqslant 0\}. \end{aligned} \tag{3.11}$$

The 'local' versions of **(A1), (A2)** ($i \in \{1,2\}$) read as follows:

(Bi$_-$) $L_- < \infty$, and for certain $\lambda \geqslant 0$, $B_-^{i,\lambda} > 0$,

(Bi$_+$) $L_+ < \infty$, and for certain $\lambda \geqslant 0$, $B_+^{i,\lambda} > 0$,

(K$_-$) $\limsup_{r \to a^+} \omega(r) < \infty$,

(K$_+$) $\limsup_{r \to b^-} \omega(r) < \infty$.

We are now ready for the statement of our new theorem.

THEOREM 3.2. *Let $-\infty \leqslant a < b \leqslant +\infty$ be fixed numbers, M be a function satisfying* **(M)**, *sets $\mathscr{R}$ be defined by (3.11), $d\mu(x) = \mathrm{e}^{-\varphi(x)}dx$. Suppose further that φ, ω are such measurable functions that ω is locally bounded and at least one of conditions:* **(B1$_-$)** *or* **(B2$_-$)** *or* **(K$_-$)** *holds, and at least one of conditions:* **(B1$_+$)** *or* **(B2$_+$)** *or* **(K$_+$)** *holds.*

Then there exist constants C_1, C_2 such that the inequality

$$\int_a^b M(\omega(r)|u(r)|)\,\mu(dr) \leqslant C_1 \int_a^b M(|u'(r)|)\,\mu(dr) + C_2 \int_a^b M(|u(r)|)\mu(dr) \quad (3.12)$$

holds for every u in the class $W^{1,M}_{\mathscr{R}}((a,b), \mathrm{e}^{-\varphi})$, where

a) $\mathscr{R} = \mathscr{R}^i_-(a,b)$ when **(Bi$_-$)** *and* **(K$_+$)** *hold,*

b) $\mathscr{R} = \mathscr{R}^i_+(a,b)$; when **(K$_-$)** *and* **(Bi$_+$)** *hold,*

c) $\mathscr{R} = \mathscr{R}^i_-(a,b) \cap \mathscr{R}^j_+(a,b)$, $i,j \in \{1,2\}$ when **(Bi$_-$)** *and* **(Bj$_+$)** *hold,*

d) $\mathscr{R} = W^{1,1}_{loc}((a,b))$ when **(K$_-$)** *and* **(K$_+$)** *hold.*

3.2.3. Optimality of the sets $\mathscr{R}$

At first let us note that all local sets $\mathscr{R}$ in (3.11) are dilation invariant (where by a dilation invariant set we mean such one that λu belong to $\mathscr{R}$ for every $u \in \mathscr{R}$ and $\lambda > 0$) under the condition **(M)**. This follows trivially from formula (2.2).

We are now to discuss the optimality of $\mathscr{R}$. For this purpose we will consider separately the following cases:

$$(-1)^{i+1}\varphi' < 0 \text{ next to } a \text{ and } \mathscr{R} = \mathscr{R}^i_-(a,b), \quad (3.13)$$

$$(-1)^{i+1}\varphi' > 0 \text{ next to } b \text{ and } \mathscr{R} = \mathscr{R}^i_+(a,b), \quad (3.14)$$

$$(-1)^{i+1}\varphi' > 0 \text{ next to } a \text{ and } \mathscr{R} = \mathscr{R}^i_-(a,b), \quad (3.15)$$

$$(-1)^{i+1}\varphi' < 0 \text{ next to } b \text{ and } \mathscr{R} = \mathscr{R}^i_+(a,b). \quad (3.16)$$

Note that when $W^{1,M}_{\mathscr{R}}((a,b), \mathrm{e}^{-\varphi}) = W^{1,M}((a,b), \mathrm{e}^{-\varphi})$, then the set $\mathscr{R}$ is maximal for (3.12) to hold. This is the case of (3.13) and (3.14), as then $\mathscr{R}$ is the same as $W^{1,1}_{loc}((a,b))$.

In the cases (3.15) and (3.16), we show that under the additional assumption

$$\int_I |\varphi'(r)|dr = \infty, \quad (3.17)$$

where I is some neighborhood of a in case of (3.15), and of b in case of (3.16), the choice of $\mathscr{R}$ is also optimal.

Let us consider the case of (3.15). If (3.17) holds (with $p=a$), then the desired inequality cannot hold for any $u \in W^{1,M}((a,b),e^{-\varphi}))\setminus\mathscr{R}$. Also, in this case we have

$$W^{1,M}_{\mathscr{R}}((a,b),\mathrm{e}^{-\varphi}) = \{u \in W^{1,M}((a,b),\mathrm{e}^{-\varphi}) : \liminf_{r\to a}|W^u(r)| = 0\}, \tag{3.18}$$

with

$$\mathscr{R} = \{u \in W^{1,1}_{loc}((a,b)) : \liminf_{r\to a}|W^u(r)| = 0\}.$$

Indeed, if there existed $u \in W^{1,M}((a,b),\mathrm{e}^{-\varphi})\setminus\mathscr{R}$, it would necessarily satisfy the inequality $|W^u(r)| > c > 0$, with some $c>0$, for every r sufficiently close to a. This inequality would in turn be equivalent to

$$M(\omega(r)|u(r)|) > c|\varphi'(r)|e^{\varphi(r)},$$

when r is sufficiently close to a. It would imply

$$\int_I M(\omega(r)|u(r)|)e^{-\varphi(r)}dr \geqslant c\int_I |\varphi'(r)|dr = \infty,$$

(where I is some neighborhood of a). It shows that (3.12) cannot hold with such u. This also implies (3.18).

The case of (3.16) (and $p=b$) is treated similarly.

We arrive at the following result, summarizing our considerations.

PROPOSITION 3.2. *Under the assumptions of Theorem 3.2 we have*

i) *Inequality (3.12) holds for every $u \in W^{1,M}((a,b),\mathrm{e}^{-\varphi})$, with constants independent of u in the following cases:*

a) **(Bi$_-$)** *and* **(K$_+$)***, when (3.13) holds,*

b) **(K$_-$)** *and* **(Bi$_+$)***, when (3.14) holds,*

c) **(Bi$_-$)** *and* **(Bj$_+$)***, when (3.13) and (3.14) hold,*

d) **(K$_-$)** *and* **(K$_+$)***.*

ii) *If* **(Bi$_-$)**, **(K$_+$)**, *(3.15) hold and additionally $\int_I |\varphi'(r)|dr = \infty$, where I is some neighborhood of a, then inequality (3.12) holds for every $u \in W^{1,M}((a,b),\mathrm{e}^{-\varphi})$ such that $\liminf_{r\to a}|W^u(r)| = 0$, with constants independent of u. Moreover, if $u \in W^{1,M}((a,b),\mathrm{e}^{-\varphi})$ and $\liminf_{r\to a}|W^u(r)| > 0$, then inequality (3.12) cannot hold.*

iii) *If* **(K$_-$)** *and* **(Bi$_+$)**, *(3.16) hold and additionally $\int_I |\varphi'(r)|dr = \infty$, where I is some neighborhood of b, then inequality (3.12) holds for every $u \in W^{1,M}((a,b),\mathrm{e}^{-\varphi})$ such that $\liminf_{r\to b}|W^u(r)| = 0$, with constants independent of u. Moreover, it cannot hold if $u \in W^{1,M}((a,b),\mathrm{e}^{-\varphi})$ and $\liminf_{r\to b}|W^u(r)| > 0$.*

3.3. Results in Orlicz norms

Let us remark that the inequality

$$\int_a^b M(\omega(r)|u(r)|)\,\mu(dr) \leqslant C\left(\int_a^b M(|u'(r)|)\,\mu(dr) + \int_a^b M(|u(r)|)\mu(dr)\right),\ \ u\in\mathscr{R}, \tag{3.19}$$

where $\mathscr{R}$ is some dilation invariant set, implies the corresponding inequality for Orlicz norms:

$$\|\omega u\|_{L^M((a,b),\mu)} \leqslant (2C+1)\left(\|u\|_{L^M((a,b),\mu)} + \|u'\|_{L^M((a,b),\mu)}\right), \tag{3.20}$$

valid also for every $u\in\mathscr{R}$.

The corresponding result reads as follows. For readers' convenience we submit its proof in the last section.

THEOREM 3.3. *Let M be an arbitrary N-function, let μ be an arbitrary Radon measure on (a,b), and let $\mathscr{R}$ be such a dilation invariant set of locally absolutely continuous functions that inequality (3.19) holds for every element of $\mathscr{R}$. Then also inequality (3.20) holds for every function u belonging to $\mathscr{R}$.*

REMARK 3.3. As indicated in the Introduction, the detailed analysis of inequalities (1.4) for particular weights of power, power-logarithmic, and power-exponential type is included in our separate paper [23].

4. Proofs of theorems

Proof of Theorem 3.1. By density argument, it suffices to prove the inequality for $u\in\mathscr{R}^i(a,b)\cap W^{1,M}((a,b),\mathrm{e}^{-\varphi})$. Let $u\in\mathscr{R}^1(a,b)$ (resp. $u\in\mathscr{R}^2(a,b)$) and let $a_n\downarrow a$, $b_n\uparrow b$ be the sequences from the definition of $\mathscr{R}^1(a,b)$ (resp. $\mathscr{R}^2(a,b)$). We set

$$J_n := \int_{a_n}^{b_n} M(\omega(r))|u(r)|)\mu(dr), \qquad G_n := \int_{a_n}^{b_n} M(|u(r)|)\mu(dr),$$

$$H_n := \int_{a_n}^{b_n} M(|u'(r)|)\mu(dr).$$

We have: $J_n = -\int_{a_n}^{b_n} h^u(r)\left[\mathrm{e}^{-\varphi(r)}\right]' dr$, where

$$h^u(r) = \frac{M(\omega(r)|u(r)|)}{\varphi'(r)}.$$

Under our assumptions, it is well defined for every $r\in(a,b)$. Since u is $W^{1,1}_{loc}((a,b))$ and M is locally Lipschitz, we infer that $h^u\in W^{1,1}_{loc}((a,b))$ and

$$\begin{aligned}(h^u)'(r) &= \frac{d}{dr}\left(\frac{1}{\varphi'(r)}\right)M(\omega(r)|u(r)|)\\ &\quad+\frac{1}{\varphi'(r)}M'(\omega(r)|u(r)|)\left(\omega'(r)|u(r)|+\omega(r)u'(r)\operatorname{sgn}u(r)\right),\end{aligned} \tag{4.1}$$

in the sense of distributions and almost everywhere. Moreover, by the ACL-continuity property of Sobolev functions, h is absolutely continuous on each interval $[s,R]\subseteq(a,b)$ (see e.g. [33], Theorems 1 and 2, Sec. 1.1.3). Integrating by parts we get that for every n,

$$J_n=\int_{a_n}^{b_n}(h^u)'(r)\mathrm{e}^{-\varphi(r)}dr-\theta_n, \tag{4.2}$$

where

$$\theta_n=h^u(b_n)\mathrm{e}^{-\varphi(b_n)}-h^u(a_n)\mathrm{e}^{-\varphi(a_n)}. \tag{4.3}$$

Now insert the expression (4.1) into (4.2), getting

$$\begin{aligned} J_n &= -\int_{a_n}^{b_n}\Big(\frac{\varphi''(r)}{(\varphi'(r))^2}\Big)M(\omega(r)|u(r)|)\mu(dr)\\ &\quad+\int_{a_n}^{b_n}\frac{\omega'(r)}{\varphi'(r)}M'(\omega(r)|u(r)|)|u(r)|\mu(dr)\\ &\quad+\int_{a_n}^{b_n}\frac{\omega(r)}{\varphi'(r)}M'(\omega(r)|u(r)|)\left(u'(r)\mathrm{sgn}\,u(r)\right)\mu(dr)-\theta_n\\ &=: -I_n+II_n+III_n-\theta_n. \end{aligned}$$

From now on the proofs of the cases: **(A1)** and **(A2)** are given separately.

PART 1. THE CASE OF **(A1)**.

Integral II_n is now split into integrals over three sets (some of them can be empty):

$$\begin{aligned} F\setminus F^{\lambda_0} &= \{r:\omega(r)\neq 0,\omega'(r)\varphi'(r)>0,|\frac{\omega'(r)}{\varphi'(r)}|\leqslant\lambda_0\},\\ G &= \{r:\omega(r)\neq 0,\omega'(r)\varphi'(r)<0\},\\ F^{\lambda_0} &= \{r:\omega(r)\neq 0,\omega'(r)\varphi'(r)>0,|\frac{\omega'(r)}{\varphi'(r)}|>\lambda_0\}, \end{aligned}$$

and by (2.2) the estimate can be continued as

$$\begin{aligned} II_n &\leqslant D_M\int_{a_n}^{b_n}\frac{\omega'(r)}{\varphi'(r)}\frac{M(\omega(r)|u(r)|)}{\omega(r)}\chi_{F\setminus F^{\lambda_0}}(r)\mu(dr)\\ &\quad+d_M\int_{a_n}^{b_n}\frac{\omega'(r)}{\varphi'(r)}\frac{M(\omega(r)|u(r)|)}{\omega(r)}\chi_G(r)\mu(dr)\\ &\quad+D_M\int_{a_n}^{b_n}\frac{\omega'(r)}{\varphi'(r)}\frac{M(\omega(r)|u(r)|)}{\omega(r)}\chi_{F^{\lambda_0}}(r)\mu(dr)\\ &=: IV_n+V_n+VI_n. \end{aligned}$$

Hence we have arrived at

$$J_n+I_n-V_n-VI_n\leqslant IV_n+III_n-\theta_n. \tag{4.4}$$

Observe that on the other hand

$$J_n + I_n - V_n - VI_n \geqslant B^{1,\lambda_0}(a,b)J_n. \tag{4.5}$$

We will now estimate IV_n and III_n.

To this end, we use Lemma 2.1 with $s_1 = \omega(r)|u(r)|$, and $s_2 = |u(r)|/\delta_1$ for IV_n, $s_1 = \omega(r)|u(r)|$, $s_2 = |u'(r)|/\delta_2$ for III_n, where $\delta_1, \delta_2 > 0$ are arbitrary positive numbers, to be specified later on. Therefore one has (see (2.3))

$$\begin{aligned}
IV_n &\leqslant D_M\lambda_0 \int_{a_n}^{b_n} \frac{M(\omega(r)|u(r)|)}{\omega(r)}\mu(dr) \\
&= \delta_1 D_M \lambda_0 \int_{\{r\in(a_n,b_n):u(r)\neq 0\}} \frac{M(\omega(r)|u(r)|)}{\omega(r)|u(r)|}\frac{|u(r)|}{\delta_1}\mu(dr) \qquad (4.6)\\
&\leqslant \frac{\delta_1 D_M(D_M-1)\lambda_0}{d_M}\int_{a_n}^{b_n} M(\omega(r)|u(r)|)\mu(dr) \\
&\quad + \frac{\delta_1 D_M \bar{c}(\frac{1}{\delta_1})\lambda_0}{d_M}\int_{a_n}^{b_n} M(|u(r)|)\mu(dr) \\
&= \frac{\delta_1 D_M(D_M-1)\lambda_0}{d_M}J_n + \frac{\delta_1 D_M \bar{c}(\frac{1}{\delta_1})\lambda_0}{d_M}G_n; \\
III_n &\leqslant \delta_2 D_M \int_{a_n}^{b_n} \frac{\omega(r)}{|\varphi'(r)|}\frac{M(\omega(r)|u(r)|)}{\omega(r)|u(r)|}\cdot\frac{|u'(r)|}{\delta_2}\mu(dr) \qquad (4.7)\\
&\leqslant \frac{\delta_2 D_M(D_M-1)}{d_M}\int_{a_n}^{b_n}\frac{\omega(r)}{|\varphi'(r)|}M(\omega(r)|u(r)|)\mu(dr) \\
&\quad + \frac{\delta_2 D_M \bar{c}(\frac{1}{\delta_2})}{d_M}\int_{a_n}^{b_n}\frac{\omega(r)}{|\varphi'(r)|}M(|u'(r)|)\mu(dr) \\
&\leqslant \frac{\delta_2 D_M(D_M-1)L(a,b)}{d_M}J_n + \frac{\delta_2 D_M \bar{c}(\frac{1}{\delta_2})L(a,b)}{d_M}H_n.
\end{aligned}$$

Adding them together, we get

$$\begin{aligned}
IV_n + III_n &\leqslant \frac{D_M(D_M-1)}{d_M}(\delta_1\lambda_0 + \delta_2 L(a,b))J_n \\
&\quad + \frac{\delta_1 D_M \bar{c}(\frac{1}{\delta_1})}{d_M}\lambda_0 G_n + \frac{\delta_2 D_M \bar{c}(\frac{1}{\delta_2})}{d_M}L(a,b)H_n.
\end{aligned} \tag{4.8}$$

Combining estimates (4.4), (4.5), and (4.8) and rearranging, we end up with

$$\begin{aligned}
&\left(B^{1,\lambda_0}(a,b) - \frac{D_M(D_M-1)}{d_M}(\delta_1\lambda_0 + \delta_2 L(a,b))\right)\cdot J_n \\
&\leqslant \frac{\delta_1 D_M \bar{c}(\frac{1}{\delta_1})}{d_M}\lambda_0 G_n + \frac{\delta_2 D_M \bar{c}(\frac{1}{\delta_2})}{d_M}L(a,b)H_n - \theta_n.
\end{aligned} \tag{4.9}$$

Now we let $n \to \infty$. Since $J_n \to J,\ G_n \to G,\ H_n \to H$, and $\theta_n \to \alpha \geqslant 0$ (see (4.3)), we obtain the estimate

$$(B-(A_1\delta_1+A_2\delta_2))J \leqslant K_1\delta_1\,\bar{c}\left(\frac{1}{\delta_1}\right)+K_2\delta_2\bar{c}\left(\frac{1}{\delta_2}\right),$$

where

$$\begin{aligned} B &= B^{1,\lambda_0}(a,b), & A_1 &= \frac{D_M(D_M-1)\lambda_0}{d_M}, & A_2 &= \frac{D_M(D_M-1)L}{d_M},\\ L &= L(a,b), & K_1 &= \frac{D_M}{d_M}\lambda_0 G, & K_2 &= \frac{D_M}{d_M}LH. \end{aligned}$$

This can be written as (provided $A_1\delta_1+A_2\delta_2<B$)

$$J \leqslant \frac{1}{B-(A_1\delta_1+A_2\delta_2)}\left(K_1\delta_1\,\bar{c}(\frac{1}{\delta_1})+K_2\delta_2\bar{c}(\frac{1}{\delta_2})\right). \tag{4.10}$$

To conclude, consider separately the cases $\lambda_0>0$ and $\lambda_0=0$.

When $\lambda_0>0$, then we can choose for example $\delta_1=\frac{B}{4A_1}, \delta_2=\frac{B}{4A_2}$, which are both well-defined and positive in this case.

When $\lambda_0=0$, then $K_1=0$ and (4.10) is just

$$J \leqslant \frac{K_2\delta_2\bar{c}\left(\frac{1}{\delta_2}\right)}{B-A_2\delta_2}.$$

For $\delta_2=\frac{d_MB}{D_M^2L}$ we have $B-A_2\delta_2=\frac{B}{D_M}>0$, and we get

$$\int_a^b M(\omega(r)|u(r)|)\,\mu(dr) \leqslant \bar{c}\left(\frac{D_M^2L(a,b)}{d_MB}\right)\int_a^b M(|u'(r)|)\,\mu(dr).$$

PART 2. THE CASE OF **(A2)**.

Similarly as before, integral II_n is split into integrals over three sets:

$$\begin{aligned} G\setminus G^{\lambda_0} &= \{r:\omega(r)\neq 0, \omega'(r)\varphi'(r)<0, |\frac{\omega'(r)}{\varphi'(r)}|\leqslant \lambda_0\},\\ F &= \{r:\omega(r)\neq 0, \omega'(r)\varphi'(r)>0\},\\ G^{\lambda_0} &= \{r:\omega(r)\neq 0, \omega'(r)\varphi'(r)<0, |\frac{\omega'(r)}{\varphi'(r)}|> \lambda_0\}, \end{aligned}$$

and, by (2.2), the estimate continues as

$$\begin{aligned} II_n \geqslant\ & D_M\int_{a_n}^{b_n}\frac{\omega'(r)}{\varphi'(r)}\frac{M(\omega(r)|u(r)|)}{\omega(r)}\chi_{G\setminus G^{\lambda_0}}(r)\,\mu(dr)\\ &+d_M\int_{a_n}^{b_n}\frac{\omega'(r)}{\varphi'(r)}\frac{M(\omega(r)|u(r)|)}{\omega(r)}\chi_F(r)\,\mu(dr)\\ &+D_M\int_{a_n}^{b_n}\frac{\omega'(r)}{\varphi'(r)}\frac{M(\omega(r)|u(r)|)}{\omega(r)}\chi_{G^{\lambda_0}}(r)\,\mu(dr)\\ =\ & IV_n'+V_n'+VI_n'. \end{aligned}$$

We have arrived at

$$-J_n = I_n - II_n - III_n + \theta_n \leqslant I_n - IV'_n - V'_n - VI'_n - III_n + \theta_n,$$

and so

$$B^{2,\lambda_0}(a,b)J_n \leqslant -J_n - I_n + V'_n + VI'_n \leqslant -IV'_n - III_n + \theta_n. \tag{4.11}$$

Observe that (4.6) holds for $-IV'_n$ instead of IV_n, and (4.7) holds for $-III_n$ instead of III_n. Therefore we get:

$$-IV'_n \leqslant \frac{\delta_1 D_M(D_M-1)\lambda_0}{d_M}J_n + \frac{\delta_1 D_M \bar{c}(\frac{1}{\delta_1})\lambda_0}{d_M}G_n$$

$$-III_n \leqslant \frac{\delta_2 D_M(D_M-1)L(a,b)}{d_M}J_n + \frac{\delta_2 D_M \bar{c}(\frac{1}{\delta_2})L(a,b)}{d_M}H_n.$$

Consequently,

$$\begin{aligned} -IV'_n - III_n \leqslant{}& \frac{D_M(D_M-1)}{d_M}(\delta_1\lambda_0 + \delta_2 L(a,b))J_n \\ &+ \frac{\delta_1 D_M \bar{c}(\frac{1}{\delta_1})}{d_M}\lambda_0 G_n + \frac{\delta_2 D_M \bar{c}(\frac{1}{\delta_2})}{d_M}L(a,b)H_n. \end{aligned} \tag{4.12}$$

Combining (4.11) and (4.12) and rearranging gives

$$\begin{aligned} &\left(B^{2,\lambda_0}(a,b) - \frac{D_M(D_M-1)}{d_M}(\delta_1\lambda_0 + \delta_2 L(a,b))\right)\cdot J_n \\ &\leqslant \frac{\delta_1 D_M \bar{c}(\frac{1}{\delta_1})}{d_M}\lambda_0 G_n + \frac{\delta_2 D_M \bar{c}(\frac{1}{\delta_2})}{d_M}L(a,b)H_n + \theta_n. \end{aligned}$$

Conclusion of the proof under assumption **(A2)** is identical with this under assumption **(A1)**. □

REMARK 4.1. To obtain the constants listed in Remark 3.2, when $\lambda_0 \neq 0$ we can choose for example

$$\delta_1 = \frac{d_M B}{D_M^2}\left(\frac{\lambda_0}{L+\lambda_0}\right)^{\frac{1}{D'_M}}\frac{1}{\lambda_0}, \quad \delta_2 = \frac{d_M B}{D_M^2}\left(\frac{L}{L+\lambda_0}\right)^{\frac{1}{D'_M}}\frac{1}{L},$$

where B stands for $B^{1,\lambda_0}(a,b)$ or $B^{2,\lambda_0}(a,b)$, as needed. This is an admissible choice because $A_1\delta_1 + A_2\delta_2 = \frac{B}{D'_M}((\frac{\lambda_0}{L+\lambda_0})^{\frac{1}{D'_M}} + (\frac{L}{L+\lambda_0})^{\frac{1}{D'_M}}) < B$, which in turn follows from the inequality $\gamma(x^\gamma + (1-x)^\gamma) < 1$, for $x \in [0,1], \gamma \in (0,1)$. We omit the technical problem of optimizing the right-hand side of (4.10) over δ_1, δ_2.

Proof of Proposition 3.1 *(sketch).* We only consider the proof under the assumption **(A1)**. Let's pick up the proof of Theorem 3.1 from formula (4.10). In our case $M(\lambda) = \lambda^p$ and so we have

$$A_1 = (p-1)\lambda_0, \quad A_2 = (p-1)L, \quad K_1 = \lambda_0 G, \quad K_2 = LH.$$

Therefore (4.10) becomes

$$J \leqslant \frac{\lambda_0 G\delta_1^{1-p} + LH\delta_2^{1-p}}{B-(p-1)(\lambda_0\delta_1 + L\delta_2)} =: r(\delta_1,\delta_2),$$

making sense for $(\delta_1,\delta_2) \in \Delta := \{(x,y) : x,y > 0, (p-1)(\lambda_0 x + Ly) < B\}$. We will be done if we minimize $r(\delta_1,\delta_2)$ over the region Δ. To this goal, substitute $\bar{\delta}_1 = \frac{(p-1)\lambda_0\delta_1}{B}$, $\bar{\delta}_2 = \frac{(p-1)L\delta_2}{B}$. Then

$$r(\delta_1,\delta_2) = \bar{r}(\bar{\delta}_1,\bar{\delta}_2) := \frac{E_1\bar{\delta}_1^{1-p} + E_2\bar{\delta}_2^{1-p}}{1-(\bar{\delta}_1+\bar{\delta}_2)},$$

where

$$E_1 = \left(\frac{\lambda_0}{B}\right)^p (p-1)^{p-1}G, \qquad E_2 = \left(\frac{L}{B}\right)^p (p-1)^{p-1}H,$$

$$(\bar{\delta}_1,\bar{\delta}_2) \in \bar{\Delta} := \{(x,y) : x+y < 1, x,y > 0\}.$$

Inside the region, the function $\bar{r}(x,y)$ has a single critical point

$$x_0 = \left(\frac{p-1}{p}\right)\frac{E_1^{1/p}}{E_1{}^{1/p} + E_2^{1/p}}, \qquad y_0 = \left(\frac{p-1}{p}\right)\frac{E_2^{1/p}}{E_1{}^{1/p} + E_2^{1/p}},$$

and the value $r(x_0,y_0) = \left(\frac{p}{B}\right)^p (\lambda_0 G^{1/p} + LH^{1/p})^p$ is minimal in Δ. It is enough to take $\bar{\delta}_1 = x_0, \bar{\delta}_2 = y_0$. $\square$

Proof of Theorem 3.2. It suffices to show (3.12) for $u \in \mathscr{R}$. We prove the theorem only in two cases: 1) when conditions **(B1$_-$)** and **(B1$_+$)** hold, 2) when **(K$_-$)** and **(B1$_+$)** hold. The proofs in all remaining cases follow by similar arguments.

Suppose then that **(B1$_-$)** and **(B1$_+$)** hold. We find numbers r_0, R_0 such that $a < r_0 < R_0 < b$ and $L(a,r_0) < \infty$, $B^{1,\lambda_0}(a,r_0) > 0$, and also $L(R_0,b) < \infty$, $B^{1,\lambda_1}(R_0,b) > 0$, for some $\lambda_0, \lambda_1 \in [0,\infty)$.

We pick two additional points: $r_1 = \frac{a+r_0}{2}$, $R_1 = \frac{R_0+b}{2}$ and consider a smooth partition of unity η_1,η_2,η_3 subordinate to the covering of (a,b) by (a,r_0), (r_1,R_1), and (R_0,b). In particular $0 \leqslant \eta_i(r) \leqslant 1$, $\eta_1(r)+\eta_2(r)+\eta_3(r) \equiv 1$, $\operatorname{supp}\eta_1 \subset [a,r_0)$, $\operatorname{supp}\eta_2 \subset (r_1,R_1)$, $\operatorname{supp}\eta_3 \subset (R_0,b]$. These points and mappings are fixed from now on and they do not depend on u. We have

$$I := \int_{(a,b)} M(\omega(r)|u(r)|)\mu(dr) = \int_{(a,b)} M((\eta_1(r)+\eta_2(r)+\eta_3(r))\omega(r)|u(r)|)\mu(dr).$$

From the convexity of M

$$M(a+b+c)=M(3\cdot\frac{1}{3}(a+b+c))\leqslant 3^{D_M}M\left(\frac{a+b+c}{3}\right)\leqslant 3^{D_M-1}(M(a)+M(b)+M(c)),$$

therefore

$$I\leqslant\sum_{i=1}^{3}3^{D_M-1}\int_{(a,b)}M(\eta_i(r)\omega(r)|u(r)|)\mu(dr)=:I_1+I_2+I_3.$$

As ω is locally bounded, $0\leqslant\eta_i\leqslant 1$, the middle integral can be estimated by

$$I_2\leqslant B\int_{(r_1,R_1)}M(|u(r)|)\mu(dr), \tag{4.13}$$

where $B=3^{D_M-1}\bar{c}\left(\sup_{r\in(r_1,R_1)}|\omega(r)|\right)$. Moreover, as $u\in\mathscr{R}^1_-(a,b)$, the function $u_1(r)=u(r)\eta_1(r)$ vanishes in a neighborhood of r_0, and it follows that $u_1\in\mathscr{R}^1(a,r_0)$. From Theorem 3.1 applied to u_1 on (a,r_0) we get

$$I_1\leqslant\bar{C}_1\int_{(a,r_0)}M(|u_1(r)|)\mu(dr)+\bar{C}_2\int_{(a,r_0)}M(|u_1'(r)|)\mu(dr),$$

where $\bar{C_1},\bar{C_2}$ are absolute constants not depending on u.

Further, we have

$$\begin{aligned}|u_1(r)|&\leqslant|u(r)|;\\ |u_1'(r)|&\leqslant|u(r)\eta_1'(r)|+|u'(r)\eta_1(r)|\leqslant|u(r)|\cdot\sup_{s\in(a,r_0)}|\eta_1'(s)|+|u'(r)|,\end{aligned}$$

and so we get

$$I_1\leqslant\widetilde{C_1}\int_{(a,r_0)}M(|u(r)|)\mu(dr)+\widetilde{C_2}\int_{(a,r_0)}M(|u'(r)|)\mu(dr), \tag{4.14}$$

with constants again not depending on u.

By similar arguments we obtain the estimate

$$I_3\leqslant\widetilde{D_1}\int_{(a,r_0)}M(|u(r)|)\mu(dr)+\widetilde{D_2}\int_{(a,r_0)}M(|u'(r)|)\mu(dr). \tag{4.15}$$

Using (4.13), (4.14) and (4.15) we finally obtain the estimate (3.12), with some constants C_1,C_2 not depending on u.

The argument in the case when **($\mathbf{K}_-$)** and **($\mathbf{B1}_+$)** hold is simpler. Fix $a<R_0<b$ such that $L(R_0,b)<\infty$, $B^{1,\lambda_0}(R_0,b)>0$ and take $r_0=\frac{a+R_0}{2}$. Then proceed as before, considering the covering of (a,b) by two intervals (a,R_0) and (r_0,b). We are done. □

Proof of Theorem 3.3. Take $0\neq u\in\mathscr{R}$ and consider $\widetilde{u}=\frac{u}{\|u\|_{L^M((a,b),\mu)}+\|u'\|_{L^M((a,b),\mu)}}$. For short, write $\|u\|_M$ instead of $\|u\|_{L^M((a,b),\mu)}$. Since $\mathscr{R}$ is dilation invariant, $\widetilde{u}$ belongs to $\mathscr{R}$ once u does.

Inequality (3.19) for $\widetilde{u}$ reads

$$\int M(|\omega\widetilde{u}|)d\mu \leqslant C\left(\int M(\frac{|u|}{\|u\|_M+\|u'\|_M}))d\mu + \int M(\frac{|u'|}{\|u\|_M+\|u'\|_M})d\mu\right)$$
$$\leqslant C\left(\int M(\frac{|u|}{\|u\|_M})d\mu + \int M(\frac{|u'|}{\|u'\|_M})d\mu\right) \leqslant 2C$$

(in the last inequality, we have used property (2.5) of Orlicz functionals). Since for any $f \in L^M$ one has $\|f\|_M \leqslant \int M(|f|)d\mu + 1$ (see (9.4) and (9.20) of [25]), this gives

$$\|\omega\widetilde{u}\|_M \leqslant 2C+1,$$

and consequently (3.20). This ends the proof of the theorem. □

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