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NONLINEAR INTEGRAL INEQUALITIES INVOLVING MAXIMA OF THE UNKNOWN SCALAR FUNCTIONS

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Abstract. This paper deals with some nonlinear integral inequalities that involve the maximum of the unknown scalar function of one variable. The considered inequalities are generalizations of the classical integral inequality of Gronwall–Bellman. The importance of these integral inequalities is due to their wide applications in qualitative investigations of differential equations with "maxima", and it is illustrated by some direct applications.

1. Introduction

In the past few years, a number of integral inequalities was established by many scholars, which are motivated by certain applications such as existence, uniqueness, continuous dependence, comparison, perturbation, boundedness and stability of solutions of differential and integral equations (see, for example, [4, 7, 12, 13, 15] and the references cited therein). Among these integral inequalities, we cite the famous Gronwall inequality and its various generalizations [1, 2, 5, 10, 9, 8].

In the last few decades, great attention has been paid to automatic control systems and their applications to computational mathematics and modeling. Many problems in control theory correspond to the maximal deviation of the regulated quantity (see [14]). Such kind of problems could be adequately modeled by differential equations that contain the maxima operator. Note that such equations involving "maxima" of the unknown function are called *differential equations with "maxima"*, see [3, 6]. In his survey [11], A. D. Mishkis also points out the necessity to study differential equations with "maxima".

The purpose of this paper is to establish some new nonlinear integral inequalities in the case when the "maxima" of the unknown scalar function is involved in the integral. Several cases depending on the type of nonlinearity are considered. These inequalities are mathematical tools in the theory of differential equations with "maxima". Their importance is illustrated by some direct applications obtaining bounds of the solutions.

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2. Main results

Let $h > 0$ be a constant and suppose t_0 and T are fixed points with $0 \leqslant t_0 < T \leqslant \infty$.

DEFINITION 2.1. The function $\alpha \in C^1([t_0,T),\mathbb{R}_+)$ is said to be from the class $\mathscr{F}$ if it is nondecreasing and satisfies $\alpha(t) \leqslant t$ for $t \in [t_0,T)$.

Let $\alpha_i, \beta_j \in \mathscr{F}$ for $i = 1,2,\ldots,n$ and $j = 1,2,\ldots,m$. Denote

$$J = \min\left\{ \min_{1\leqslant i\leqslant n} \alpha_i(t_0),\ \min_{1\leqslant j\leqslant m} \beta_j(t_0) \right\}.$$

2.1. Constant additive term

In this subsection, we discuss the case of a constant k in the inequality (2.1) below. For this purpose, we define the following functions.

DEFINITION 2.2. The function $\omega \in C(\mathbb{R}_+,\mathbb{R}_+)$ is said to be from Ω_1 if

(i) $\omega(x) > 0$ for $x > 0$ and ω is a nondecreasing function;

(ii) $\int^\infty \frac{\mathrm{d}x}{\omega(x)} = \infty$.

In the case when the nonlinear functions under the integrals in the inequality (2.1) below are from the set Ω_1, we obtain the following result.

THEOREM 2.3. *Let the following conditions be fulfilled:*

(A$_1$) $\phi \in C([J-h,t_0],\mathbb{R}_+)$.

(A$_2$) $k > 0$.

(A$_3$) $\alpha_i, \beta_j \in \mathscr{F}$ *for* $i = 1,2,\ldots,n,\ j = 1,2,\ldots,m$.

(A$_4$) $f_i, g_j \in C([J,T),\mathbb{R}_+)$ *for* $i = 1,2,\ldots,n,\ j = 1,2,\ldots,m$.

(A$_5$) $\omega_i, \tilde{\omega}_j \in \Omega_1$ *for* $i = 1,2,\ldots,n,\ j = 1,2,\ldots,m$.

(A$_6$) $\psi \in C^1(\mathbb{R}_+,\mathbb{R}_+)$ *is increasing,* $\psi(0) = 0$*, and* $\lim_{t\to\infty}\psi(t) = \infty$.

(A$_7$) $u \in C([J-h,T),\mathbb{R}_+)$ *satisfies for some* $p \geqslant 0$ *the inequalities*

$$\psi\Big(u(t)\Big) \leqslant k + \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} f_i(s)u^p(s)\omega_i\Big(u(s)\Big)\mathrm{d}s \qquad (2.1)$$

$$+ \sum_{j=1}^{m} \int_{\beta_j(t_0)}^{\beta_j(t)} g_j(s)u^p(s)\tilde{\omega}_j\Big(\max_{\xi\in[s-h,s]} u(\xi)\Big)\mathrm{d}s \quad \textit{for} \quad t \in [t_0,T),$$

$$u(t) \leqslant \phi(t) \quad \textit{for} \quad t \in [J-h,t_0]. \qquad (2.2)$$

Then for $t_0 \leqslant t \leqslant t_1$, the inequality

$$u(t) \leqslant \psi^{-1}\left(\Psi^{-1}\left(W^{-1}\left(W\big(\Psi(M)\big)+A(t)\right)\right)\right) \tag{2.3}$$

holds, where

$$\Psi(r)=\int_{r_0}^{r}\frac{\mathrm{d}s}{\left[\psi^{-1}(s)\right]^{p}}, \quad 0<r_0<k, \tag{2.4}$$

$$W(r)=\int_{r_1}^{r}\frac{\mathrm{d}s}{q\big(\psi^{-1}(\Psi^{-1}(s))\big)}, \quad 0<r_1<\Psi(M), \tag{2.5}$$

$$q(t)=\max\left\{\max_{1\leqslant i\leqslant n}\omega_i(t),\ \max_{1\leqslant j\leqslant m}\tilde{\omega}_j(t)\right\}, \tag{2.6}$$

$$A(t)=\sum_{i=1}^{n}\int_{\alpha_i(t_0)}^{\alpha_i(t)}f_i(s)\mathrm{d}s+\sum_{j=1}^{m}\int_{\beta_j(t_0)}^{\beta_j(t)}g_j(s)\mathrm{d}s, \tag{2.7}$$

$$M=\max\left\{k,\ \psi\Big(\max_{s\in[J-h,t_0]}\phi(s)\Big)\right\}, \tag{2.8}$$

$$\begin{aligned} t_1 = \sup\Big\{\tau\in[t_0,T):\ & W\big(\Psi(M)\big)+A(t)\in \operatorname{Dom}\big(W^{-1}\big),\\ & W^{-1}\Big(W\big(\Psi(M)\big)+A(t)\Big)\in\operatorname{Dom}\big(\Psi^{-1}\big) \quad \textit{and}\\ & \Psi^{-1}\Big(W^{-1}\Big(W\big(\Psi(M)\big)+A(t)\Big)\Big)\in\operatorname{Dom}\big(\psi^{-1}\big) \quad \textit{for} \quad t\in[t_0,\tau]\Big\}.\end{aligned}$$

Proof. Define a function $z:[J-h,T)\to\mathbb{R}_+$ by

$$z(t)=\begin{cases} M+\displaystyle\sum_{i=1}^{n}\int_{\alpha_i(t_0)}^{\alpha_i(t)}f_i(s)u^p(s)\omega_i\big(u(s)\big)\mathrm{d}s \\ \quad +\displaystyle\sum_{j=1}^{m}\int_{\beta_j(t_0)}^{\beta_j(t)}g_j(s)u^p(s)\tilde{\omega}_j\Big(\max_{\xi\in[s-h,s]}u(\xi)\Big)\mathrm{d}s, & t\in[t_0,T),\\ M, & t\in[J-h,t_0].\end{cases}$$

The function z is nondecreasing. Since $\psi(u(t))\leqslant\psi(\max_{s\in[J-h,t_0]}\phi(s))\leqslant M=z(t)$ for $t\in[J-h,t_0]$ by (2.2) and (2.8) and $\psi(u(t))\leqslant z(t)$ for $t\in[t_0,T)$ by (2.1) and (2.8), the inequality

$$u(t)\leqslant\psi^{-1}\big(z(t)\big) \quad \text{holds for} \quad t\in[J-h,T). \tag{2.9}$$

Note that $\max_{\xi\in[s-h,s]}\psi^{-1}\big(z(\xi)\big)=\psi^{-1}\big(z(s)\big)$ for $s\in[\beta_j(t_0),\beta_j(T))$, $j=1,2,\dots,m$. Then from inequality (2.1) and the definition of the function q, we get for $t\in[t_0,T)$

$$\begin{aligned} z(t)\leqslant\ & M+\sum_{i=1}^{n}\int_{\alpha_i(t_0)}^{\alpha_i(t)} f_i(s)\Big(\psi^{-1}\big(z(s)\big)\Big)^p\omega_i\Big(\psi^{-1}\big(z(s)\big)\Big)\mathrm{d}s\\ &+\sum_{j=1}^{m}\int_{\beta_j(t_0)}^{\beta_j(t)} g_j(s)\Big(\psi^{-1}\big(z(s)\big)\Big)^p\tilde{\omega}_j\Big(\psi^{-1}\big(z(s)\big)\Big)\mathrm{d}s\\ \leqslant\ & M+\sum_{i=1}^{n}\int_{\alpha_i(t_0)}^{\alpha_i(t)} f_i(s)\Big(\psi^{-1}\big(z(s)\big)\Big)^p q\Big(\psi^{-1}\big(z(s)\big)\Big)\mathrm{d}s\\ &+\sum_{j=1}^{m}\int_{\beta_j(t_0)}^{\beta_j(t)} g_j(s)\Big(\psi^{-1}\big(z(s)\big)\Big)^p q\Big(\psi^{-1}\big(z(s)\big)\Big)\mathrm{d}s=:K(t), \end{aligned}\tag{2.10}$$

where the function $K:[t_0,T)\to[M,\infty)$ is nondecreasing and satisfies $K(t_0)=M$. Differentiate the function K and use its monotonicity (observe Definition 2.1) and (2.10) to obtain

$$\begin{aligned} K'(t)=&\sum_{i=1}^{n} f_i\big(\alpha_i(t)\big)\Big[\psi^{-1}\big(z(\alpha_i(t))\big)\Big]^p q\Big(\psi^{-1}\big(z(\alpha_i(t))\big)\Big)\alpha_i'(t)\\ &+\sum_{j=1}^{m} g_j\big(\beta_j(t)\big)\Big[\psi^{-1}\big(z(\beta_j(t))\big)\Big]^p q\Big(\psi^{-1}\big(z(\beta_j(t))\big)\Big)\beta_j'(t)\\ \leqslant&\sum_{i=1}^{n} f_i\big(\alpha_i(t)\big)\Big[\psi^{-1}\big(K(\alpha_i(t))\big)\Big]^p q\Big(\psi^{-1}\big(z(\alpha_i(t))\big)\Big)\alpha_i'(t)\\ &+\sum_{j=1}^{m} g_j\big(\beta_j(t)\big)\Big[\psi^{-1}\big(K(\beta_j(t))\big)\Big]^p q\Big(\psi^{-1}\big(z(\beta_j(t))\big)\Big)\beta_j'(t)\\ \leqslant&\Big[\psi^{-1}\big(K(t)\big)\Big]^p\Bigg\{\sum_{i=1}^{n} f_i(\alpha_i(t))q\Big(\psi^{-1}\big(z(\alpha_i(t))\big)\Big)\alpha_i'(t)\\ &+\sum_{j=1}^{m} g_j(\beta_j(t))q\Big(\psi^{-1}\big(z(\beta_j(t))\big)\Big)\beta_j'(t)\Bigg\}. \end{aligned}\tag{2.11}$$

From (2.4) and (2.11), we have that

$$\begin{aligned} (\Psi\circ K)'(t)=\frac{K'(t)}{\Big[\psi^{-1}\big(K(t)\big)\Big]^p}\leqslant&\sum_{i=1}^{n} f_i\big(\alpha_i(t)\big)q\Big(\psi^{-1}\big(z(\alpha_i(t))\big)\Big)\alpha_i'(t)\\ &+\sum_{j=1}^{m} g_j\big(\beta_j(t)\big)q\Big(\psi^{-1}\big(z(\beta_j(t))\big)\Big)\beta_j'(t). \end{aligned}\tag{2.12}$$

Integrate (2.12) from t_0 to $t\in[t_0,t_1]$ and change the variables to get

$$\begin{aligned} \Psi\Big(K(t)\Big)\leqslant\Psi(M)&+\sum_{i=1}^{n}\int_{\alpha_i(t_0)}^{\alpha_i(t)} f_i(s)q\Big(\psi^{-1}\big(z(s)\big)\Big)\mathrm{d}s\\ &+\sum_{j=1}^{m}\int_{\beta_j(t_0)}^{\beta_j(t)} g_j(s)q\Big(\psi^{-1}\big(z(s)\big)\Big)\mathrm{d}s=:K_1(t), \end{aligned}\tag{2.13}$$

where the function K_1 is nondecreasing and satisfies $K_1(t_0) = \Psi(M)$ and, due to (2.10) and (2.13),

$$z(t) \leqslant K(t) \leqslant \Psi^{-1}\big(K_1(t)\big) \quad \text{holds for} \quad t \in [t_0, t_1). \tag{2.14}$$

Differentiate the function K_1 and use its monotonicity (observe Definition 2.1) and (2.14) to obtain

$$\begin{aligned} K_1'(t) &= \sum_{i=1}^{n} f_i(\alpha_i(t)) q\Big(\psi^{-1}\big(z(\alpha_i(t))\big)\Big)\alpha_i'(t) \\ &\quad + \sum_{j=1}^{m} g_j(\beta_j(t)) q\Big(\psi^{-1}\big(z(\beta_j(t))\big)\Big)\beta_j'(t) \\ &\leqslant q\Big(\psi^{-1}\Big(\Psi^{-1}\big(K_1(t)\big)\Big)\Big)\Big\{\sum_{i=1}^{n} f_i(\alpha_i(t))\alpha_i'(t) + \sum_{j=1}^{m} g_j(\beta_j(t))\beta_j'(t)\Big\}. \end{aligned} \tag{2.15}$$

From (2.15) and (2.5), we get

$$\begin{aligned} (W \circ K_1)'(t) &= \frac{K_1'(t)}{q\Big(\psi^{-1}\Big(\Psi^{-1}\big(K_1(t)\big)\Big)\Big)} \\ &\leqslant \sum_{i=1}^{n} f_i(\alpha_i(t))\alpha_i'(t) + \sum_{j=1}^{m} g_j(\beta_j(t))\beta_j'(t). \end{aligned} \tag{2.16}$$

Integrate (2.16) from t_0 to $t \in [t_0, t_1]$ and change the variables to get

$$W\big(K_1(t)\big) = W\big(\Psi(M)\big) + A(t), \tag{2.17}$$

where the function A is defined by (2.7). Since W^{-1} is increasing and since, due to (2.9) and (2.14), $u(t) \leqslant \psi^{-1}(z(t)) \leqslant \psi^{-1}\Big(\Psi^{-1}\big(K_1(t)\big)\Big)$, (2.17) implies the required inequality (2.3). □

COROLLARY 2.4. *Let $k = 0$ and $\phi(t) \equiv 0$. Suppose* (A_3)–(A_7) *hold. Then for $t_0 \leqslant t \leqslant t_2$, the inequality*

$$u(t) \leqslant \psi^{-1}\Big(\Psi^{-1}\Big(W^{-1}\big(A(t)\big)\Big)\Big)$$

holds, where

$$\begin{aligned} t_2 = \sup\Big\{\tau \in [t_0, T):\ & A(t) \in \mathrm{Dom}\left(W^{-1}\right),\ W^{-1}\big(A(t)\big) \in \mathrm{Dom}\left(\Psi^{-1}\right) \\ & \textit{and}\ \Psi^{-1}\Big(W^{-1}\big(A(t)\big)\Big) \in \mathrm{Dom}\left(\psi^{-1}\right)\ \textit{for}\ t \in [t_0, \tau]\Big\}. \end{aligned}$$

Proof. This claim follows from Theorem 2.3 by choosing an arbitrary $\varepsilon > 0$, letting $k = \phi(t) = \varepsilon$, and taking the limit as $\varepsilon \to 0$. □

Note that the inequalities (2.1), (2.2) could have another type of solution as follows, which is simpler than (2.3) but the used integral function is more complicated.

THEOREM 2.5. *Suppose* (A_1)–(A_7) *hold. Then for* $t_0 \leqslant t \leqslant t_3$, *the inequality*

$$u(t) \leqslant \psi^{-1}\left(\Psi_1^{-1}\left(\Psi_1(M) + A(t)\right)\right) \tag{2.18}$$

holds, where Ψ_1^{-1} *is the inverse function of*

$$\Psi_1(r) = \int_{r_3}^{r} \frac{\mathrm{d}s}{\left[\psi^{-1}(s)\right]^p q\left(\psi^{-1}(s)\right)}, \quad 0 < r_3 < k, \tag{2.19}$$

the functions q *and* A *and the constant* M *are defined by* (2.6), (2.7), *and* (2.8), *respectively, and*

$$t_3 = \sup\left\{\tau \in [t_0, T): \ \Psi_1(M) + A(t) \in \mathrm{Dom}\left(\Psi_1^{-1}\right) \ \text{and} \right.$$
$$\left. \Psi_1^{-1}\left(\Psi_1(M) + A(t)\right) \in \mathrm{Dom}\left(\psi^{-1}\right) \ \text{for} \ t \in [t_0, \tau]\right\}.$$

Proof. Following the proof of Theorem 2.3, we obtain the inequalities (2.10) and (2.11). From (2.11) and Definition 2.1, we conclude

$$\begin{aligned} K'(t) \leqslant \left[\psi^{-1}(K(t))\right]^p q\left(\psi^{-1}(K(t))\right)\Bigg\{ &\sum_{i=1}^{n} f_i(\alpha_i(t))\alpha_i'(t) \\ &+ \sum_{j=1}^{m} g_j(\beta_j(t))\beta_j'(t)\Bigg\}. \end{aligned} \tag{2.20}$$

From (2.19) and (2.20), we have that

$$\begin{aligned} (\Psi_1 \circ K)'(t) &= \frac{K'(t)}{\left[\psi^{-1}(K(t))\right]^p q\left(\psi^{-1}(K(t))\right)} \\ &\leqslant \sum_{i=1}^{n} f_i\left(\alpha_i(t)\right)\alpha_i'(t) + \sum_{j=1}^{m} g_j\left(\beta_j(t)\right)\beta_j'(t). \end{aligned} \tag{2.21}$$

Integrate inequality (2.21) from t_0 to $t \in [t_0, t_3]$ and change the variables to get

$$\Psi_1\left(K(t)\right) \leqslant \Psi_1(M) + \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} f_i(s)\mathrm{d}s + \sum_{j=1}^{m} \int_{\beta_j(t_0)}^{\beta_j(t)} g_j(s)\mathrm{d}s = \Psi_1(M) + A(t). \tag{2.22}$$

By (2.9), (2.10), and (2.22), we obtain the required inequality (2.18). □

In the case when $p = 0$, both solutions of inequalities (2.1), (2.2) given in Theorem 2.3 and Theorem 2.5 coincide.

COROLLARY 2.6. *Suppose* (A_1)–(A_6) *hold and assume*

(A'_7) $u \in C([J-h,T),\mathbb{R}_+)$ *satisfies the inequalities*

$$\psi\Big(u(t)\Big) \leqslant k + \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} f_i(s)\omega_i\Big(u(s)\Big)\mathrm{d}s + \sum_{j=1}^{m} \int_{\beta_j(t_0)}^{\beta_j(t)} g_j(s)\tilde{\omega}_j\Big(\max_{\xi\in[s-h,s]} u(\xi)\Big)\mathrm{d}s \quad \textit{for} \quad t\in[t_0,T),$$
$$u(t) \leqslant \phi(t) \quad \textit{for} \quad t\in[J-h,t_0].$$

Then for $t_0 \leqslant t \leqslant t_4$, *the inequality*

$$u(t) \leqslant \psi^{-1}\Big(W^{-1}\Big(W(M)+A(t)\Big)\Big)$$

holds, where W^{-1} *is the inverse function of*

$$W(r) = \int_{r_4}^{r} \frac{\mathrm{d}s}{q\big(\psi^{-1}(s)\big)}, \quad 0 < r_4 < k,$$

the functions q *and* A *and the constant* M *are defined by* (2.6), (2.7), *and* (2.8), *respectively, and*

$$t_4 = \sup\Big\{\tau\in[t_0,T):\ W(M)+A(t)\in \mathrm{Dom}\left(W^{-1}\right) \textit{ and } W^{-1}\Big(W(M)+A(t)\Big)\in \mathrm{Dom}\left(\psi^{-1}\right) \textit{ for } t\in[t_0,\tau]\Big\}.$$

In the case when the left part of the considered inequality is linear, i.e., $\psi(x)=x$, we obtain the following particular case of Theorem 2.3.

COROLLARY 2.7. *Suppose* (A_1)–(A_5) *hold and assume*

(A''_7) $u \in C([J-h,T),\mathbb{R}_+)$ *satisfies the inequalities*

$$u(t) \leqslant k + \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} f_i(s)\omega_i\Big(u(s)\Big)\mathrm{d}s + \sum_{j=1}^{m} \int_{\beta_j(t_0)}^{\beta_j(t)} g_j(s)\tilde{\omega}_j\Big(\max_{\xi\in[s-h,s]} u(\xi)\Big)\mathrm{d}s \quad \textit{for} \quad t\in[t_0,T),$$
$$u(t) \leqslant \phi(t) \quad \textit{for} \quad t\in[J-h,t_0].$$

Then for $t_0 \leqslant t \leqslant t_5$, *the inequality*

$$u(t) \leqslant W^{-1}\Big(W(M)+A(t)\Big)$$

holds, where W^{-1} *is the inverse function of*

$$W(r) = \int_{r_5}^{r} \frac{\mathrm{d}s}{q(s)}, \quad r_5 > 0, \tag{2.23}$$

the functions q *and* A *and the constant* M *are defined by* (2.6)*,* (2.7)*, and* (2.8)*, respectively, and*

$$t_5 = \sup\left\{\tau \in [t_0, T): \; W(M) + A(t) \in \mathrm{Dom}\left(W^{-1}\right) \textit{ for } t \in [t_0, \tau]\right\}.$$

2.2. Monotone additive term

In this subsection, we solve an inequality in which the constant k from Subsection 2.1 is replaced by a monotonic function. For this purpose, we introduce the following set of functions.

DEFINITION 2.8. The function $\psi \in \mathrm{C}^1(\mathbb{R}_+, \mathbb{R}_+)$ is said to be from Λ if

(i) ψ is an increasing function;

(ii) $t\psi(x) \geqslant \psi\big(tx\big)$ for $0 \leqslant t \leqslant 1$.

REMARK 2.9. Note that the functions $\psi(x) = x$ and $\psi(x) = x^n$, where $n > 1$, are from Λ.

DEFINITION 2.10. The function $\omega \in \Omega_1$ is said to be from Ω_2 if

$$\omega(tx) \geqslant t\omega(x) \quad \text{for} \quad 0 \leqslant t \leqslant 1.$$

THEOREM 2.11. *Let the following conditions be fulfilled:*

(B_1) $\phi \in \mathrm{C}([J-h, t_0], [0, \tilde{k}])$*, where* $\tilde{k} = k(t_0)$.

(B_2) $k \in \mathrm{C}([t_0, T), [1, \infty))$ *is nondecreasing.*

(B_3) $\alpha_i, \beta_j \in \mathscr{F}$ *for* $i = 1, 2, \ldots, n, \; j = 1, 2, \ldots, m$.

(B_4) $f_i, g_j \in \mathrm{C}([J, T), \mathbb{R}_+)$ *for* $i = 1, 2, \ldots, n, \; j = 1, 2, \ldots, m$.

(B_5) $\omega_i, \tilde{\omega}_j \in \Omega_2$ *for* $i = 1, 2, \ldots, n, \; j = 1, 2, \ldots, m$.

(B_6) $\psi \in \Lambda$.

(B_7) $u \in \mathrm{C}([J-h, T), \mathbb{R}_+)$ *satisfies for some* $p \geqslant 0$ *the inequalities*

$$\begin{aligned} \psi\Big(u(t)\Big) \leqslant k(t) &+ \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} f_i(s) u^p(s) \omega_i\Big(u(s)\Big)\mathrm{d}s \\ &+ \sum_{j=1}^{m} \int_{\beta_j(t_0)}^{\beta_j(t)} g_j(s) u^p(s) \tilde{\omega}_j\bigg(\max_{\xi \in [s-h, s]} u(\xi)\bigg)\mathrm{d}s, \quad t \in [t_0, T), \end{aligned} \tag{2.24}$$

$$u(t) \leqslant \phi(t), \quad t \in [J-h, t_0]. \tag{2.25}$$

Then for $t_0 \leqslant t \leqslant t_6$, *the inequality*

$$u(t) \leqslant k(t)\psi^{-1}\Big(\Psi^{-1}\Big(W^{-1}\Big(W\big(\Psi(1)\big)+A_1(t)\Big)\Big)\Big) \tag{2.26}$$

holds, where the functions Ψ *and* W *are defined by* (2.4) *and* (2.5), *respectively, and*

$$A_1(t)=\sum_{i=1}^{n}\int_{\alpha_i(t_0)}^{\alpha_i(t)} f_i(s)\big(k(s)\big)^p \mathrm{d}s+\sum_{j=1}^{m}\int_{\beta_j(t_0)}^{\beta_j(t)} g_j(s)\big(k(s)\big)^p \mathrm{d}s, \tag{2.27}$$

$$\begin{aligned} t_6=\sup\Big\{\tau\in[t_0,T): \ & W\big(\Psi(1)\big)+A_1(t)\in \mathrm{Dom}\big(W^{-1}\big),\\ & W^{-1}\Big(W\big(\Psi(1)\big)+A_1(t)\Big)\in \mathrm{Dom}\big(\Psi^{-1}\big) \ \textit{and}\\ & \Psi^{-1}\Big(W^{-1}\Big(W\big(\Psi(1)\big)+A_1(t)\Big)\Big)\in \mathrm{Dom}\big(\psi^{-1}\big) \ \textit{for} \ t\in[t_0,\tau]\Big\}. \end{aligned}$$

Proof. From (2.24), (2.25), (B_2), (B_6), and $0\leqslant \frac{1}{k(t)}\leqslant 1$, we obtain

$$\begin{aligned} \psi\left(\frac{u(t)}{k(t)}\right) \leqslant\ & 1+\sum_{i=1}^{n}\int_{\alpha_i(t_0)}^{\alpha_i(t)} f_i(s)u^p(s)\omega_i\left(\frac{u(s)}{k(s)}\right)\mathrm{d}s \\ & +\sum_{j=1}^{m}\int_{\beta_j(t_0)}^{\beta_j(t)} g_j(s)u^p(s)\tilde{\omega}_j\left(\frac{\max_{\xi\in[s-h,s]}u(\xi)}{k(s)}\right)\mathrm{d}s, \quad t\in[t_0,T), \end{aligned} \tag{2.28}$$

$$\frac{u(t)}{k(t_0)} \leqslant \frac{\phi(t)}{k(t_0)} \leqslant 1, \quad t\in[J-h,t_0]. \tag{2.29}$$

Let $s\in[\beta_j(t_0),\beta_j(T))$, where $1\leqslant j\leqslant m$ is arbitrary. From the monotonicity of the function k in $[t_0,T)$, we obtain the inequality

$$\frac{\max_{\xi\in[s-h,s]}u(\xi)}{k(s)}=\frac{u(\xi_1)}{k(s)}\leqslant\frac{u(\xi_1)}{k(\xi_1)}\leqslant\max_{\xi\in[s-h,s]}\frac{u(\xi)}{k(\xi)},$$

where $\xi_1\in[s-h,s]$. Define a function $v\in C([J-h,T),\mathbb{R}_+)$ by

$$v(t)=\begin{cases} \dfrac{u(t)}{k(t)} & \text{for} \quad t\in[t_0,T) \\ \dfrac{u(t)}{k(t_0)} & \text{for} \quad t\in[J-h,t_0]. \end{cases}$$

Then inequalities (2.28) and (2.29) can be rewritten as

$$\begin{aligned} \psi\big(v(t)\big) \leqslant\ & 1+\sum_{i=1}^{n}\int_{\alpha_i(t_0)}^{\alpha_i(t)} f_i(s)k^p(s)v^p(s)\omega_i\big(v(s)\big)\mathrm{d}s \\ & +\sum_{j=1}^{m}\int_{\beta_j(t_0)}^{\beta_j(t)} g_j(s)k^p(s)v^p(s)\tilde{\omega}_j\Big(\max_{\xi\in[s-h,s]}v(\xi)\Big)\mathrm{d}s, \quad t\in[t_0,T), \end{aligned} \tag{2.30}$$

$$v(t)\leqslant 1, \quad t\in[J-h,t_0]. \tag{2.31}$$

Theorem 2.3, applied to the inequalities (2.30) and (2.31), implies the validity of inequality (2.26). □

In the case when Theorem 2.5 is applied instead of Theorem 2.3 in the last part of the proof of Theorem 2.11, we obtain the following result.

THEOREM 2.12. *Suppose* (B_1)–(B_7) *hold. Then for* $t_0 \leqslant t \leqslant t_7$*, the inequality*

$$u(t) \leqslant k(t)\psi^{-1}\left(\Psi_1^{-1}\left(\Psi_1(1)+A_1(t)\right)\right)$$

holds, where Ψ_1 *and* A_1 *are defined by* (2.19) *and* (2.27)*, respectively, and*

$$t_7 = \sup\Bigg\{\tau \in [t_0,T): \ \Psi_1(1)+A_1(t) \in \mathrm{Dom}\left(\Psi_1^{-1}\right) \ \text{and} \\ \Psi_1^{-1}\left(\Psi_1(1)+A(t)\right) \in \mathrm{Dom}\left(\psi^{-1}\right) \ \text{for} \ t \in [t_0,\tau]\Bigg\}.$$

In the case when $p=0$ and the left part of the considered inequality is linear, i.e., $\psi(x)=x$, the results of Theorem 2.11 and Theorem 2.12 coincide, and we obtain the following result.

COROLLARY 2.13. *Suppose* (B_1)–(B_5) *hold and assume*

(B_7') $u \in C([J-h,T),\mathbb{R}_+)$ *satisfies the inequalities*

$$u(t) \leqslant k(t)+\sum_{i=1}^{n}\int_{\alpha_i(t_0)}^{\alpha_i(t)} f_i(s)\omega_i\left(u(s)\right)\mathrm{d}s \\ +\sum_{j=1}^{m}\int_{\beta_j(t_0)}^{\beta_j(t)} g_j(s)\tilde{\omega}_j\left(\max_{\xi\in[s-h,s]} u(\xi)\right)\mathrm{d}s, \quad t \in [t_0,T), \\ u(t) \leqslant \phi(t), \quad t \in [J-h,t_0].$$

Then for $t_0 \leqslant t \leqslant t_8$*, the inequality*

$$u(t) \leqslant k(t)W^{-1}\left(W(1)+A(t)\right)$$

holds, where the functions W *and* A *are defined by* (2.23) *and* (2.7)*, respectively, and*

$$t_8 = \sup\left\{\tau \in [t_0,T): \ W(1)+A(t) \in \mathrm{Dom}\left(W^{-1}\right) \ \text{for} \ t \in [t_0,\tau]\right\}.$$

2.3. Arbitrary additive term

In this subsection, we solve a nonlinear inequality in which the constant k of Subsection 2.1 is replaced by an arbitrary function. In this case, we define the following set of functions.

DEFINITION 2.14. The function $\omega \in \Omega_2$ is said to be from Ω_3 if

$$\omega(x)+\omega(y) \geqslant \omega(x+y).$$

REMARK 2.15. Note that the functions $\omega(x)=\sqrt{x}$ and $\omega(x)=x$ are from Ω_3.

THEOREM 2.16. *Let the following conditions be fulfilled:*

(C_2) $k \in C([J-h,T),\mathbb{R}_+)$.

(C_3) $\alpha_i,\beta_j \in \mathscr{F}$ *for* $i=1,2,\dots,n,\ j=1,2,\dots,m$.

(C_4) $f_i,g_j \in C([J,T),\mathbb{R}_+)$ *for* $i=1,2,\dots,n,\ j=1,2,\dots,m$.

(C_5) $\omega_i,\tilde{\omega}_j \in \Omega_3$ *for* $i=1,2,\dots,n,\ j=1,2,\dots,m$.

(C_6) $\mu \in C([t_0,T),[1,\infty))$ *is nondecreasing.*

(C_7) $u \in C([J-h,T),\mathbb{R}_+)$ *satisfies the inequalities*

$$\begin{aligned} u(t) \leqslant k(t)+\mu(t)\Bigg\{ &\sum_{i=1}^{n}\int_{\alpha_i(t_0)}^{\alpha_i(t)} f_i(s)\omega_i\Big(u(s)\Big)\mathrm{d}s \\ &+\sum_{j=1}^{m}\int_{\beta_j(t_0)}^{\beta_j(t)} g_j(s)\tilde{\omega}_j\Big(\max_{\xi\in[s-h,s]} u(\xi)\Big)\mathrm{d}s\Bigg\}, \quad t\in[t_0,T), \end{aligned} \tag{2.32}$$

$$u(t) \leqslant k(t), \quad t\in[J-h,t_0]. \tag{2.33}$$

Then for $t_0 \leqslant t \leqslant t_9$, *the inequality*

$$u(t) \leqslant k(t)+\mathscr{M}(t)e(t)W^{-1}\Big(W(1)+A_2(t)\Big) \tag{2.34}$$

holds, where the function W *is defined by* (2.23),

$$\begin{aligned} e(t) = 1 &+ \sum_{i=1}^{n}\int_{\alpha_i(t_0)}^{\alpha_i(t)} f_i(s)\omega_i\Big(k(s)\Big)\mathrm{d}s \\ &+\sum_{j=1}^{m}\int_{\beta_j(t_0)}^{\beta_j(t)} g_j(s)\tilde{\omega}_j\Big(\max_{\xi\in[s-h,s]} k(\xi)\Big)\mathrm{d}s, \quad t\in[t_0,T), \end{aligned} \tag{2.35}$$

$$A_2(t) = \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} f_i(s)\mathscr{M}(s)\mathrm{d}s + \sum_{j=1}^{m} \int_{\beta_j(t_0)}^{\beta_j(t)} g_j(s)\mathscr{M}(s)\mathrm{d}s,$$

$$\mathscr{M}(t) = \begin{cases} \mu(t) & for \quad t \in [t_0, T), \\ \mu(t_0) & for \quad t \in [J-h, t_0], \end{cases}$$

$$t_9 = \sup\Big\{\tau \in [t_0, T) : \ W(1) + A_2(t) \in \mathrm{Dom}\big(W^{-1}\big) \ for \ \ t \in [t_0, \tau]\Big\}.$$

Proof. Define a function $z : [J-h, T) \to \mathbb{R}_+$ by

$$z(t) = \begin{cases} \displaystyle\sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} f_i(s)\omega_i\big(u(s)\big)\mathrm{d}s \\ \displaystyle + \sum_{j=1}^{m} \int_{\beta_j(t_0)}^{\beta_j(t)} g_j(s)\tilde{\omega}_j\Big(\max_{\xi \in [s-h,s]} u(\xi)\Big)\mathrm{d}s, & t \in [t_0, T), \\ 0, & t \in [J-h, t_0]. \end{cases}$$

From (2.32) and the definition of the function z, we obtain for $s \in [\beta_j(t_0), \beta_j(T))$ and $1 \leqslant j \leqslant m$

$$u(t) \leqslant k(t) + \mathscr{M}(t)z(t) \quad \text{for} \quad t \in [J-h, T]. \tag{2.36}$$

Since the function $\mathscr{M}$ is nondecreasing on $[J-h, T)$, we obtain

$$\max_{\xi \in [s-h,s]} u(\xi) \leqslant \max_{\xi \in [s-h,s]} k(\xi) + \mathscr{M}(s) \max_{\xi \in [s-h,s]} z(\xi). \tag{2.37}$$

From (2.36), (2.37), (C_3), and (C_5), we get for $t \in [t_0, T)$

$$\begin{aligned} \int_{\alpha_i(t_0)}^{\alpha_i(t)} f_i(s)\omega_i\Big(u(s)\Big)\mathrm{d}s &\leqslant \int_{\alpha_i(t_0)}^{\alpha_i(t)} f_i(s)\omega_i\Big(k(s) + \mathscr{M}(s)z(s)\Big)\mathrm{d}s \\ &\leqslant \int_{\alpha_i(t_0)}^{\alpha_i(t)} f_i(s)\omega_i\Big(k(s)\Big)\mathrm{d}s + \int_{\alpha_i(t_0)}^{\alpha_i(t)} f_i(s)\mathscr{M}(s)\omega_i\Big(z(s)\Big)\mathrm{d}s \end{aligned} \tag{2.38}$$

and

$$\begin{aligned} &\int_{\beta_j(t_0)}^{\beta_j(t)} g_j(s)\tilde{\omega}_j\Big(\max_{\xi \in [s-h,s]} u(\xi)\Big)\mathrm{d}s \\ &\quad \leqslant \int_{\beta_j(t_0)}^{\beta_j(t)} g_j(s)\tilde{\omega}_j\Big(\max_{\xi \in [s-h,s]} k(\xi)\Big)\mathrm{d}s + \int_{\beta_j(t_0)}^{\beta_j(t)} g_j(s)\mathscr{M}(s)\tilde{\omega}_j\Big(\max_{\xi \in [s-h,s]} z(\xi)\Big)\mathrm{d}s. \end{aligned} \tag{2.39}$$

From the definition of the function z and (2.36), (2.38), and (2.39), it follows that

$$\begin{aligned} z(t) \leqslant e(t) &+ \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} f_i(s)\mathscr{M}(s)\omega_i\Big(z(s)\Big)\mathrm{d}s \\ &+ \sum_{j=1}^{m} \int_{\beta_j(t_0)}^{\beta_j(t)} g_j(s)\mathscr{M}(s)\tilde{\omega}_j\Big(\max_{\xi \in [s-h,s]} z(\xi)\Big)\mathrm{d}s, \quad t \in [t_0, T), \end{aligned} \tag{2.40}$$

$$z(t) \leqslant 0, \quad t \in [J-h, t_0], \tag{2.41}$$

where the function e is defined by (2.35). Note that $e:[t_0,T)\to[1,\infty)$ is nondecreasing and $e(t_0)=1$. Corollary 2.13 applied to inequalities (2.40) and (2.41) implies the required inequality (2.34). $\square$

3. Applications

In this section, we consider the differential equation with "maxima"

$$px^{p-1}x' = F\Big(t, x(t), \max_{s\in[\sigma(t),\tau(t)]} x(s)\Big) \quad \text{for} \quad t\in[t_0,T), \tag{3.1}$$

with the initial condition

$$x(t)=\varphi(t) \quad \text{for} \quad t\in[\tau(t_0)-h,t_0], \tag{3.2}$$

where $\varphi:[\tau(t_0)-h,t_0]\to\mathbb{R}$, $F:[t_0,T)\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, and p is a natural number.

THEOREM 3.1. (Upper bound) *Let the following conditions be fulfilled:*

(H_1) $\varphi\in C([\tau(t_0)-h,t_0],\mathbb{R})$.

(H_2) $\tau,\sigma\in\mathscr{F}$ *and there exists a constant* h *with* $0<\tau(t)-\sigma(t)\leqslant h$ *for* $t\geqslant t_0$.

(H_3) $F\in C([t_0,T)\times\mathbb{R}\times\mathbb{R},\mathbb{R})$ *satisfies*

$$\big|F(t,u,v)\big| \leqslant Q(t)\big|u\big|^q + R(t)\big|v\big|^q \quad \textit{for} \quad u,v\in\mathbb{R} \quad \textit{and} \quad t\geqslant t_0,$$

where $Q,R\in C([t_0,T),\mathbb{R}_+)$ *and* $q\in(0,p)$.

(H_4) $x:[\tau(t_0)-h,T)\to\mathbb{R}$ *is a solution of the initial value problem* (3.1), (3.2).

Then x *satisfies the inequality*

$$\big|x(t)\big| \leqslant \sqrt[p-q]{M^{\frac{p-q}{p}} + \frac{p-q}{p}\int_{t_0}^{t}\Big[Q(s)+R(s)\Big]\mathrm{d}s} \quad \textit{for} \quad t\in[t_0,T), \tag{3.3}$$

where $M=\sup_{s\in[\tau(t_0)-h,t_0]}\Big(\varphi(s)\Big)^p$.

Proof. The function x satisfies the integral problem

$$\begin{aligned}\big(x(t)\big)^p &= \big(\varphi(t_0)\big)^p + \int_{t_0}^{t} F\Big(s, x(s), \max_{\xi\in[\sigma(s),\tau(s)]} x(\xi)\Big)\mathrm{d}s \quad \text{for} \quad t\in[t_0,T),\\ x(t) &= \varphi(t) \quad \text{for} \quad t\in[\tau(t_0)-h,t_0].\end{aligned}$$

Then for the norm of the solution x, we obtain

$$\begin{aligned}\big|x(t)\big|^p &\leqslant \big|\varphi(t_0)\big|^p + \int_{t_0}^t \Big|F\Big(s, x(s), \max_{\xi\in[\sigma(s),\tau(s)]} x(\xi)\Big)\Big|\mathrm{d}s\\ &\leqslant \big|\varphi(t_0)\big|^p + \int_{t_0}^t \Big(Q(s)\big|x(s)\big|^q + R(s)\Big|\max_{\xi\in[\sigma(s),\tau(s)]} x(\xi)\Big|^q\Big)\mathrm{d}s\\ &\leqslant \big|\varphi(t_0)\big|^p + \int_{t_0}^t Q(s)\big|x(s)\big|^q\mathrm{d}s\\ &\quad + \int_{t_0}^t R(s)\Big(\max_{\xi\in[\sigma(s),\tau(s)]}\big|x(\xi)\big|\Big)^q\mathrm{d}s \quad \text{for} \quad t\in[t_0,T),\end{aligned} \tag{3.4}$$

$$\big|x(t)\big| = \big|\varphi(t)\big| \quad \text{for} \quad t\in[\tau(t_0)-h, t_0]. \tag{3.5}$$

Change the variable $s=\tau^{-1}(\eta)$ in the second integral of (3.4), use the inequality $\max_{\xi\in[\sigma(s),\tau(s)]}|x(\xi)| \leqslant \max_{\xi\in[\tau(s)-h,\tau(s)]}|x(\xi)|$ for $s\in[t_0,T)$ that follows from (H_2), and obtain

$$\begin{aligned}\big|x(t)\big|^p &\leqslant \big|\varphi(t_0)\big|^p + \int_{t_0}^t Q(s)\big|x(s)\big|^q\mathrm{d}s\\ &\quad + \int_{\tau(t_0)}^{\tau(t)} R(\tau^{-1}(\eta))(\tau^{-1})'(\eta)\Big(\max_{\xi\in[\eta-h,\eta]}\big|x(\xi)\big|\Big)^q\mathrm{d}\eta.\end{aligned} \tag{3.6}$$

Note that the conditions of Corollary 2.6 are satisfied for

$$\begin{gathered} u(t)=|x(t)|, \quad n=1, \quad \alpha_1(t)=t, \quad m=1, \quad \beta_1=\tau,\\ k=|\varphi(t_0)|^p, \quad f_1=Q, \quad g_1=(R\circ\tau^{-1})(\tau^{-1})' \text{ on } [\tau(t_0),T),\\ \psi(x)=x^p, \quad \psi^{-1}(x)=\sqrt[p]{x}, \quad \mathrm{Dom}(\psi^{-1})=\mathbb{R}_+,\\ \omega_1(x)=\tilde\omega_1(x)=x^q, \quad W(r)=\int_0^r s^{-\frac{q}{p}}\mathrm{d}s=\frac{p}{p-q}r^{\frac{p-q}{p}},\\ W^{-1}(r)=\sqrt[p-q]{\Big(\frac{p-q}{p}r\Big)^p}, \quad \mathrm{Dom}(W^{-1})=\mathbb{R}_+.\end{gathered}$$

According to Corollary 2.6, from (3.6) and (3.5), we obtain (3.3). □

COROLLARY 3.2. *Let* $\varphi(t)\equiv 0$. *Suppose* (H_2)–(H_4) *hold. Then the solution* x *of the initial value problem* (3.1), (3.2) *satisfies the inequality*

$$\big|x(t)\big| \leqslant \sqrt[p-q]{\frac{p-q}{p}\int_{t_0}^t \Big[Q(s)+R(s)\Big]\mathrm{d}s} \quad \textit{for} \quad t\in[t_0,T).$$

REMARK 3.3. Note that in the case of $p=q=1$ in (H_3), we use the functions $\psi(x)=\psi^{-1}(x)=x$ and $W(r)=\ln r$, $W^{-1}(r)=e^r$, apply Corollary 2.6, and obtain

$$|x(t)| \leqslant e^{\ln M+\int_{t_0}^t\Big[Q(s)+R(s)\Big]\mathrm{d}s} = Me^{\int_{t_0}^t\Big[Q(s)+R(s)\Big]\mathrm{d}s} \quad \text{for} \quad t\in[t_0,T),$$

which gives us the known result about uniqueness of the solution of a first-order differential equation with Lipschitz-continuous right-hand side.

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