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THE CONVEXITY AND THE CONCAVITY DERIVED FROM NEWTON'S INEQUALITY

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Abstract. By Newton's inequality, a sequence $\{a_i\}_{i=0}^n$ of nonnegative real numbers is unimodal if its generating function $\sum_{i=0}^n a_i x^i$ has only real zeros. This paper is devoted to show that there exist two indices s and t with $s \leqslant t$, such that $a_0, a_1, \ldots, a_{s-1}, a_s$ and $a_t, a_{t+1}, \ldots, a_n$ are convex, while $a_{s-1}, a_s, \ldots, a_t, a_{t+1}$ is concave.

1. Introduction

Let $a_0, a_1, a_2, \ldots$ be a sequence of nonnegative real numbers. It is called *unimodal* if $a_0 \leqslant a_1 \leqslant \cdots \leqslant a_{m-1} \leqslant a_m \geqslant a_{m+1} \geqslant \cdots$ for certain m, where the index m is called *mode*. The sequence $\{a_i\}_{i\geqslant 0}$ is called *log-concave* if for all $i \geqslant 1$, $a_{i-1}a_{i+1} \leqslant a_i^2$ and called *strictly log-concave* if for all $i \geqslant 1$, $a_{i-1}a_{i+1} < a_i^2$ ([6]). It is easy to verify that if a sequence of positive numbers is strictly log-concave, then it is unimodal and has at most two consecutive modes. A sequence $\{a_i\}_{i\geqslant 0}$ of nonnegative real numbers is called *concave*(resp. *convex*) if for $i \geqslant 1$, $a_{i-1} + a_{i+1} \leqslant 2a_i$(resp. $a_{i-1} + a_{i+1} \geqslant 2a_i$). By the arithmetic-geometric mean inequality, the concavity implies the log-concavity. Unimodality problems often arise in many branches of mathematics. See articles [2, 3, 7] and references therein.

A well-known result of Newton states the following (see, e.g., [4]):

NEWTON'S INEQUALITY. *If all the zeros of a polynomial $f(x) = \sum_{i=0}^n a_i x^i$ are real, then the coefficients of the polynomial $f(x)$ satisfy*

$$a_i^2 \geqslant a_{i-1}a_{i+1}\left(1+\frac{1}{i}\right)\left(1+\frac{1}{n-i}\right), \quad 1 \leqslant i \leqslant n-1.$$

It should be mentioned that the coefficients of $f(x)$ need not to be nonnegative. By Newton's inequality, a sequence $\{a_i\}_{i=0}^n$ of nonnegative real numbers is strictly log-concave and is therefore unimodal with at most two modes if its generating function $\sum_{i=0}^n a_i x^i$ has only real zeros. That is, the coefficients of a polynomial with only non-positive zeros form a bell-shaped sequence. This paper is devoted to study the convexity and the concavity derived from Newton's inequality, which is a further description of the previous bell-shaped sequence. Our main result is as follows.

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THEOREM 1. *Let $a_0, a_1, \ldots, a_n$ be a sequence of nonnegative real numbers. Suppose that its generating function $\sum_{i=0}^{n} a_i x^i$ has only real zeros. Then there exist two indices s and t with $s \leqslant t$, such that $a_0, a_1, \ldots, a_{s-1}, a_s$ and $a_t, a_{t+1}, \ldots, a_n$ are convex, while $a_{s-1}, a_s, \ldots, a_t, a_{t+1}$ is concave.*

The proof of Theorem 1 will be given in Section 2, where we see that every mode of the sequence $\{a_i\}_{i=0}^{n}$ lies between s and t. Section 3 gives some remarks.

2. Proof of Theorem 1

Assume first that the sequence $\{a_i\}_{i=0}^{n}$ has only one mode m. Clearly, $a_0 < a_1 < \cdots < a_{m-1} < a_m > a_{m+1} > \cdots > a_n$ and $a_{m-1} + a_{m+1} \leqslant 2a_m$. For $i \neq m$, any three adjacent terms a_{i-1}, a_i, a_{i+1} satisfy either $a_{i-1} + a_{i+1} \leqslant 2a_i$ or $a_{i-1} + a_{i+1} \geqslant 2a_i$. Now we will show that the sequence $\{a_i\}_{i=0}^{n}$ changes the convexity/concavity at most once on each monotonicity interval.

The increasing segment: Suppose that $a_{i-2} < a_{i-1} < a_i < a_{i+1}$ and $a_i + a_{i-2} \leqslant 2a_{i-1}$, where $2 \leqslant i \leqslant m-1$. Now define $g(x) = f(x)(1-x)$, i.e.,

$$g(x) = a_0 + (a_1 - a_0)x + (a_2 - a_1)x^2 + \cdots + (a_n - a_{n-1})x^n - a_n x^{n+1}.$$

Using Newton's inequality, we have

$$(a_{i-1} - a_{i-2})(a_{i+1} - a_i) < (a_i - a_{i-1})^2 \tag{1}$$

since $g(x)$ has only real zeros. For $a_{i-2} < a_{i-1} < a_i < a_{i+1}$ and $a_i + a_{i-2} \leqslant 2a_{i-1}$, we have by (1)

$$\frac{a_{i+1} - a_i}{a_i - a_{i-1}} < \frac{a_i - a_{i-1}}{a_{i-1} - a_{i-2}} \leqslant 1,$$

which implies $a_{i-1} + a_{i+1} < 2a_i$. Therefore the subsequence $a_{s-1}, a_s, \ldots, a_m, a_{m+1}$ is concave, where

$$s = \min\{i : a_{i-1} + a_{i+1} \leqslant 2a_i \ \text{ and } \ 1 \leqslant i \leqslant m\}.$$

On the other hand, for $j = s-1$, $a_{j-1} + a_{j+1} \geqslant 2a_j$. Then by (1),

$$1 \leqslant \frac{a_{j+1} - a_j}{a_j - a_{j-1}} < \frac{a_j - a_{j-1}}{a_{j-1} - a_{j-2}},$$

which implies $a_{j-2} + a_j > 2a_{j-1}$. Repeating the previous process, we get $a_{i-1} + a_{i+1} \geqslant 2a_i$ for $1 \leqslant i \leqslant s-1$, i.e., $a_0, a_1, \ldots, a_{s-1}, a_s$ is convex. So there exists an index s such that $a_0, a_1, \ldots, a_{s-1}, a_s$ is convex and $a_{s-1}, a_s, \ldots, a_m, a_{m+1}$ is concave.

The decreasing segment: Suppose that $a_{i-2} > a_{i-1} > a_i > a_{i+1}$ and $a_{i-2} + a_i \geqslant 2a_{i-1}$, where $m+2 \leqslant i \leqslant n-1$. Define $h(x) = f(x)(x-1)$. It follows that

$$h(x) = -a_0 + (a_0 - a_1)x + (a_1 - a_2)x^2 + \cdots + (a_{n-1} - a_n)x^n + a_n x^{n+1}$$

and

$$(a_{i-2} - a_{i-1})(a_i - a_{i+1}) < (a_{i-1} - a_i)^2. \tag{2}$$

Since $a_{i-2} > a_{i-1} > a_i > a_{i+1}$ and $a_{i-2} + a_i \geqslant 2a_{i-1}$, we have by (2)

$$1 \leqslant \frac{a_{i-2} - a_{i-1}}{a_{i-1} - a_i} < \frac{a_{i-1} - a_i}{a_i - a_{i+1}},$$

which implies $a_{i-1} + a_{i+1} > 2a_i$. Hence the subsequence $a_{t-1}, a_t, \ldots, a_{n-1}, a_n$ is convex, where

$$t = \min\{i : a_{i-1} + a_{i+1} \geqslant 2a_i \ \text{and} \ m+1 \leqslant i \leqslant n-1\}.$$

On the other hand, for $j = t-1$, $a_{j-1} + a_{j+1} \leqslant 2a_j$. Then by (2),

$$\frac{a_{j-2} - a_{j-1}}{a_{j-1} - a_j} < \frac{a_{j-1} - a_j}{a_j - a_{j+1}} \leqslant 1,$$

which implies $a_{j-2} + a_j < 2a_{j-1}$. Repeating the previous process, we get $a_{i-1} + a_{i+1} \leqslant 2a_i$ for $m \leqslant i \leqslant t-1$, i.e., $a_{m-1}, a_m, \ldots, a_{t-1}, a_t$ is concave. Thus $a_{m-1}, a_m, \ldots, a_{t-1}, a_t$ is concave and $a_{t-1}, a_t, \ldots, a_n$ is convex.

In summary, there exist two indices s, t such that $a_0, a_1, \ldots, a_s, a_{s+1}$ and $a_{t-1}, a_t, \ldots, a_n$ are convex, while $a_s, a_{s+1}, \ldots, a_{m-1}, a_m, a_{m+1}, \ldots, a_{t-1}, a_t$ is concave.

For the case that the sequence $\{a_i\}_{i=0}^n$ has two modes m and $m+1$, we have $a_{m-1} + a_{m+1} \leqslant 2a_m$ and $a_m + a_{m+2} \leqslant 2a_{m+1}$. Then using Newton's inequality similarly, we find two indices s^* and t^* such that: $a_0, a_1, \ldots, a_{s^*-1}, a_{s^*}$ and $a_{t^*-1}, a_{t^*}, \ldots, a_n$ are convex, while $a_{s^*-1}, a_{s^*}, \ldots, a_m, a_{m+1}, \ldots, a_{t^*-1}, a_{t^*}$ is concave, where

$$\begin{aligned} s^* &= \min\{i : a_{i-1} + a_{i+1} \leqslant 2a_i \ \text{and} \ 1 \leqslant i \leqslant m\}, \\ t^* &= \min\{i : a_{i-1} + a_{i+1} \geqslant 2a_i \ \text{and} \ m+2 \leqslant i \leqslant n-1\}. \end{aligned}$$

3. Concluding remarks

We have shown that the nonnegative sequences whose generating functions have only real zeros can change their convexity/concavity at most once on each monotonicity interval. For example, the sequences: $\{1,3,1\}$ is only concave, $\{1,10,20\}$ is only convex, and $\{6,41,89,60\}$ is first concave and then convex. A further example is the binomial sequence $\{\binom{n}{i}\}_{i=0}^n$. Its generating function $\sum_{i=0}^n \binom{n}{i} x^i = (1+x)^n$. Then by Theorem 1, the subsequences $\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{\lfloor \frac{n-\sqrt{n+2}}{2} \rfloor}, \binom{n}{\lfloor \frac{n-\sqrt{n+2}}{2} \rfloor + 1}$ and $\binom{n}{\lceil \frac{n+\sqrt{n+2}}{2} \rceil - 1}, \binom{n}{\lceil \frac{n+\sqrt{n+2}}{2} \rceil}, \ldots, \binom{n}{n-1}, \binom{n}{n}$ are convex, while the subsequence $\binom{n}{\lceil \frac{n-\sqrt{n+2}}{2} \rceil - 1}, \binom{n}{\lceil \frac{n-\sqrt{n+2}}{2} \rceil}, \ldots, \binom{n}{\lfloor \frac{n+\sqrt{n+2}}{2} \rfloor}, \binom{n}{\lfloor \frac{n+\sqrt{n+2}}{2} \rfloor + 1}$ is concave. Here the "inflection points" about convexity/concavity (i.e., the indices in Theorem 1) are obtained by noting that

$$2\binom{n}{i} - \binom{n}{i-1} - \binom{n}{i+1} = \frac{n!}{(i+1)!(n-i+1)!}(-4i^2 + 4ni - n^2 + n + 2)$$

and the function $H(i) = -4i^2 + 4ni - n^2 + n + 2$ has two zeros $\frac{n-\sqrt{n+2}}{2}$ and $\frac{n+\sqrt{n+2}}{2}$.

Now let $\{a_n(i)\}_{i=1}^n$ be a triangular array of nonnegative numbers, $n = 1, 2, \ldots$. Denote by X_n a random variable which is defined as

$$P(X_n = i) = p_n(i) = \frac{a_n(i)}{\sum_{j=1}^n a_n(j)}$$

and denote $g_n(x) = \sum_{i=1}^{n} p_n(i)x^i$. Let $\tilde{X}_n = (X_n - \mathrm{E}(X_n))/\sqrt{\mathrm{Var}(X_n)}$, where $\mathrm{E}(X_n)$ and $\mathrm{Var}(X_n)$ represent the mean and the variance of a random variable X_n respectively. A well-known result due to Bender [1] states that if $g_n(x)$ has only real zeros for all n, and $\sqrt{\mathrm{Var}(X_n)} \to \infty$ as n tends to infinity, then $\tilde{X}_n \to \mathscr{N}(0,1)$. For example, the rows of the triangular array of the Stirling numbers of the second kind is asymptotically normal([5]). Note that the standard normal distribution $\mathscr{N}(0,1)$ has the probability density function $f(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$. It is easy to see that the second derivative function $f''(x) < 0$ on $(-\infty,-1)$ and $(1,+\infty)$, while $f''(x) > 0$ on $(-1,1)$. Hence the inflection points about convexity/concavity of $\mathscr{N}(0,1)$ is -1 and 1. This implies that as n tends to infinity, the inflection points of the rows of a triangular array satisfying Bender's assumption is asymptotically fixed. An exercise left to the readers is to consider the inflection points of the binomial distribution $\{\binom{n}{i}p^i(1-p)^{n-i}\}_{i=0}^{n} (0 < p < 1)$.

At the end, we point out that Theorem 1 does not hold in general if the sequence $\{a_i\}_{i=0}^{n}$ is strictly log-concave only. For example, the strictly log-concave sequence $\{1,3,7,10,14\}$, whose generating function does not have real zeros only, is first convex, then concave and finally convex on its increasing interval.

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