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SOME INEQUALITIES INVOLVING UNITARILY INVARIANT NORMS

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Abstract. This paper aims to present some inequalities for unitarily invariant norms. We first give inverses of Young and Heinz type inequalities for scalars. Then we use these inequalities to establish some inequalities for unitarily invariant norms.

1. Introduction

Let $M_{m,n}$ be the space of $m\times n$ complex matrices and $M_n = M_{n,n}$. Let $\|\cdot\|$ denote any unitarily invariant norm on M_n. So, $\|UAV\| = \|A\|$ for all $A\in M_n$ and for all unitary matrices $U,\ V\in M_n$. For $A=(a_{ij})\in M_n$, the Hilbert-Schmidt norm of A is defined by $\|A\|_2 = \sqrt{\sum_{j=1}^{n} s_j^2(A)}$, where $s_1(A)\geqslant\cdots\geqslant s_n(A)$ are the singular values of A, that is, the eigenvalues of the positive semidefinite matrix $|A|=\sqrt{AA^*}$, arranged in decreasing order and repeated according to multiplicity. Note that $\|A\|_2=\sqrt{tr(AA^*)}$, where tr is the usual trace functional. It is known that the Hilbert-Schmidt norm is unitarily invariant, and it is evident that each unitarily invariant norm is a symmetric gauge function of singular values [1, p. 54-55].

The scalar Young inequality says that if $a,\ b\geqslant 0$ and $0\leqslant v\leqslant 1$, then

$$a^v b^{1-v}\leqslant va+(1-v)b \tag{1.1}$$

with equality if and only if $a=b$. The scalar Heinz inequality says that if $a,\ b\geqslant 0$ and $0\leqslant v\leqslant 1$,

$$a^v b^{1-v}+a^{1-v}b^v\leqslant a+b. \tag{1.2}$$

Hirzallah and Kittaneh [2] obtained a refinement of the Young inequality (1.1) as follows:

$$\left(a^v b^{1-v}\right)^2+r_0^2(a-b)^2\leqslant (va+(1-v)b)^2, \tag{1.3}$$

where $r_0=\min\{v,\ 1-v\}$. Kittaneh and Manasrah [3] obtained a refinement of the Heinz inequality (1.2):

$$\left(a^v b^{1-v}+a^{1-v}b^v\right)^2+2r_0(a-b)^2\leqslant (a+b)^2, \tag{1.4}$$

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where $r_0 = \min\{v,\ 1-v\}$.

Kosaki [4] and Bhatia and Parthasarathy [5] proved that if $A,\ B,\ X \in M_n$ such that A and B are positive semidefinite and if $0 \leqslant v \leqslant 1$, then

$$\left\|A^v X B^{1-v}\right\|_2^2 \leqslant \|vAX + (1-v)XB\|_2^2. \tag{1.5}$$

Based on the refined Young inequality (1.3), Hirzallah and Kittaneh [2] proved that if $A,\ B,\ X \in M_n$ such that A and B are positive semidefinite and if $0 \leqslant v \leqslant 1$, then

$$\left\|A^v X B^{1-v}\right\|_2^2 + r_0^2 \|AX - XB\|_2^2 \leqslant \|vAX + (1-v)XB\|_2^2. \tag{1.6}$$

Bhatia and Davis [6] proved that if $A,\ B,\ X \in M_n$ such that A and B are positive semidefinite and if $0 \leqslant v \leqslant 1$, then

$$2\left\|A^{1/2} X B^{1/2}\right\| \leqslant \left\|A^v X B^{1-v} + A^{1-v} X B^v\right\| \leqslant \|AX + XB\|. \tag{1.7}$$

Kittaneh and Manasrah [3] obtained an improvement of the second inequality in (1.7) for the Hilbert-Schmidt norm, which can be stated as follows:

$$\left\|A^v X B^{1-v} + A^{1-v} X B^v\right\|_2^2 + 2r_0 \|AX - XB\|_2^2 \leqslant \|AX + XB\|_2^2, \tag{1.8}$$

where $r_0 = \min\{v,\ 1-v\}$.

In this paper, we present the reverses of the inequalities (1.3) and (1.4). Based on these inequalities, we establish the reverses of the inequalities (1.6) and (1.8) and give some other inequalities for unitarily invariant norms.

2. Reverses of Young and Heinz type inequalities

In this section, we present reverses of the improved Young inequality (1.3) and the refined Heinz inequality (1.4). To achieve this goal, we need the following lemma [7, p. 137]; for the reader's convenience, we will give a sketch of its proof.

LEMMA 2.1. *If $a,\ b \geqslant 0$ and $0 \leqslant v \leqslant 1$, then*

$$va^2 + (1-v)b^2 \leqslant \left(a^v b^{1-v}\right)^2 + s_0 (a-b)^2, \tag{2.1}$$

where $s_0 = \max\{v,\ 1-v\}$.

Proof. If $v = \frac{1}{2}$, the inequality becomes equality. If $v > \frac{1}{2}$, then by (1.1), we have

$$\begin{aligned}
\left(a^v b^{1-v}\right)^2 &+ s_0 (a-b)^2 - va^2 - (1-v)b^2 \\
&= \left(a^v b^{1-v}\right)^2 + (2v-1)b^2 - 2vab \\
&= \left(a^v b^{1-v}\right)^2 + (2v-1)b^2 + (2-2v)ab - 2ab \\
&\geqslant \left(a^v b^{1-v}\right)^2 + \left(a^{1-v} b^v\right)^2 - 2ab \geqslant 0.
\end{aligned}$$

If $v < \frac{1}{2}$, then by (1.1), we have

$$\begin{aligned}\left(a^{v}b^{1-v}\right)^{2} + s_{0}(a-b)^{2} - va^{2} - (1-v)b^{2} &= \left(a^{v}b^{1-v}\right)^{2} + (1-2v)a^{2} - 2(1-v)ab \\ &= \left(a^{v}b^{1-v}\right)^{2} + (1-2v)a^{2} + 2vab - 2ab \\ &\geqslant \left(a^{v}b^{1-v}\right)^{2} + \left(a^{1-v}b^{v}\right)^{2} - 2ab \geqslant 0.\end{aligned}$$

Hence, for all $v \in [0,1]$,

$$va^{2} + (1-v)b^{2} \leqslant \left(a^{v}b^{1-v}\right)^{2} + s_{0}(a-b)^{2}.$$

This completes the proof. □

The following result gives a reverse of the inequality (1.3).

THEOREM 2.1. *If* $a,\ b \geqslant 0$ *and* $0 \leqslant v \leqslant 1$, *then*

$$(va + (1-v)b)^{2} \leqslant \left(a^{v}b^{1-v}\right)^{2} + s_{0}^{2}(a-b)^{2}, \tag{2.2}$$

where $s_{0} = \max\{v,\ 1-v\}$.

Proof. If $v = \frac{1}{2}$, the inequality becomes equality. If $v > \frac{1}{2}$, then we have

$$\begin{aligned}(va + (1-v)b)^{2} - s_{0}^{2}(a-b)^{2} &= (va + (1-v)b)^{2} - v^{2}(a-b)^{2} \\ &= (1-2v)b^{2} + 2vab \\ &= va^{2} + (1-v)b^{2} + 2vab - vb^{2} - va^{2}.\end{aligned}$$

By Lemma 2.1, we have

$$va^{2} + (1-v)b^{2} + 2vab - vb^{2} - va^{2} \leqslant \left(a^{v}b^{1-v}\right)^{2}$$

and so

$$(va + (1-v)b)^{2} \leqslant \left(a^{v}b^{1-v}\right)^{2} + v^{2}(a-b)^{2}.$$

If $v < \frac{1}{2}$, then

$$\begin{aligned}(va + (1-v)b)^{2} - s_{0}^{2}(a-b)^{2} &= (va + (1-v)b)^{2} - (1-v)^{2}(a-b)^{2} \\ &= (2v-1)a^{2} + 2(1-v)ab \\ &= va^{2} + (1-v)b^{2} + 2(1-v)ab - (1-v)a^{2} \\ &\quad - (1-v)b^{2}.\end{aligned}$$

By Lemma 2.1, we have

$$va^{2} + (1-v)b^{2} + 2(1-v)ab - (1-v)a^{2} - (1-v)b^{2} \leqslant \left(a^{v}b^{1-v}\right)^{2}$$

and so

$$(va + (1-v)b)^{2} \leqslant \left(a^{v}b^{1-v}\right)^{2} + (1-v)^{2}(a-b)^{2}.$$

Hence, for all $v \in [0,1]$,

$$(va + (1-v)b)^2 \leqslant \left(a^v b^{1-v}\right)^2 + s_0^2 (a-b)^2 .$$

This completes the proof. □

It is easy to see that the left- and right-hand sides of the inequality (2.1) are greater than or equal to the corresponding sides of the inequality (2.2), respectively. Therefore, neither the inequality (2.1) nor the inequality (2.2) is uniformly better than the other.

By Lemma 2.1, we have the following result, i.e., a reverse inequality of (1.4).

THEOREM 2.2. *If* $a,\ b \geqslant 0$ *and* $0 \leqslant v \leqslant 1$*, then*

$$(a+b)^2 \leqslant \left(a^v b^{1-v} + a^{1-v} b^v\right)^2 + 2s_0 (a-b)^2 , \tag{2.3}$$

where $s_0 = \max\{v,\ 1-v\}$.

Proof. By Lemma 2.1, we have

$$\begin{aligned}(a+b)^2 - \left(a^v b^{1-v} + a^{1-v} b^v\right)^2 &= a^2 + b^2 - \left(a^v b^{1-v}\right)^2 - \left(a^{1-v} b^v\right)^2 \\ &= va^2 + (1-v)b^2 - \left(a^v b^{1-v}\right)^2 + (1-v)a^2 \\ &\quad + vb^2 - \left(a^{1-v} b^v\right)^2 \\ &\leqslant s_0 (a-b)^2 + s_0 (a-b)^2 \\ &= 2s_0 (a-b)^2 .\end{aligned}$$

This completes the proof. □

3. Some inequalities for unitarily invariant norms

In this section, we first establish reverses of the inequalities (1.6) and (1.8) and then give some other inequalities for unitarily invariant norms, which are based on the inequalities (2.1)-(2.3).

By Theorem 2.1, we have a reverse inequality of (1.6).

THEOREM 3.1. *Let* $A,\ B,\ X \in M_n$ *such that* A *and* B *are positive semidefinite. If* $0 \leqslant v \leqslant 1$*, then*

$$\|vAX + (1-v)XB\|_2^2 \leqslant \left\|A^v X B^{1-v}\right\|_2^2 + s_0^2 \|AX - XB\|_2^2 ,$$

where $s_0 = \max\{v,\ 1-v\}$.

Proof. Since every positive semidefinite matrix is unitarily diagonalizable, it follows that there are unitary matrices $U,\ V \in M_n$ such that

$$A = U\Lambda_1 U^*, \quad B = V\Lambda_2 V^*,$$

where

$$\Lambda_1 = diag(\lambda_1, \cdots, \lambda_n), \quad \Lambda_2 = diag(\mu_1, \cdots, \mu_n), \quad \lambda_i,\ \mu_i \geqslant 0, \quad i = 1, \cdots, n.$$

Let $Y = U^*XV = [y_{ij}]$, then

$$vAX + (1-v)XB = U(v\Lambda_1 Y + (1-v)Y\Lambda_2)V^* = U[(v\lambda_i + (1-v)\mu_j)y_{ij}]V^*,$$

$$AX - XB = U[(\lambda_i - \mu_j)y_{ij}]V^*$$

and

$$A^vXB^{1-v} = U\left[\lambda_i^v \mu_j^{1-v} y_{ij}\right]V^*.$$

By the inequality (2.2), we have

$$\begin{aligned}\|vAX + (1-v)XB\|_2^2 &= \sum_{i,j=1}^{n} (v\lambda_i + (1-v)\mu_j)^2 \left|y_{ij}\right|^2 \\ &\leqslant \sum_{i,j=1}^{n} \left(\lambda_i^v \mu_j^{1-v}\right)^2 \left|y_{ij}\right|^2 + s_0^2 \sum_{i,j=1}^{n} (\lambda_i - \mu_j)^2 \left|y_{ij}\right|^2 \\ &= \left\|A^vXB^{1-v}\right\|_2^2 + s_0^2 \|AX - XB\|_2^2.\end{aligned}$$

This completes the proof. □

By Theorem 2.2, we have the following result, i.e., a reverse inequality of (1.8).

THEOREM 3.2. *Let A, B, $X \in M_n$ such that A and B are positive semidefinite. If $0 \leqslant v \leqslant 1$, then*

$$\|AX + XB\|_2^2 \leqslant \left\|A^vXB^{1-v} + A^{1-v}XB^v\right\|_2^2 + 2s_0 \|AX - XB\|_2^2,$$

where $s_0 = \max\{v,\ 1-v\}$.

Proof. Since every positive semidefinite matrix is unitarily diagonalizable, it follows that there are unitary matrices U, $V \in M_n$ such that

$$A = U\Lambda_1 U^*, \quad B = V\Lambda_2 V^*,$$

where

$$\Lambda_1 = diag(\lambda_1, \cdots, \lambda_n), \quad \Lambda_2 = diag(\mu_1, \cdots, \mu_n), \quad \lambda_i,\ \mu_i \geqslant 0, i = 1, \cdots, n.$$

Let $Y = U^*XV = [y_{ij}]$, we have

$$A^vXB^{1-v} + A^{1-v}XB^v = U\left(\Lambda_1^v Y \Lambda_2^{1-v} + \Lambda_1^{1-v} Y \Lambda_2^v\right)V^*.$$

Therefore,

$$\left\|A^vXB^{1-v} + A^{1-v}XB^v\right\|_2^2 = \sum_{i,j=1}^{n} \left(\lambda_i^v \mu_j^{1-v} + \lambda_i^{1-v} \mu_j^v\right)^2 \left|y_{ij}\right|^2.$$

By the inequality (2.3), we obtain

$$\begin{aligned}\|AX + XB\|_2^2 &= \sum_{i,j=1}^{n} (\lambda_i + \mu_j)^2 \left|y_{ij}\right|^2 \\ &\leqslant \sum_{i,j=1}^{n} \left(\lambda_i^v \mu_j^{1-v} + \lambda_i^{1-v} \mu_j^v\right)^2 \left|y_{ij}\right|^2 + 2s_0 \sum_{i,j=1}^{n} (\lambda_i - \mu_j)^2 \left|y_{ij}\right|^2 \\ &= \left\|A^vXB^{1-v} + A^{1-v}XB^v\right\|_2^2 + 2s_0 \|AX - XB\|_2^2.\end{aligned}$$

This completes the proof. □

Kittaneh proved in [8] that if $A,\ B,\ X\in M_n$ such that A and B are positive semidefinite and if $0\leqslant v\leqslant 1$, then

$$\left\|A^vXB^{1-v}+A^{1-v}XB^v\right\| - 4r_0\left\|A^{1/2}XB^{1/2}\right\|+2r_0\left\|AX+XB\right\| \leqslant \left\|AX+XB\right\|, \tag{3.1}$$

where $r_0=\min\{v,\ 1-v\}$. This is a refinement of the second inequality in (1.7).

Here, we have a reverse inequality of (3.1).

THEOREM 3.3. *Let $A,\ B,\ X\in M_n$ such that A and B are positive semidefinite. If $0\leqslant v\leqslant 1$, then*

$$\left\|AX+XB\right\| \leqslant \left\|A^vXB^{1-v}+A^{1-v}XB^v\right\| - 4s_0\left\|A^{1/2}XB^{1/2}\right\|+2s_0\left\|AX+XB\right\|,$$

where $s_0=\max\{v,\ 1-v\}$.

Proof. If

$$\left\|A^vXB^{1-v}+A^{1-v}XB^v\right\| - 4s_0\left\|A^{1/2}XB^{1/2}\right\|+2s_0\left\|AX+XB\right\| < \left\|AX+XB\right\|, \tag{3.2}$$

then it follows from (3.1) and (3.2) that

$$2\left\|A^vXB^{1-v}+A^{1-v}XB^v\right\| - 4\left\|A^{1/2}XB^{1/2}\right\|+2\left\|AX+XB\right\| < 2\left\|AX+XB\right\|.$$

So,

$$\left\|A^vXB^{1-v}+A^{1-v}XB^v\right\| < 2\left\|A^{1/2}XB^{1/2}\right\|,$$

which contradicts the first inequality in (1.7). This completes the proof. □

In view of the inequalities (1.7) and (3.1), one may ask whether it is true that

$$2\left\|A^{1/2}XB^{1/2}\right\| - 4r_0\left\|A^{1/2}XB^{1/2}\right\|+2r_0\left\|AX+XB\right\| \leqslant \left\|A^vXB^{1-v}+A^{1-v}XB^v\right\|,$$

where $r_0=\min\{v,\ 1-v\}$.

This is refuted by the following example

$$A=\begin{pmatrix}5&1\\1&1\end{pmatrix},\quad X=\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad B=\begin{pmatrix}4&1\\1&2\end{pmatrix},$$

$$r_0=v=\tfrac{1}{4},\quad \|\cdot\|=\|\cdot\|_2.$$

Let $A,\ B,\ X\in M_n$ such that A and B are positive semidefinite. If $0\leqslant v\leqslant 1$, then the following inequality holds (see [6]):

$$\left\|A^vXB^{1-v}\right\| \leqslant v\left\|AX\right\|+(1-v)\left\|XB\right\|. \tag{3.3}$$

Recently, Kittaneh and Manasrah [3] presented a refinement of (3.3):

$$\left\|A^{v}XB^{1-v}\right\|+r_0\left(\sqrt{\|AX\|}-\sqrt{\|XB\|}\right)^2\leqslant v\|AX\|+(1-v)\|XB\|, \tag{3.4}$$

where $r_0=\min\{v,\ 1-v\}$.

Next, we ask whether the reverse of (3.4):

$$v\|AX\|+(1-v)\|XB\|\leqslant\left\|A^{v}XB^{1-v}\right\|+s_0\left(\sqrt{\|AX\|}-\sqrt{\|XB\|}\right)^2, \tag{3.5}$$

is true or not, where $s_0=\max\{v,\ 1-v\}$.

It is not always true; we give two examples as follows:

$$A=\begin{pmatrix}5 & -2\\ -2 & 5\end{pmatrix},\quad X=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\quad B=\begin{pmatrix}3 & -2\\ -2 & 2\end{pmatrix},\quad s_0=v=\tfrac{9}{10},$$

then we have

$$v\|AX\|_2+(1-v)\|XB\|_2=7.3125$$

and

$$\left\|A^{v}XB^{1-v}\right\|_2+s_0\left(\sqrt{\|AX\|_2}-\sqrt{\|XB\|_2}\right)^2=7.4798.$$

However, if we choose

$$A=\begin{pmatrix}2 & -2\\ -2 & 4\end{pmatrix},\quad X=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\quad B=\begin{pmatrix}4 & -2\\ -2 & 2\end{pmatrix},\quad s_0=v=\tfrac{3}{5},$$

then we have

$$v\|AX\|_2+(1-v)\|XB\|_2=5.2915$$

and

$$\left\|A^{v}XB^{1-v}\right\|_2+s_0\left(\sqrt{\|AX\|_2}-\sqrt{\|XB\|_2}\right)^2=4.9112.$$

It should be pointed out that although the inequality (3.5) doesn't hold, the following inequality is true:

$$v\|AX\|+(1-v)\|XB\|\leqslant\|AX\|^{v}\|XB\|^{1-v}+s_0\left(\sqrt{\|AX\|}-\sqrt{\|XB\|}\right)^2,$$

which is a direct consequence of the inequality (2.1). Note that this inequality is clearly weaker than the inequality (3.5) by the following result [9]:

LEMMA 3.1. *Let A, B, $X\in M_n$ such that A and B are positive semidefinite. If $0\leqslant v\leqslant 1$, then*

$$\left\|A^{v}XB^{1-v}\right\|\leqslant\|AX\|^{v}\|XB\|^{1-v}.$$

Let A, B, $X \in M_n$ such that A and B are positive semidefinite. In view of the inequality (2.1), one may ask whether the following result is true or not:

$$\begin{aligned}\left\|A^vXB^{1-v}+A^{1-v}XB^v\right\|_2^2 \geqslant\ & \|vAX+(1-v)XB\|_2^2 \\ & +\|(1-v)AX+vXB\|_2^2 \\ & +2\left(\|AXB\|_2^2-s_0\|AX-XB\|_2^2\right),\end{aligned}$$

where $0 \leqslant v \leqslant 1$ and $s_0=\max\{v,\ 1-v\}$.

The inequality above is not always true, as the next counterexample shows:

$$A=\begin{pmatrix}25 & 0\\ 0 & 4\end{pmatrix},\quad X=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\quad B=\begin{pmatrix}16 & 0\\ 0 & 9\end{pmatrix},\quad v=\tfrac{1}{2}=s_0.$$

We obtain that

$$\left\|A^vXB^{1-v}+A^{1-v}XB^v\right\|_2^2=\left\|2A^{1/2}B^{1/2}\right\|_2^2=1744$$

and

$$\begin{aligned}&\|vAX+(1-v)XB\|_2^2+\|(1-v)AX+vXB\|_2^2+2\left(\|AXB\|_2^2-s_0\|AX-XB\|_2^2\right)\\ &\qquad=\tfrac{1}{2}\|A+B\|_2^2+2\|AB\|_2^2-\|A-B\|_2^2=323411.\end{aligned}$$

Next, we give another improvement of the inequality (3.3).

THEOREM 3.4. *Let A, B, $X \in M_n$ such that A and B are positive semidefinite. If $0 \leqslant v \leqslant 1$, then*

$$\left\|A^vXB^{1-v}\right\|^2+r_0^2(\|AX\|-\|XB\|)^2 \leqslant (v\|AX\|+(1-v)\|XB\|)^2$$

where $r_0=\min\{v,\ 1-v\}$.

Proof. By Lemma 3.1 and the inequality [3]:

$$\left(a^vb^{1-v}\right)^2+r_0(a-b)^2 \leqslant va^2+(1-v)b^2,$$

we have

$$\begin{aligned}\left\|A^vXB^{1-v}\right\|^2+r_0^2(\|AX\|-\|XB\|)^2 &\leqslant \left(\|AX\|^v\|XB\|^{1-v}\right)^2+r_0^2(\|AX\|-\|XB\|)^2\\ &\leqslant (v\|AX\|+(1-v)\|XB\|)^2.\end{aligned}$$

This completes the proof. □

Finally, we give an inequality involving the trace norm and the Hilbert-Schmidt norm. To do this, we need the following lemma (see [1, p. 48]).

LEMMA 3.2. *Let A, $B \in M_n$. Then*

$$\sum_{j=1}^{n} s_j(AB) \leqslant \sum_{j=1}^{n} s_j(A)s_j(B).$$

THEOREM 3.5. *Let A, $B \in M_n$ be positive semidefinite. If $0 \leqslant v \leqslant 1$, then*

$$tr(vA+(1-v)B) \leqslant \|A^v\|_2 \left\|B^{1-v}\right\|_2 + s_0\left(trA+trB-tr\left|A^{1/2}B^{1/2}\right|\right),$$

where $s_0 = \max\{v,\ 1-v\}$.

Proof. By the inequality (2.1), we have

$$vs_j(A)+(1-v)s_j(B) \leqslant s_j^v(A)s_j^{1-v}(B)+s_0\left(s_j^{1/2}(A)-s_j^{1/2}(B)\right)^2$$

for $j=1,\cdots,n$. Thus, by Lemma 3.2 and the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
& tr(vA+(1-v)B) \\
&= vtrA+(1-v)trB \\
&= \sum_{j=1}^{n}\left(vs_j(A)+(1-v)s_j(B)\right) \\
&\leqslant \sum_{j=1}^{n} s_j(A^v)s_j\left(B^{1-v}\right)+s_0\sum_{j=1}^{n}\left(s_j^{1/2}(A)-s_j^{1/2}(B)\right)^2 \\
&= \sum_{j=1}^{n} s_j(A^v)s_j\left(B^{1-v}\right)+s_0\sum_{j=1}^{n}\left(s_j(A)+s_j(B)\right)-2s_0\sum_{j=1}^{n}s_j^{1/2}(A)s_j^{1/2}(B) \\
&\leqslant \left(\sum_{j=1}^{n}s_j^2(A^v)\right)^{1/2}\left(\sum_{j=1}^{n}s_j^2\left(B^{1-v}\right)\right)^{1/2}+s_0\left(trA+trB-\sum_{j=1}^{n}s_j\left(A^{1/2}B^{1/2}\right)\right) \\
&= \|A^v\|_2\left\|B^{1-v}\right\|_2+s_0\left(trA+trB-tr\left|A^{1/2}B^{1/2}\right|\right).
\end{aligned}$$

This completes the proof. □

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