Mathematical
Inequalities
& Applications
Volume 15, Number 4 (2012), 787–810 doi:10.7153/mia-15-68

POTENTIAL INEQUALITY REVISITED I: GENERAL CASE

NEVEN ELEZOVIĆ, JOSIP PEČARIĆ AND MARJAN PRALJAK

(*Communicated by Ivan Perić*)

Abstract. The main subject of this paper is a detailed study of potential inequality, which was introduced in [5]. Original inequality is extended to the case of general convex and concave functions. Various functionals connected with this inequality are defined and some improvements or refinements of known inequalities are given. Special attention is given to exponential convexity of such functionals. The inequalities obtained here are of general nature. They will be specified and studied in more details with concrete examples of involved kernels in our forthcoming papers.

1. Introduction

M. Rao and H. Šikić introduced potential inequality in [5] (their formulation of it is given below) and used it as a powerful tool which generates Hardy's inequality and many other inequalities as special cases. This paper will be the main reference in our work. However, they define convex and concave function in a rather unusual way. For example, for a convex function φ, function $-\varphi$ is not concave in their sense, so the main theorem cannot be applied directly to concave functions. Their theory applies well to functions which behave like power functions $x \mapsto x^p$, and this seems to be sufficient enough for main applications.

The purpose of this paper is to revisit potential inequality in a more general setting. We shall derive identities and corresponding inequalities in a general form which can be applied to convex and concave function defined in the standard way. As a consequence of this approach, the scale of possible applications is considerably larger and the derived formulas are much more complete.

For the convenience of the reader, let us briefly describe the Rao-Šikić approach to potential inequality.

We say that $N(x,dy)$ is a *(positive) kernel* on X if $N : X \times \mathscr{B}(X) \to [0,+\infty]$ is a mapping such that, for every $x \in X$, $A \mapsto N(x,A)$ is a σ-finite measure, and, for every $A \in \mathscr{B}(X)$, $x \mapsto N(x,A)$ is a measurable function. For a measurable function f, the *potential of f with respect to N at a point $x \in X$* is

$$(Nf)(x) = \int_X f(y)N(x,dy),$$

Mathematics subject classification (2010): 26D15.
Keywords and phrases: Potential inequality, convex function, exponential convexity.

Paper MIA-15-68

whenever the integral exists. The class of functions that have the potential at every point is denoted by $\mathscr{POT}(N)$.

For a measure μ on $(X,\mathscr{B}(X))$ and a measurable set $C \in \mathscr{B}(X)$ we will denote by $\hat{N}_C\mu$ the measure defined by

$$(\hat{N}_C\mu)(dy) = \int_C N(x,dy)\mu(dx).$$

If $C = X$ we will omit the subscript, i. e. $\hat{N}\mu$ will denote the measure $\hat{N}_X\mu$.

DEFINITION 1. ([5]) Let N be a positive kernel on X and $\mathscr{R} \subset \mathscr{POT}(N)$. We say that N *satisfies the strong maximum principle on* $\mathscr{R}$ (with constant $M \geqslant 1$) if

$$(Nf)(x) \leqslant Mu + N[f^+\mathbf{1}_{\{(Nf)\geqslant u\}}](x)$$

holds for every $x \in X$, $f \in \mathscr{R}$ and $u \geqslant 0$.

The main result in [5] which we want to generalize is stated for convex and concave functions, but the definition given there of these classes of functions is not the standard one. In order to understand the frame, we will restate the definitions from [5] and call these functions RS-convex and RS-concave.

DEFINITION 2. ([5]) Function $\Phi : [0,+\infty) \to [0,+\infty)$ is an *RS-convex function* if there exists a (positive) Borel σ-finite measure η on $[0,+\infty)$ such that

$$\Phi(\tau) = \int_0^\tau \varphi(t)dt, \qquad \text{for every } \tau \in [0,+\infty),$$

where

$$\varphi(t) = \eta([0,t]), \qquad \text{for every } t \in [0,+\infty).$$

DEFINITION 3. Function $\Phi : [0,+\infty) \to [0,+\infty)$ is an *RS-concave function* if there exists a (positive) Borel measure η on $[0,+\infty)$ such that

$$\eta([a,b]) \text{ is finite}, \qquad \text{for every } 0 < a \leqslant b < +\infty$$

and

$$\Phi(\tau) = \int_0^\tau \varphi(t)dt, \qquad \text{for every } \tau \in [0,+\infty),$$

where φ satisfies the following properties:

$$0 \leqslant \varphi(t) < +\infty, \qquad \text{for every } t > 0, \qquad \lim_{t\to 0+} t\varphi(t) = 0,$$

and

$$\varphi(t) - \varphi(s) = -\eta((s,t]), \qquad \text{for every } 0 < s < t < +\infty.$$

Therefore, RS-convex function is convex in the usual sense and has additional properties - it is increasing and satisfies $\Phi(0) = 0$. Similarly, RS-concave function is concave in the usual sense, increasing and satisfies $\Phi(0) = 0$. Prototypes for both of the classes and main examples used in applications are the power functions $\Phi(t) = t^p$, $0 < p < \infty$.

THEOREM 1. (The potential inequality for convex functions, [5]) *Let Φ be an RS-convex function. Let $N(x,dy)$ be a kernel on X which satisfies the strong maximum principle on $\mathscr{R} \subset \mathscr{POT}(N)$, with constant M. Then, for every $f \in \mathscr{R}$,*

$$\Phi\left[\frac{s}{M}\right] \leqslant \frac{1}{M} N\left[f^+ \varphi(s)\right],$$

where $s = (Nf)^+$.

COROLLARY 2. *Let Φ and $N(x,dy)$ be as in Theorem 1. Then for each σ-finite measure μ on $(X, \mathscr{B}(X))$, and for every $f \in \mathscr{R}$,*

$$\int_X \Phi\left(\frac{s}{M}\right) d\mu \leqslant \frac{1}{M} \int_X f^+ \varphi(s) d(\hat{N}\mu).$$

where $s = (Nf)^+$. In particular, if $\Phi(\tau) = \tau^p$, $p \geqslant 1$, then

$$\int_X s^p d\mu \leqslant pM^{p-1} \int_X f^+ s^{p-1} d(\hat{N}\mu).$$

THEOREM 3. (The potential inequality for concave functions, [5]) *Let Φ be an RS-concave function. Let $N(x,dy)$ be a kernel on X which satisfies the maximum principle on the set of nonnegative functions (i. e. $\mathscr{R} = X^+$), with constant M. Then, for every nonnegative $f : X \to [0, +\infty)$,*

$$\Phi\left[\frac{s}{M}\right] \geqslant \frac{1}{M} N\left[f \varphi(s)\right],$$

where $s = (Nf)^+ = Nf$.

COROLLARY 4. *Let Φ and $N(x,dy)$ be as in Theorem 3. Then for each σ-finite measure μ on $(X, \mathscr{B}(X))$, and for every nonnegative $f : X \to [0, +\infty)$,*

$$\int_X \Phi\left(\frac{s}{M}\right) d\mu \geqslant \frac{1}{M} \int_X f \varphi(s) d(\hat{N}\mu).$$

where $s = (Nf)^+$. In particular, if $\Phi(\tau) = \tau^p$, $0 < p < 1$, then

$$\int_X s^p d\mu \geqslant pM^{p-1} \int_X f s^{p-1} d(\hat{N}\mu).$$

2. Potential inequality revisited

Our goal is to generalize the results given above to the class of convex and concave functions defined in the standard way (see [3], Definition 1.1), without any additional restrictions.

Let Φ be a convex or a concave function and φ its right-continuous derivative, i. e.

$$\varphi(u) = \lim_{\tau \searrow u} \frac{\Phi(\tau) - \Phi(u)}{\tau - u}.$$

The positive measure generated by φ is denoted by $d\varphi(u)$. Of course, if Φ is differentiable function, then φ is its derivative.

Integration by parts gives

$$\begin{aligned}\Phi(\tau) - \Phi(z) &= \int_z^\tau \varphi(u)du \\ &= \tau\varphi(\tau) - z\varphi(z) - \int_z^\tau u d\varphi(u) \\ &= \int_z^\tau (\tau - u)d\varphi(u) + \varphi(z)(\tau - z). \end{aligned} \tag{1}$$

THEOREM 5. (The potential inequality for convex functions) *Let $\Phi : (0,+\infty) \to \mathbb{R}$ be a convex function and $N(x,dy)$ a positive kernel on X which satisfies the strong maximum principle on $\mathscr{R}$ with constant M. Let $f \in \mathscr{R}$, $x \in X$ and $z > 0$ be such that $z \leqslant (Nf)(x)/M$ and denote by B_z the set*

$$B_z = \big\{y \in X : (Nf)(y) \geqslant z\big\}.$$

Then

$$\begin{aligned}\Phi\Big(\frac{1}{M}(Nf)(x)\Big) - \Phi(z) \leqslant \frac{1}{M}N[f^+\varphi(Nf)\mathbf{1}_{B_z}](x) \\ +\frac{1}{M}\varphi(z)N[f - f^+\mathbf{1}_{B_z}](x) - z\varphi(z).\end{aligned}$$

Proof. Let $\tau(x) = \frac{1}{M}(Nf)(x)$. Using (1) and the strong maximum principle, since $d\varphi(u)$ is a positive measure, we get

$$\begin{aligned}\Phi(\tau(x)) - \Phi(z) &= \int_z^{\tau(x)} (\tau(x) - u)d\varphi(u) + \varphi(z)(\tau(x) - z) \\ &\leqslant \int_z^{\tau(x)} \frac{1}{M}N[f^+\mathbf{1}_{\{Nf\geqslant u\}}]d\varphi(u) + \varphi(z)(\tau(x) - z)\end{aligned}$$

Applying Fubini and the fact that $f^+\mathbf{1}_{\{Nf\geqslant u\}}$ is a nonnegative function, we further get

$$\begin{aligned}&\int_z^{\tau(x)} N[f^+\mathbf{1}_{\{Nf\geqslant u\}}]d\varphi(u) \\ &\qquad = \int_z^{\tau(x)} \int_X f^+(y)\mathbf{1}_{\{(Nf)\geqslant u\}}(y)N(x,dy)d\varphi(u)\end{aligned}$$

$$
\begin{aligned}
&= \int_X N(x,dy) f^+(y) \int_z^{\tau(x)} \mathbf{1}_{\{(Nf)\geqslant u\}}(y)\, d\varphi(u) \\
&\leqslant \int_X N(x,dy) f^+(y) \int_z^{+\infty} \mathbf{1}_{\{(Nf)\geqslant u\}}(y)\, d\varphi(u) \\
&= \int_X f^+(y)[\varphi((Nf)(y)) - \varphi(z)]\mathbf{1}_{B_z}(y) N(x,dy) \\
&= N[f^+\varphi(Nf)\mathbf{1}_{B_z}](x) - \varphi(z) N[f^+\mathbf{1}_{B_z}](x).
\end{aligned}
$$

Finally, the two inequalities above, together with linearity of the potential, give the claim of the theorem. □

Let us further denote the set

$$
B = \bigcup_{z>0} B_z = \big\{x \in X : (Nf)(x) > 0\big\}.
$$

By integrating the potential inequality with respect to the variable x we can get the following, integral version of the potential inequality

COROLLARY 6. *Let the assumptions of Theorem 5 hold for a function $z : B \to (0,+\infty)$, i. e. $z(x) \leqslant (Nf)(x)/M$ for $x \in B$. Then, for $C \subset B$, $C \in \mathscr{B}(X)$, and a finite measure μ on $(X,\mathscr{B}(X))$, the following inequality holds*

$$
\begin{aligned}
&\int_C \Big(\Phi\big(\frac{1}{M}(Nf)(x)\big) - \Phi(z(x))\Big)\mu(dx) \\
&\qquad \leqslant \frac{1}{M}\int_C \int_{B_{z(x)}} f^+(y)\varphi((Nf)(y)) N(x,dy)\mu(dx) \\
&\qquad\quad - \frac{1}{M}\int_C \varphi(z(x)) N[f^+\mathbf{1}_{B_{z(x)}} - f](x)\mu(dx) - \int_C z(x)\varphi(z(x))\mu(dx).
\end{aligned}
$$

In particular, for $C = B_z$ and $z(x) \equiv z$, we get

$$
\begin{aligned}
&\int_{B_z} \Phi\big(\frac{1}{M}(Nf)(x)\big)\mu(dx) - \Phi(z)\mu(B_z) \\
&\qquad \leqslant \frac{1}{M}\int_{B_z} f^+(x)\varphi((Nf)(x))(\hat{N}_{B_z}\mu)(dx) \\
&\qquad\quad + \frac{1}{M}\varphi(z)\int_{B_z} N[f - f^+\mathbf{1}_{B_z}](x)\mu(dx) - z\varphi(z)\mu(B_z).
\end{aligned}
$$

Proof. Integrating the potential inequality with respect to the measure μ we get

$$
\begin{aligned}
\int_C \Big(\Phi\big(\frac{1}{M}(Nf)(x)\big) - \Phi(z(x))\Big)\mu(dx) &\leqslant \frac{1}{M}\int_C N[f^+\varphi(Nf)\mathbf{1}_{B_{z(x)}}]\mu(dx) \\
&+ \frac{1}{M}\int_C \varphi(z(x)) N[f - f^+\mathbf{1}_{B_{z(x)}}](x)\mu(dx) - \int_C z(x)\varphi(z(x))\mu(dx),
\end{aligned}
$$

which is the first inequality.

The second inequality follows by taking $C = B_z$ and $z(x) \equiv z$ and by applying Fubini's theorem on the first integral of the right-hand side. $\square$

REMARK 7. The first inequality in Corollary 6 is valid for a σ-finite measure μ as long as the integrals on the right-hand side are finite.

Let us denote by Φ_p the following class of functions

$$\Phi_p(\tau) = \begin{cases} \frac{\tau^p}{p(p-1)}, & p \neq 0,1, \\ -\log\tau, & p = 0, \\ \tau\log\tau, & p = 1. \end{cases} \tag{2}$$

COROLLARY 8. *Under the assumptions of Corollary 6, for $p \in \mathbb{R}\backslash\{0,1\}$ the following inequality holds*

$$\frac{1}{p(p-1)}\int_{B_z}(Nf)^p(x)\mu(dx) \leqslant \frac{M^{p-1}}{(p-1)}\int_{B_z} f^+(x)(Nf)^{p-1}(x)(\hat{N}_{B_z}\mu)(dx)$$
$$+\frac{(zM)^{p-1}}{(p-1)}\int_{B_z} N[f - f^+\mathbf{1}_{B_z}](x)\mu(dx) - \frac{(zM)^p\mu(B_z)}{p}.$$

Furthermore, for $q = p/(p-1)$ the following inequality holds

$$\frac{1}{p(p-1)}\int_{B_z}(Nf)^p d\mu \leqslant \frac{M^{p-1}}{(p-1)}\left[\int_{B_z}(f^+)^p d(\hat{N}_{B_z}\mu)\right]^{\frac{1}{p}}\left[\int_{B_z}(Nf)^p d(\hat{N}_{B_z}\mu)\right]^{\frac{1}{q}}$$
$$+\frac{(zM)^{p-1}}{(p-1)}\int_{B_z} N[f - f^+\mathbf{1}_{B_z}](x)\mu(dx) - \frac{(zM)^p\mu(B_z)}{p}.$$

Proof. Applying the second inequality from Corollary 6 for convex functions Φ_p, $p \in \mathbb{R}\backslash\{0,1\}$, and rearranging we get the first inequality. The second inequality follows from the first by applying Hölder's inequality on the first integral of the right-hand side. $\square$

COROLLARY 9. *Under the assumptions of Theorem 5, if*

(i) *f is nonnegative and $\varphi(z) \leqslant 0$*

or

(ii) *$B_z = X$ and $\varphi(z) \geqslant 0$,*

then for every $x \in B_z$ the following inequality holds

$$\Phi\left(\frac{1}{M}(Nf)(x)\right) - \Phi(z) \leqslant \frac{1}{M}N[f^+\varphi(Nf)\mathbf{1}_{B_z}](x) - z\varphi(z). \tag{3}$$

Furthermore, for a finite measure μ *on* $(X, \mathscr{B}(X))$*, the following inequality holds*

$$\int_{B_z} \Phi(\frac{1}{M}(Nf)(x))\mu(dx) - \Phi(z)\mu(B_z) \leqslant \frac{1}{M}\int_{B_z} f^+(x)\varphi((Nf)(x))(\hat{N}_{B_z}\mu)(dx) - z\varphi(z)\mu(B_z). \tag{4}$$

Proof. Under the assumptions of part (i), for nonnegative f we have

$$\varphi(z)N[f - f^+\mathbf{1}_{B_z}] = \varphi(z)N[f\mathbf{1}_{B_z^c}] \leqslant 0.$$

Under the assumptions of part (ii) we have

$$\varphi(z)N[f - f^+\mathbf{1}_{B_z}] = -\varphi(z)N[f^-\mathbf{1}_{B_z^c}] \leqslant 0.$$

Hence, in either case inequality (3) follows by potential inequality of Theorem 5.

Inequality (4) follows by integrating inequality (3) with respect to the measure μ over the set B_z and applying Fubini's theorem on the right hand side integral. □

COROLLARY 10. *Under the assumptions of Corollary 9* (i)*, for* $p < 1$, $p \neq 0$*, the following inequality holds*

$$\frac{1}{p(p-1)}\int_{B_z}(Nf)^p(x)\mu(dx) \leqslant \frac{M^{p-1}}{(p-1)}\int_{B_z} f(x)(Nf)^{p-1}(x)(\hat{N}_{B_z}\mu)(dx) - \frac{(zM)^p\mu(B_z)}{p}. \tag{5}$$

Furthermore, for $q = p/(p-1)$ *the following inequality holds*

$$\frac{1}{p(p-1)}\int_{B_z}(Nf)^p(x)\mu(dx) \leqslant \frac{M^{p-1}}{(p-1)}\left[\int_{B_z} f^p d(\hat{N}_{B_z}\mu)\right]^{\frac{1}{p}}\left[\int_{B_z}(Nf)^p d(\hat{N}_{B_z}\mu)\right]^{\frac{1}{q}} - \frac{(zM)^p\mu(B_z)}{p}. \tag{6}$$

Under the assumptions of Corollary 9 (ii)*, the above inequalities are valid, with* f *replaced by* f^+*, for* $p > 1$.

Proof. Applying Corollary 9 for convex functions Φ_p, $p \in \mathbb{R}\backslash\{0,1\}$, and rearranging we get the first inequality. The second inequality follows from the first by applying Hölder's inequality on the right-hand side integral. □

3. Limiting cases of potential inequality

In this section we will give various forms of potential inequality based on the limiting behaviour of φ at zero. If Theorem 5 or Corollary 9 (ii) hold for $z > 0$, then they hold for every z', $0 < z' \leqslant z$. Letting $z' \to 0$ we can get further inequalities.

In the following corollaries we will assume that either φ is nonnegative, or that for every $x \in B$ there exists a function $g_x \in L^1(N(x,\cdot))$ such that $|f^+\varphi(Nf)| \leqslant g_x$. In either case, by the monotone convergence theorem in the former and by the dominated convergence theorem in the latter, we have

$$\lim_{z \searrow 0} N[f^+\varphi(Nf)\mathbf{1}_{B_z}] = N[f^+\varphi(Nf)\mathbf{1}_B]$$

since $f^+\varphi(Nf)\mathbf{1}_{B_z} \to f^+\varphi(Nf)\mathbf{1}_B$ pointwise, when $z \to 0$.

THEOREM 11. *Under the assumptions of Theorem 5, if $\varphi(0+)$ is finite, then for every $x \in B$ we have*

$$\Phi(\frac{1}{M}(Nf)(x)) - \Phi(0+) \leqslant \frac{1}{M}N[f^+\varphi(Nf)\mathbf{1}_B](x) + \frac{1}{M}\varphi(0+)N[f - f^+\mathbf{1}_B](x).$$

Furthermore, if μ is a finite measure on $(X, \mathscr{B}(X))$, then the following inequality holds

$$\int_B \Phi(\frac{1}{M}(Nf)(x))\mu(dx) - \Phi(0+)\mu(B) \leqslant \frac{1}{M}\int_B f^+(x)\varphi((Nf)(x))(\hat{N}_B\mu)(dx) + \frac{1}{M}\varphi(0+)\int_B N[f - f^+\mathbf{1}_B](x)\mu(dx).$$

Proof. Since $\varphi(0+)$ is finite and $\lim_{z'\to 0} z'\varphi(z') = 0$, the first inequality follows from Theorem 5.

The second inequality follows by integrating the first with respect to the measure μ over the set B and applying Fubini's theorem on the first integral of the righ-hand side. □

COROLLARY 12. *Let the assumptions of Theorem 5 hold and let $\varphi(0+)$ be finite. Then*

(i) *for every $x \in B$*

$$\Phi(\frac{1}{M}(Nf)(x)) - \Phi(0+) \leqslant \frac{1}{M}N[f^+\varphi((Nf)^+)](x) - \frac{1}{M}\varphi(0+)(Nf^-)(x).$$

(*ii*) *if* f *is nonnegative, then for every* $x \in X$ *we have*

$$\Phi(\frac{1}{M}(Nf)(x)) - \Phi(0+) \leqslant \frac{1}{M}N[f\varphi(Nf)](x).$$

(*iii*) *if* f *is nonnegative,* $\Phi(0+) = 0$ *and* μ *is a* σ*-finite measure on* $(X, \mathscr{B}(X))$*, the following inequality holds*

$$\int_X \Phi(\frac{1}{M}(Nf)(x))\mu(dx) \leqslant \frac{1}{M}\int_X f(x)\varphi((Nf)(x))(\hat{N}\mu)(dx).$$

Proof. (i) For $x \in B$, the last term on the right hand side of the potential inequality from Theorem 5 disappears because $\lim_{z\to 0} z\varphi(z) = 0$. Furthermore, since $\varphi(0+)$ is finite and $B^c = \{(Nf)^+ = 0\}$, we have

$$N[f^+\varphi((Nf)^+)\mathbf{1}_{B^c}] = \varphi(0+)N[f^+\mathbf{1}_{B^c}].$$

Finally, adding and subtracting the term $\frac{1}{M}\varphi(0+)N[f^+\mathbf{1}_{B^c}](x)$ on the right hand side of the potential inequality and rearranging, we get the inequality in part (i).

(ii) For $x \in B$ the inequality follows from part (i). On the other hand, since f is nonnegative, $(Nf)(x) = 0$ iff $f \equiv 0$ $N(x,dy)$-a.e. Therefore, for $x \in X \backslash B$, we also have $N[f\varphi(Nf)](x) = 0$, i. e. the inequality holds trivially, with zero on both sides.

(iii) The inequality follows by integrating the inequality from part (ii) with respect to the measure μ and applying Fubini's theorem on the integral from the right hand side. □

COROLLARY 13. *Under the assumptions of Theorem 11, for* $p > 1$ *the following inequality holds*

$$\int_B (Nf)^p(x)\mu(dx) \leqslant pM^{p-1}\int_B f^+(x)(Nf)^{p-1}(x)(\hat{N}_B\mu)(dx).$$

Furthermore, for $q = p/(p-1)$ *the following inequality holds*

$$\int_B (Nf)^p(x)\mu(dx) \leqslant pM^{p-1}\left[\int_B (f^+)^p d(\hat{N}_B\mu)\right]^{\frac{1}{p}}\left[\int_B (Nf)^p d(\hat{N}_B\mu)\right]^{\frac{1}{q}}.$$

Proof. The first inequality holds since convex functions Φ_p, $p > 1$, satisfy the assumptions of Theorem 11 with $\Phi_p(0+) = \varphi_p(0+) = 0$. The second inequality follows from the first by applying Hölder's inequality on the right-hand side integral. □

REMARK 14. By Corollary 12, we see that Corollary 13 remains true if we replace B with X.

THEOREM 15. *Under the assumptions of Theorem 5, if f is nonnegative and $\lim_{z\to 0} z\varphi(z) = 0$, then*

$$\Phi(\frac{1}{M}(Nf)(x)) - \Phi(0+) \leqslant \frac{1}{M} N[f\varphi(Nf)\mathbf{1}_B](x).$$

Furthermore, for a finite measure μ on $(X, \mathscr{B}(X))$, the following inequality holds

$$\int_B \Phi(\frac{1}{M}(Nf)(x))\mu(dx) - \Phi(0+)\mu(B) \leqslant \frac{1}{M}\int_B f(x)\varphi((Nf)(x))(\hat{N}_B\mu)(dx).$$

Proof. By the maximum principle

$$N[f\mathbf{1}_{\{Nf<z\}}] + N[f\mathbf{1}_{\{Nf\geqslant z\}}] = N(f) \leqslant Mz + N[f\mathbf{1}_{\{Nf\geqslant z\}}],$$

i. e.

$$\frac{1}{M} N[f1_{B_z^c}] \leqslant z.$$

Therefore,

$$\begin{aligned}\left|\lim_{z\to 0}\frac{1}{M}\varphi(z)N[f - f\mathbf{1}_{B_z}]\right| &= \left|\lim_{z\to 0}\frac{1}{M}\varphi(z)N[f\mathbf{1}_{B_z^c}]\right| \\ &\leqslant \lim_{z\to 0}|z\varphi(z)| = 0.\end{aligned}$$

Hence, the first inequality follows from Theorem 5. The second inequality follows by integrating the first with respect to the measure μ over the set B and applying Fubini's theorem on the right hand side integral. □

COROLLARY 16. *Under the assumptions of Theorem 15, for $p > 0$, $p \neq 1$, the following inequality holds*

$$\frac{1}{p(p-1)}\int_B (Nf)^p(x)\mu(dx) \leqslant \frac{M^{p-1}}{(p-1)}\int_B f(x)(Nf)^{p-1}(x)(\hat{N}_B\mu)(dx).$$

Furthermore, for $q = p/(p-1)$ the following inequality holds

$$\frac{1}{p(p-1)}\int_B (Nf)^p(x)\mu(dx) \leqslant \frac{M^{p-1}}{(p-1)}\left[\int_B f^p d(\hat{N}_B\mu)\right]^{\frac{1}{p}}\left[\int_B (Nf)^p d(\hat{N}_B\mu)\right]^{\frac{1}{q}} \quad (7)$$

Proof. The first inequality holds since convex functions Φ_p, $p > 0$, satisfy the assumptions of Theorem 15 with $\Phi_p(0+) = 0$. The second inequality follows from the first by applying Hölder's inequality on the right-hand side integral. □

Notice that p and $q = p/(p-1)$ satisfy

$$\begin{array}{rcl} p > 1 & \Longleftrightarrow & q > 1 \\ 0 < p < 1 & \Longleftrightarrow & q < 0 \\ p < 0 & \Longleftrightarrow & 0 < q < 1 \end{array} \quad (8)$$

When one or both of the measures μ and $\hat{N}_C\mu$ is bounded by the other up to a multiplicative constant, then we can state further inequalities. Let K_1 and K_2 be positive constants, if they exist, such that

$$K_1\mu \leqslant \hat{N}_C\mu \tag{9}$$

and

$$\hat{N}_C\mu \leqslant K_2\mu. \tag{10}$$

COROLLARY 17. *Let the assumptions of Corollary 8 hold. If N and μ satisfy (10) with $C = B_z$, then for $p > 1$*

$$\int_{B_z} (Nf)^p d\mu \leqslant pK_2 M^{p-1} \left[\int_{B_z} (f^+)^p d\mu\right]^{\frac{1}{p}} \left[\int_{B_z} (Nf)^p d\mu\right]^{\frac{1}{q}} + p(zM)^{p-1} \int_{B_z} N[f - f^+ \mathbf{1}_{B_z}](x)\mu(dx) - (p-1)(zM)^p \mu(B_z).$$

When N and μ satisfy both (9) and (10) with $C = B_z$, then for $0 < p < 1$

$$\int_{B_z} (Nf)^p d\mu \geqslant pK_1^{1/p} K_2^{1/q} M^{p-1} \left[\int_{B_z} (f^+)^p d\mu\right]^{\frac{1}{p}} \left[\int_{B_z} (Nf)^p d\mu\right]^{\frac{1}{q}} + p(zM)^{p-1} \int_{B_z} N[f - f^+ \mathbf{1}_{B_z}](x)\mu(dx) - (p-1)(zM)^p \mu(B_z),$$

while for $p < 0$

$$\int_{B_z} (Nf)^p d\mu \leqslant pK_1^{1/q} K_2^{1/p} M^{p-1} \left[\int_{B_z} (f^+)^p d\mu\right]^{\frac{1}{p}} \left[\int_{B_z} (Nf)^p d\mu\right]^{\frac{1}{q}} + p(zM)^{p-1} \int_{B_z} N[f - f^+ \mathbf{1}_{B_z}](x)\mu(dx) - (p-1)(zM)^p \mu(B_z).$$

Proof. The inequalities follow directly from (9), (10) and Corollary 8, taking into account properties (8). □

COROLLARY 18. *Let the assumptions of Corollary 13 hold. If N and μ satisfy (10) with $C = B$, then for $p > 1$*

$$\left[\int_B (Nf)^p d\mu\right]^{\frac{1}{p}} \leqslant pK_2 M^{p-1} \left[\int_B (f^+)^p d\mu\right]^{\frac{1}{p}}.$$

Proof. The inequality follows directly from (10) and Corollary 13, taking into account properties (8). □

COROLLARY 19. *Let the assumptions of Corollary 16 hold. If N and μ satisfy (10) with $C = B$, then for $p > 1$*

$$\left[\int_B (Nf)^p d\mu\right]^{\frac{1}{p}} \leqslant pK_2 M^{p-1} \left[\int_B f^p d\mu\right]^{\frac{1}{p}}$$

If N and μ satisfy (9) and (10) with $C = B$, then for $0 < p < 1$

$$\left[\int_B (Nf)^p d\mu\right]^{\frac{1}{p}} \geqslant pK_1^{1/p} K_2^{1/q} M^{p-1} \left[\int_B f^p d\mu\right]^{\frac{1}{p}}$$

Proof. The inequalities follow directly from (9), (10) and Corollary 16, taking into account properties (8). □

COROLLARY 20. *Let the assumptions of Corollary 16 hold. If N and μ satisfy (10) with $C = B$, then for $p > 0$, $p \neq 1$,*

$$\frac{1}{p(p-1)} \left[\int_B (Nf)^p d\mu\right]^{\frac{1}{p}} \leqslant \frac{K_2^{1/q} M^{p-1}}{p-1} \left[\int_B f^p d(\hat{N}_B \mu)\right]^{\frac{1}{p}}$$

Proof. The inequality follows directly from (10) and Corollary 16, taking into account properties (8). □

REMARK 21. *Concave case.* When Φ is concave, $d\varphi(u)$ is a negative measure and the inequalities in Theorem 5, Corollary 6, Theorem 11 and Theorem 15 are reversed. The reversed inequality in Corollary 9 (i) holds if $\varphi(z) \geqslant 0$ and in Corollary 9(ii) if $\varphi(z) \leqslant 0$.

4. Exponential Convexity

In this section, the well-known results on exponential convexity will be applied to functionals derived from the potential inequality. Let us recall briefly definition and main properties of exponential convexity.

DEFINITION 4. A function $\psi : I \to \mathbb{R}$ is exponentially convex on an interval I if it is continuous and

$$\sum_{i,j=1}^{n} \xi_i \xi_j \psi(x_i + x_j) \geqslant 0$$

for all $n \in \mathbb{N}$, all choices $\xi_i \in \mathbb{R}$ and $x_i + x_j \in I$, $1 \leqslant i, j \leqslant n$.

From the definition, one can easily see that an exponentially convex function ψ is nonnegative. Moreover, if there exists $x \in I$ such that $\psi(x) = 0$, then ψ is identically equal to zero.

PROPOSITION 22. *For a $\psi : I \to \mathbb{R}$, the following statements are equivalent*

(i) ψ *is exponentially convex,*

(ii) ψ *is continuous and*

$$\sum_{i,j=1}^{n} \xi_i \xi_j \psi\Big(\frac{x_i + x_j}{2}\Big) \geqslant 0$$

for all $n \in \mathbb{N}$, *all choices* $\xi_i \in \mathbb{R}$ *and* $x_i \in I$, $1 \leqslant i \leqslant n$.

COROLLARY 23. *If* $\psi : I \to \mathbb{R}$ *is exponentially convex, then*

(i) *the matrix* $[\psi(\frac{x_i+x_j}{2})]_{i,j=1}^{n}$ *is positive semidefinite, so*

$$\det\Big[\psi\Big(\frac{x_i + x_j}{2}\Big)\Big]_{i,j=1}^{n} \geqslant 0$$

for all $n \in \mathbb{N}$ *and* $x_i \in I$, $1 \leqslant i \leqslant n$.

(ii) ψ *is a log-convex function, i. e.*

$$\psi(\lambda x + (1-\lambda)y) \leqslant \psi^{\lambda}(x)\psi^{1-\lambda}(y), \quad \textit{for all } x, y \in I, \lambda \in [0,1].$$

COROLLARY 24. *Function* ψ *is log-convex on an interval* I *if and only if for all* $a, b, c \in I$, $a < b < c$, *the following inequality holds*

$$\big[\psi(b)\big]^{c-a} \leqslant \big[\psi(a)\big]^{c-b}\big[\psi(c)\big]^{b-a}.$$

COROLLARY 25. *If* ψ *is a positive log-convex function on an interval* I, *and* $p, q, r, s \in I$ *are such that* $p \leqslant r$, $q \leqslant s$, $p \neq q$ *and* $r \neq s$, *then*

$$\left(\frac{\psi(p)}{\psi(q)}\right)^{\frac{1}{p-q}} \leqslant \left(\frac{\psi(r)}{\psi(s)}\right)^{\frac{1}{r-s}}.$$

Let us define linear functionals $A_1 = A_{1;f,N,z,x}$ and $A_2 = A_{2;f,N,z,\mu}$ with

$$\begin{aligned} A_1(\Phi) &= \frac{1}{M} N[f^+ \varphi(Nf)\mathbf{1}_{B_z}](x) + \frac{1}{M}\varphi(z) N[f - f^+\mathbf{1}_{B_z}](x) \\ &\quad - \Phi\big(\frac{1}{M}(Nf)(x)\big) + \Phi(z) - z\varphi(z) \\ A_2(\Phi) &= \frac{1}{M}\int_{B_z} f^+(x)\varphi((Nf)(x))(\hat{N}_{B_z}\mu)(dx) \\ &\quad + \frac{1}{M}\varphi(z)\int_{B_z} N[f - f^+\mathbf{1}_{B_z}](x)\mu(dx) \\ &\quad - \int_{B_z}\Phi\big(\frac{1}{M}(Nf)(x)\big)\mu(dx) + \Phi(z)\mu(B_z) - z\varphi(z)\mu(B_z). \end{aligned}$$

Linear functionals A_k, $k = 1,2$, depend on the choices of function f, kernel N and points x and z, but if these choices are clear from context, we will omit them from the notation.

Similarly, we define linear functionals $A_k = A_{k;f,N,x,\mu}$, $k = 3,4,5,6$, with

$$\begin{aligned}
A_3(\Phi) &= \frac{1}{M}N[f^+\varphi(Nf)\mathbf{1}_B](x) + \frac{1}{M}\varphi(0+)N[f - f^+\mathbf{1}_B](x) \\
&\quad -\Phi(\frac{1}{M}(Nf)(x)) + \Phi(0+),
\end{aligned}$$

$$\begin{aligned}
A_4(\Phi) &= \frac{1}{M}\int_B f^+(x)\varphi((Nf)(x))(\hat{N}_B\mu)(dx) \\
&\quad +\frac{1}{M}\varphi(0+)\int_B N[f - f^+\mathbf{1}_B](x)\mu(dx) \\
&\quad -\int_B \Phi(\frac{1}{M}(Nf)(x))\mu(dx) + \Phi(0+)\mu(B),
\end{aligned}$$

$$A_5(\Phi) = \frac{1}{M}N[f\varphi(Nf)\mathbf{1}_B](x) - \Phi(\frac{1}{M}(Nf)(x)) + \Phi(0+),$$

$$\begin{aligned}
A_6(\Phi) &= \frac{1}{M}\int_B f(x)\varphi((Nf)(x))(\hat{N}_B\mu)(dx) \\
&\quad -\int_B \Phi(\frac{1}{M}(Nf)(x))\mu(dx) + \Phi(0+)\mu(B),
\end{aligned}$$

We also define functions $\psi_k : I_k \to \mathbb{R}_+$ by

$$\psi_k(p) = A_k(\Phi_p) \tag{11}$$

with $I_1 = I_2 = \mathbb{R}$, $I_3 = I_4 = (1,+\infty)$ and $I_5 = I_6 = (0,\infty)$. By Corollaries 8, 13 and 16, functions ψ_k, $k = 1,\dots,6$, are, indeed, well-defined and nonnegative. It is straightforward to check that all of the functions ψ_k are continuous.

LEMMA 26. *For each $k \in \{1,2,\dots,6\}$, the function ψ_k is exponentially convex.*

Proof. Let $n \in \mathbb{N}$, $\xi_i \in \mathbb{R}$ and $p_i \in I_k$, $1 \leqslant i \leqslant n$, be arbitrary. Define the function Φ by

$$\Phi(\tau) = \sum_{i,j=1}^{n} \xi_i\xi_j\Phi_{\frac{p_i+p_j}{2}}(\tau).$$

Since

$$\Phi''(\tau) = \sum_{i,j=1}^{n} \xi_i\xi_j\tau^{\frac{p_i+p_j}{2}-2} = \Big(\sum_{i=1}^{n}\xi_i\tau^{\frac{p_i}{2}-1}\Big)^2 \geqslant 0,$$

the function Φ is convex.

Furthermore, if $k=3$ or 4, we have

$$\varphi(0+) = \Big| \sum_{i,j=1}^{n} \xi_i \xi_j \varphi_{\frac{p_i+p_j}{2}}(0+) \Big| < +\infty,$$

so Φ satisfies the assumptions of Corollary 13. Similarly, if $k=5$ or 6, the function Φ satisfies the assumptions of Corollary 16 since

$$\lim_{z\to 0} z\varphi(z) = \lim_{z\to 0} \sum_{i,j=1}^{n} z\varphi_{\frac{p_i+p_j}{2}}(z) = 0.$$

Hence, by Corollaries 8, 13 and 16 and by continuity of ψ_k, we have, for each k,

$$0 \leqslant A_k(\Phi) = \sum_{i,j=1}^{n} \xi_i \xi_j A_k\Big(\Phi_{\frac{p_i+p_j}{2}}\Big) = \sum_{i,j=1}^{n} \xi_i \xi_j \psi_k\Big(\frac{p_i+p_j}{2}\Big),$$

i. e., ψ_k is exponentially convex. □

COROLLARY 27. *For ψ_k, $k=1,\dots,6$, defined by (11) the following statements hold*

(*i*) *For all $n\in\mathbb{N}$ and $p_i \in I_k$, $1\leqslant i\leqslant n$ the matrix $[\psi_k(\frac{p_i+p_j}{2})]_{i,j=1}^n$ is positive semidefinite, so*

$$\det\Big[\psi_k\Big(\frac{p_i+p_j}{2}\Big)\Big]_{i,j=1}^{n} \geqslant 0.$$

(*ii*) *For $p,s,t\in I_k$ we have*

$$\psi_k(p) \geqslant \big[\psi_k(s)\big]^{\frac{t-p}{t-s}} \big[\psi_k(t)\big]^{\frac{p-s}{t-s}} \qquad \text{if } p<s<t \quad \text{or} \quad s<t<p$$

$$\psi_k(p) \leqslant \big[\psi_k(s)\big]^{\frac{t-p}{t-s}} \big[\psi_k(t)\big]^{\frac{p-s}{t-s}} \qquad \text{if } s<p<t$$

Proof. Since the functions ψ_k are exponentially convex by Lemma 26, the inequalities in (i) follow from Corollary 23(i), while inequalities in (ii) follow from Corollary 24. □

Notice that the first set of inequalities in Corollary 27(ii) are refinements of the first inequalities in Corollaries 8, 13 and 16. Indeed, the latter inequalities, in the notation introduced in this section, are

$$0 \leqslant \psi_k(p), \qquad k=2,4,6, p\in I_k\backslash\{0,1\},$$

while the right-hand sides of inequalities in Corollary 27(ii) are nonnegative. The same results from Corollary 27 can be used to refine the second inequalities in Corollaries 8, 13 and 16 as well.

COROLLARY 28.

(i) *Let the assumptions of Corollary 8 hold. For $p,s,t \in \mathbb{R}$, $p<s<t$ or $s<t<p$, $p \neq 0,1$ and $q = p/(p-1)$ we have*

$$\left[\psi_2(s)\right]^{\frac{t-p}{t-s}}\left[\psi_2(t)\right]^{\frac{p-s}{t-s}} \leqslant \frac{1}{(p-1)M}\left[\int_{B_z}(f^+)^p d(\hat{N}_{B_z}\mu)\right]^{\frac{1}{p}}\left[\int_{B_z}(Nf)^p d(\hat{N}_{B_z}\mu)\right]^{\frac{1}{q}} + \frac{z^{p-1}}{(p-1)M}\int_{B_z} N[f-f^+\mathbf{1}_{B_z}](x)d\mu - \frac{z^p\mu(B_z)}{p} - \frac{1}{p(p-1)M^p}\int_{B_z}(Nf)^p d\mu.$$

(ii) *Let the assumptions of Corollary 13 hold. For $p,s,t \in (1,+\infty)$, $p<s<t$ or $s<t<p$, and $q = p/(p-1)$ we have*

$$\left[\psi_4(s)\right]^{\frac{t-p}{t-s}}\left[\psi_4(t)\right]^{\frac{p-s}{t-s}} \leqslant \frac{1}{(p-1)M}\left[\int_B (f^+)^p d(\hat{N}_B\mu)\right]^{\frac{1}{p}}\left[\int_B (Nf)^p d(\hat{N}_B\mu)\right]^{\frac{1}{q}} - \frac{1}{p(p-1)M^p}\int_B (Nf)^p(x)\mu(dx).$$

(iii) *Let the assumptions of Corollary 16 hold. For $p,s,t \in (0,+\infty)$, $p<s<t$ or $s<t<p$, $p \neq 1$ and $q = p/(p-1)$ we have*

$$\left[\psi_6(s)\right]^{\frac{t-p}{t-s}}\left[\psi_6(t)\right]^{\frac{p-s}{t-s}} \leqslant \frac{1}{(p-1)M}\left[\int_B f^p d(\hat{N}_B\mu)\right]^{\frac{1}{p}}\left[\int_B (Nf)^p d(\hat{N}_B\mu)\right]^{\frac{1}{q}} - \frac{1}{p(p-1)M^p}\int_B (Nf)^p(x)\mu(dx).$$

Proof. We have shown in Corollary 27 (ii) that the left-hand sides of the inequalities are less than or equal to $\psi_k(p)$, $k = 2,4,6$.

On the other hand, in the proof of Corollaries 8, 13 and 16 we have shown that the right-hand sides of the inequalities are greater than or equal to $A_k(\Phi_p) = \psi_k(p)$, $k = 2,4,6$, which finishes the proof. □

Similarly, by using the inequalities from Corollary 27 (ii) we can refine inequalities from Corollaries 17-20.

COROLLARY 29. *Let the assumptions of Corollary 17 hold and let $p,s,t \in \mathbb{R}$, $p<s<t$ or $s<t<p$, $p \neq 0,1$, $q = p/(p-1)$. If the kernel N and measure μ satisfy (10) with $C = B_z$, then for $p>1$*

$$\left[\psi_2(s)\right]^{\frac{t-p}{t-s}}\left[\psi_2(t)\right]^{\frac{p-s}{t-s}} \leqslant \frac{K_2}{(p-1)M}\left[\int_{B_z}(f^+)^p d\mu\right]^{\frac{1}{p}}\left[\int_{B_z}(Nf)^p d\mu\right]^{\frac{1}{q}} + \frac{z^{p-1}}{(p-1)M}\int_{B_z} N[f-f^+\mathbf{1}_{B_z}](x)\mu(dx) - \frac{z^p\mu(B_z)}{p} - \frac{1}{p(p-1)M^p}\int_{B_z}(Nf)^p d\mu.$$

When N and μ satisfy both (9) and (10) with $C = B_z$, then for $0 < p < 1$

$$\left[\psi_2(s)\right]^{\frac{t-p}{t-s}}\left[\psi_2(t)\right]^{\frac{p-s}{t-s}} \leqslant \frac{K_1^{1/p}K_2^{1/q}}{(p-1)M}\left[\int_{B_z}(f^+)^p d\mu\right]^{\frac{1}{p}}\left[\int_{B_z}(Nf)^p d\mu\right]^{\frac{1}{q}} + \frac{z^{p-1}}{(p-1)M}\int_{B_z} N[f-f^+\mathbf{1}_{B_z}](x)\mu(dx) - \frac{z^p\mu(B_z)}{p} - \frac{1}{p(p-1)M^p}\int_{B_z}(Nf)^p d\mu,$$

while for $p < 0$

$$\left[\psi_2(s)\right]^{\frac{t-p}{t-s}}\left[\psi_2(t)\right]^{\frac{p-s}{t-s}} \leqslant \frac{K_1^{1/q}K_2^{1/p}}{(p-1)M}\left[\int_{B_z}(f^+)^p d\mu\right]^{\frac{1}{p}}\left[\int_{B_z}(Nf)^p d\mu\right]^{\frac{1}{q}} + \frac{z^{p-1}}{(p-1)M}\int_{B_z} N[f-f^+\mathbf{1}_{B_z}](x)\mu(dx) - \frac{z^p\mu(B_z)}{p} - \frac{1}{p(p-1)M^p}\int_{B_z}(Nf)^p d\mu.$$

Proof. From the proof of Corollary 17 we can see that the right hand sides of the above inequalities are greater than or equal to the right hand side of the inequality from Corollary 28 (i), hence the claim follows. □

COROLLARY 30. *Let the assumptions of Corollary 18 hold and let $p,s,t \in (1,+\infty)$, $p < s < t$ or $s < t < p$, $q = p/(p-1)$. If the kernel N and measure μ satisfy (10) with $C = B$, then*

$$\left[\psi_4(s)\right]^{\frac{t-p}{t-s}}\left[\psi_4(t)\right]^{\frac{p-s}{t-s}}\left[\int_B (Nf)^p d\mu\right]^{-\frac{1}{q}} \leqslant \frac{K_2}{(p-1)M}\left[\int_B (f^+)^p d\mu\right]^{\frac{1}{p}} - \frac{1}{p(p-1)M^p}\left[\int_B (Nf)^p d\mu\right]^{\frac{1}{p}}.$$

Proof. By multiplying both sides of the inequality with $[\int_B (Nf)^p d\mu]^{-1/q}$, from the proof of Corollary 18 we can see that the right hand sides of the above inequality is greater than or equal to the right-hand side of the inequality from Corollary 28 (ii), hence the claim follows. □

COROLLARY 31. *Let the assumptions of Corollary 19 hold and let $p,s,t \in (0,+\infty)$, $p<s<t$ or $s<t<p$, $p \neq 1$, $q = p/(p-1)$. If the kernel N and measure μ satisfy (10) with $C = B$, then for $p > 1$*

$$[\psi_6(s)]^{\frac{t-p}{t-s}} [\psi_6(t)]^{\frac{p-s}{t-s}} \left[\int_B (Nf)^p d\mu\right]^{-\frac{1}{q}} \leqslant \frac{K_2}{(p-1)M}\left[\int_B f^p d\mu\right]^{\frac{1}{p}} - \frac{1}{p(p-1)M^p}\left[\int_B (Nf)^p d\mu\right]^{\frac{1}{p}}.$$

If N and μ satisfy (9) and (10) with $C = B$, then for $0 < p < 1$

$$[\psi_6(s)]^{\frac{t-p}{t-s}} [\psi_6(t)]^{\frac{p-s}{t-s}} \left[\int_B (Nf)^p d\mu\right]^{-\frac{1}{q}} \leqslant \frac{K_1^{1/p} K_2^{1/q}}{(p-1)M}\left[\int_B f^p d\mu\right]^{\frac{1}{p}} - \frac{1}{p(p-1)M^p}\left[\int_B (Nf)^p d\mu\right]^{\frac{1}{p}}.$$

Proof. By multiplying both sides of the inequalities with $[\int_B (Nf)^p d\mu]^{-1/q}$, from the proof of Corollary 19 we can see that the right-hand sides of the above inequalities are greater than or equal to the right-hand side of the inequality from Corollary 28 (iii), hence the claim follows. □

COROLLARY 32. *Let the assumptions of Corollary 20 hold and let $p,s,t \in (0,+\infty)$, $p<s<t$ or $s<t<p$, $p \neq 1$, $q = p/(p-1)$. If the kernel N and measure μ satisfy (10) with $C = B$, then*

$$[\psi_6(s)]^{\frac{t-p}{t-s}} [\psi_6(t)]^{\frac{p-s}{t-s}} \left[\int_B (Nf)^p d\mu\right]^{-\frac{1}{q}} \leqslant \frac{K_2^{1/q}}{(p-1)M}\left[\int_B f^p d(\hat{N}_B \mu)\right]^{\frac{1}{p}} - \frac{1}{p(p-1)M^p}\left[\int_B (Nf)^p d\mu\right]^{\frac{1}{p}}.$$

Proof. By multiplying both sides of the inequality with $[\int_B (Nf)^p d\mu]^{-1/q}$, from the proof of Corollary 20 we can see that the right-hand sides of the above inequality is greater than or equal to the right-hand side of the inequality from Corollary 28 (iii), hence the claim follows. □

5. Lagrange and Cauchy type mean value theorems

In the final section of this paper we shall derive some mean value type theorems for the functionals defined above.

THEOREM 33. *Let $k \in \{1,...,6\}$ and let the potential Nf be uniformly bounded, i. e. let there be a constant $K \in \mathbb{R}$ such that $(Nf)(x) \leqslant K$ for every $x \in X$. If $\Psi \in C^2(0,K]$ with $A_k(\Psi)$ finite, $A_k(\Phi_2) \neq 0$ and the function Ψ satisfies the same limiting conditions at zero as the function Φ in Theorem 11 (for $k=3$ or 4) or Theorem 15 (for $k=5$ or 6), then there exists $\xi_k \in [0,K]$ (assuming that $\Psi''(0) = \lim_{z\to 0}\Psi''(z)$ exists when $\xi_k = 0$) such that*

$$A_k(\Psi) = \Psi''(\xi_k)A_k(\Phi_2).$$

Proof. Since Φ_2 is a convex function and $A_k(\Phi_2) \neq 0$, Theorems 5, 11, 15 and Corollary 6 imply that $A_k(\Phi_2) > 0$, $k = 1,...,6$. Let

$$l = \inf_{\tau\in(0,K]} \Psi''(\tau) \quad \text{and} \quad L = \sup_{\tau\in(0,+K]} \Psi''(\tau).$$

If $L < +\infty$, then the function $L\Phi_2 - \Psi$ is convex since

$$\frac{d^2}{d\tau^2}\Big(L\frac{\tau^2}{2} - \Psi(\tau)\Big) = L - \Psi''(\tau) \geqslant 0.$$

Since the potential Nf is uniformly bounded by K, potential inequalities of Theorems 5, 11, 15 and Corollary 6 are meaningful for functions defined on the interval $(0,K]$. Since, by the assumptions of the lemma, the convex function $L\Phi_2 - \Psi$ satisfies the assumptions of Theorems 5 (for $k=1$), 11 (for $k=3$ or 4), 15 (for $k=5$ or 6) and Corollary 6 (for $k=2$), we have

$$0 \leqslant A_k\Big(L\Phi_2 - \Psi\Big), \qquad k = 1,...,6,$$

i. e.

$$A_k(\Psi) \leqslant LA_k(\Phi_2), \qquad k = 1,...,6. \tag{12}$$

If $L = +\infty$, then inequality (12) holds trivially. Similarly, for a finite l the inequality

$$lA_k(\Phi_2) \leqslant A_k(\Psi), \qquad k = 1,...,6 \tag{13}$$

holds since $\Psi - l\Phi_2$ is convex, while for $l = -\infty$ inequality (13) holds trivially.

Finally, the existence of ξ_k, $k = 1,...,6$, follows from (12) and (13) and continuity of Ψ''. □

THEOREM 34. *Let $k \in \{1,...6\}$ and let there exist $K \in \mathbb{R}$ such that $(Nf)(x) \leqslant K$ for every $x \in X$. If Ψ and $\widetilde{\Psi}$ satisfy the assumptions of Theorem 33 and if $A_k(\Phi_2) \neq 0$ and $A_k(\widetilde{\Psi}) \neq 0$, then there exists $\xi_k \in [0,K]$ such that*

$$\frac{\Psi''(\xi_k)}{\widetilde{\Psi}''(\xi_k)} = \frac{A_k(\Psi)}{A_k(\widetilde{\Psi})}. \tag{14}$$

Proof. Let us define the function ϕ by

$$\phi(\tau) = \Psi(\tau)A_k(\widetilde{\Psi}) - \widetilde{\Psi}(\tau)A_k(\Psi).$$

The function ϕ also satisfies the assumptions of Theorem 33 and, hence, there exists $\xi_k \in [0,K]$ such that $A_k(\phi) = \phi''(\xi_k)A_k(\Phi_2)$. Since $A_k(\phi) = 0$ and $\phi''(\xi_k) = \Psi''(\xi_k)A_k(\widetilde{\Psi}) - \widetilde{\Psi}''(\xi_k)A_k(\Psi)$, equality (14) follows. □

Relation (14) allows us to define various means, because when $\Psi''/\widetilde{\Psi}''$ is an invertible function we have

$$\xi_k = \left(\frac{\Psi''}{\widetilde{\Psi}''}\right)^{-1}\left(\frac{A_k(\Psi)}{A_k(\widetilde{\Psi})}\right).$$

Specially, for $\Psi = \Phi_p$ and $\widetilde{\Psi} = \Phi_q$, recalling the definitions (2) and (11) of functions Φ_p and ψ_k, respectively, we can define means $E^k_{p,q}$ by

$$E^k_{p,q} = \left(\frac{A_k(\Phi_p)}{A_k(\Phi_q)}\right)^{\frac{1}{p-q}} = \left(\frac{\psi_k(p)}{\psi_k(q)}\right)^{\frac{1}{p-q}}$$

for $p,q \in I_k$, $p \neq q$. Moreover, we can continuously extend these means to cover the case $p = q$ as well by calculating the limits $\lim_{p\to q} E^k_{p,q}$. For $k = 1$ or 2 we get

$$E^k_{p,q} = \begin{cases} \left(\frac{A_k(\Phi_p)}{A_k(\Phi_q)}\right)^{\frac{1}{p-q}}, & p \neq q \\ \exp\left\{\frac{1-2p}{p(p-1)} - \frac{A_k(\Phi_0\Phi_p)}{A_k(\Phi_p)}\right\}, & p = q \neq 0,1 \\ \exp\left\{-1 - \frac{A_k(\Phi_0\Phi_1)}{2A_k(\Phi_1)}\right\}, & p = q = 1 \\ \exp\left\{1 - \frac{A_k(\Phi_0^2)}{2A_k(\Phi_0)}\right\}, & p = q = 0 \end{cases} \tag{15}$$

The means $E^k_{p,q}$, $k = 3,\dots,6$, have the same form, but are defined only for $p > 1$ and $q > 1$ when $k = 3$ or 4, and for $p > 0$ and $q > 0$ when $k = 5$ or 6.

COROLLARY 35. *Let* $1 \leqslant k \leqslant 6$ *and* $p,q,r,s \in I_k$ *be such that* $p \leqslant r$ *and* $q \leqslant s$. *Then*

$$E^k_{p,q} \leqslant E^k_{r,s}.$$

Proof. Due to log-convexity of the functions ψ_k (Lemma 26 and Corollary 23 (ii)) and continuity of the means E^k, the claim follows from Corollary 25. □

By using our approach, we can get Hardy-type inequalities for gradients and use these to obtain Fridriech-type inequalities. More formally, let Ω be a bounded, open and connected set in $\mathbb{R}^n$ and let $\bar{\Omega}$ be its closure. Let $f \in C^1(\bar{\Omega})$ with $\mathrm{supp}(f) \subset \Omega$ and define g by

$$g(x) = \frac{1}{\omega_n}\int_\Omega \frac{\|\nabla f(y)\|}{\|x-y\|^{n-1}}\,dy, \tag{16}$$

where ω_n is the area of the surface of the unit hypersphere S^{n-1} in $\mathbb{R}^n$. Function g is equal to $N(\|\nabla f\|)$, where N is the kernel with density

$$G(x,y)=\frac{1}{\omega_n\|x-y\|^{n-1}}. \tag{17}$$

The kernel N on $\mathbb{R}^n$ satisfies the maximum principle on the set of nonnegative functions with constant $M=1$ for $n\geqslant 3$ (see [4] and Remark 1 in [5]) and with constant $M=6$ for $n=2$ (see Proposition 13 in [5]). Hence, the restriction of N on Ω also satisfies the maximum principle, since a nonnegative function on Ω can be extended with zero outside of Ω.

Finally, define a linear functional $A=A_{6;\|\nabla f\|,N,\mu}$ for the nonnegative function $\|\nabla f\|$ and measure $\mu(dx)=dx$ by

$$A(\Phi)=\frac{1}{\omega_n M}\int_\Omega\int_\Omega\frac{\|\nabla f(x)\|\varphi(g(x))}{\|y-x\|^{n-1}}\,dy\,dx-\int_\Omega\Phi\Big(\frac{1}{M}g(x)\Big)dx. \tag{18}$$

THEOREM 36. *Let Ω be a bounded, open and connected set in $\mathbb{R}^n$, let $f\in C^1(\bar{\Omega})$ be such that* $\operatorname{supp}(f)\subset\Omega$*, let g be defined by (16) and the kernel N on $X=\Omega$ by its density (17). Then, for $p,s,t\in(0,+\infty)$, $p\neq 1$, $p<s<t$ or $s<t<p$ and $q=p/(p-1)$ we have*

$$\big[\psi(s)\big]^{\frac{t-p}{t-s}}\big[\psi(t)\big]^{\frac{p-s}{t-s}}\leqslant\frac{1}{\omega_n M(p-1)}\int_\Omega\int_\Omega\frac{\|\nabla f(x)\|g^{p-1}(x)}{\|y-x\|^{n-1}}\,dy\,dx-\frac{1}{p(p-1)M^p}\int_\Omega g^p(x)dx,$$

where, for $r\neq 1$,

$$\psi(r)=\frac{1}{\omega_n M(r-1)}\int_\Omega\int_\Omega\frac{\|\nabla f(x)\|g^{r-1}(x)}{\|y-x\|^{n-1}}\,dy\,dx-\frac{1}{r(r-1)M^p}\int_\Omega g^r(x)dx,$$

$$\psi(1)=\frac{1}{\omega_n M}\int_\Omega\int_\Omega\frac{\|\nabla f(x)\|\big(1+\log g(x)\big)}{\|y-x\|^{n-1}}\,dy\,dx-\frac{1}{M}\int_\Omega g(x)(\log\frac{g(x)}{M})dx,$$

with $M=1$ for $n\geqslant 3$ and $M=6$ for $n=2$.

Proof. For a constant function f, we have $\nabla f\equiv 0$, so the inequality is trivially satisfied. For a non-constant f, the set $B=\{N(\|\nabla f\|)>0\}$ is equal to Ω, so

$$d(\hat{N}_B\mu)(x)=d(\hat{N}\mu)(x)=\frac{1}{\omega_n}\int_\Omega\frac{dy}{\|y-x\|^{n-1}}. \tag{19}$$

Therefore, the linear functional A defined by (18) satisfies

$$A(\Phi)=\frac{1}{M}\int_\Omega\|\nabla f(x)\|\varphi(g(x))(\hat{N}\mu)(dx)-\int_\Omega\Phi\Big(\frac{1}{M}g(x)\Big)dx.$$

Taking into account Remark 7, we see that the linear functional A, restricted to the set of functions Φ for which $\Phi(0+) = 0$, satisfies the assumptions of Corollary 27 for $k = 6$, with $B = \Omega$ and f replaced with the nonnegative function $\|\nabla f\|$. Since $\psi(r) = A(\Phi_r)$ is equal to the $\psi_6(r)$ from Corollary (27), by part (ii) of that corollary we have

$$\left[\psi(s)\right]^{\frac{t-p}{t-s}}\left[\psi(t)\right]^{\frac{p-s}{t-s}} \leqslant \frac{1}{M(p-1)}\int_\Omega \|\nabla f(x)\| g^{p-1}(x)(\hat{N}\mu)(dx) - \frac{1}{p(p-1)M^p}\int_\Omega g^p(x)dx.$$

Taking into account (19), we see that this is exactly the inequality stated in the theorem. □

Since

$$\frac{1}{\omega_n}\int_\Omega \frac{dy}{\|y-x\|^{n-1}} \leqslant \frac{\operatorname{diam}(\Omega)}{2}, \tag{20}$$

we see that the kernel N and measure μ satisfy the condition (10) with constant $K_2 = \operatorname{diam}(\Omega)/2$. Using this, we can further restate the inequality from the last theorem for $p > 1$.

COROLLARY 37. *Let Ω, f, g, N, ψ and M be as in Theorem 36. Then, for $p,s,t \in (1,+\infty)$, $p < s < t$ or $s < t < p$ and $q = p/(p-1)$ we have*

$$p(p-1)M^p\left[\psi(s)\right]^{\frac{t-p}{t-s}}\left[\psi(t)\right]^{\frac{p-s}{t-s}}\left[\int_\Omega g^p(x)dx\right]^{-\frac{1}{q}} \leqslant \frac{dpM^{p-1}}{2}\left[\int_\Omega \|\nabla f(x)\|^p dx\right]^{\frac{1}{p}} - \left[\int_\Omega g^p(x)dx\right]^{\frac{1}{p}},$$

where $d = \operatorname{diam}(\Omega)$.

Proof. Applying inequality (20) on the first integral of the inequality from Theorem 36 for $p > 1$ we get

$$\left[\psi(s)\right]^{\frac{t-p}{t-s}}\left[\psi(t)\right]^{\frac{p-s}{t-s}} \leqslant \frac{d}{2M(p-1)}\int_\Omega \|\nabla f\|(x) g^{p-1}(x)(dx) - \frac{1}{p(p-1)M^p}\int_\Omega g^p(x)dx.$$

Finally, applying Hölder's inequality on the first integral from the right hand side and multiplying by $p(p-1)M^p\left[\int_\Omega g^p(x)dx\right]^{-1/q}$ we get the claim of the corollary. □

When the support of f is contained in Ω, then the well-known formula

$$f(x) = \frac{1}{\omega_n}\int_\Omega \frac{\nabla f(y)\cdot(x-y)}{\|x-y\|^n}dy \tag{21}$$

holds. Using (21) and Corollary 37 we can prove the following, Friedrichs' type inequality

COROLLARY 38. *Let Ω be a bounded, open and connected set in $\mathbb{R}^n$, $d = \operatorname{diam}(\Omega)$ and let $f \in C^1(\Omega)$ be such that* $\operatorname{supp}(f) \subset \Omega$. *Then, for $p > 1$*

$$\|f\|_{L^p(\Omega)} \leqslant C\|\nabla f\|_{L^p(\Omega)},$$

where $C = dp/2$ for $n \geqslant 3$ and $C = dp6^{p-1}/2$ for $n = 2$.

Proof. The inequality follows from Corollary 37 since the left hand side of the inequality from Corollary 37 is nonnegative and from (21) we have that

$$|f|^p \leqslant g^p. \qquad \square$$

COROLLARY 39. *Let Ω, f, g, N and ψ be as in Theorem 36. Then ψ is exponentially convex and*

(i) *For all $n \in \mathbb{N}$ and $p_i > 0$, $1 \leqslant i \leqslant n$ the matrix $[\psi(\frac{p_i+p_j}{2})]_{i,j=1}^n$ is positive semidefinite, so*

$$\det\left[\psi\left(\frac{p_i+p_j}{2}\right)\right]_{i,j=1}^n \geqslant 0.$$

(ii) *For $p,s,t \in (0,+\infty)$ we have*

$$\psi(p) \geqslant \left[\psi(s)\right]^{\frac{t-p}{t-s}} \left[\psi(t)\right]^{\frac{p-s}{t-s}} \qquad \text{if } p<s<t \quad \text{or} \quad s<t<p$$

$$\psi(p) \leqslant \left[\psi(s)\right]^{\frac{t-p}{t-s}} \left[\psi(t)\right]^{\frac{p-s}{t-s}} \qquad \text{if } s<p<t$$

Proof. Linear functional A defined by (18) and restricted to the set of functions Φ for which $\Phi(0+) = 0$ satisfies the assumptions of Lemma 26 with $k = 6$ (with $A_6 = A$ and f replaced by $\|\nabla f\|$). From the proof of Theorem 36, one can see that $\psi(p) = A(\Phi_p)$, so ψ is equal to ψ_6 from Lemma 26. Therefore, ψ is exponentially convex and the inequalities in parts (i) and (ii) follow from Corollary 27. $\square$

COROLLARY 40. *Let the assumptions of Corollary 39 hold, let the linear functional A be defined by (18) and let $\Psi \in C^2(0,+\infty)$ be a convex function for which* $\lim_{z\to 0} z\Psi'(z) = 0$ *and* $\Psi(0+) = 0$. *Then*

(i) *there exists $\xi \in [0,+\infty)$ (assuming that $\Psi''(0) = \lim_{z\to 0}\Psi''(z)$ exists when $\xi = 0$) such that $A(\Psi) = \Psi''(\xi)A(\Phi_2)$.*

(ii) *if $\widetilde{\Psi}$ satisfies the same assumptions as Ψ and $A(\Phi_2) \neq 0$, then there exist $\xi \in [0,+\infty)$ such that*

$$\frac{\Psi''(\xi)}{\widetilde{\Psi}''(\xi)} = \frac{A(\Psi)}{A(\widetilde{\Psi})}.$$

Proof. Since $f \in C^1(\bar{\Omega})$ with $\mathrm{supp}(f) \subset \Omega$, the norm of the gradient $\|\nabla f\|$ is uniformly bounded and, by (16) and (20), the potential $N(\|\nabla f\|)$ is also uniformly bounded. Hence, A, Ψ and $\widetilde{\Psi}$ satisfy the assumptions of Theorems 33 and 34 with $k = 6$ and the claims follow.

Using the last corollary with $\Psi = \Phi_p$ and $\widetilde{\Psi} = \Phi_q$, we can define means $E_{p,q}$ for $p, q \in (0, +\infty)$, $p \neq q$. We can extend these means continuously to cover the cases $p = q$ as well, getting the same expression as in (15) for $p, q \in (0, +\infty)$, with $E^k_{p,q}$ replaced by $E_{p,q}$ and A_k by A.

COROLLARY 41. *Let the means E be defined as above. Then, for $p_1, p_2, p_3, p_4 \in (0, +\infty)$ such that $p_1 \leqslant p_3$ and $p_2 \leqslant p_4$, we have*

$$E_{p_1,p_2} \leqslant E_{p_3,p_4}.$$

Proof. The function $\psi(p) = A(\Phi_p)$ is exponentially convex by Corollary 39. Hence, the claim follows by Corollary 25 and continuity of the means E. □

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(Received December 2, 2010)

Neven Elezović
Faculty of Electrical Engineering and Computing
University of Zagreb
Unska 3
10000 Zagreb, Croatia
e-mail: neven.elez@fer.hr

Josip Pečarić
Faculty of Textile Technology
University of Zagreb
Prilaz baruna Filipovića 28a
10000 Zagreb, Croatia
e-mail: pecaric@element.hr

Marjan Praljak
Faculty of Food Technology and Biotechnology
University of Zagreb
Pierottijeva 6
10000 Zagreb, Croatia
e-mail: mpraljak@pbf.hr