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OPERATOR INEQUALITIES FOR J–CONTRACTIONS

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Abstract. A selfadjoint involutive matrix J endows $\mathbb{C}^n$ with an indefinite inner product $[\cdot,\cdot]$ given by $[x,y] = \langle Jx,y\rangle$, $x,y \in \mathbb{C}^n$. Characterizations of the J-chaotic order $\mathrm{Log}(A) \geqslant^J \mathrm{Log}(B)$ are presented for J-selfadjoint matrices A,B with positive eigenvalues, in terms of operator functions involving the α-power mean and the J-relative entropy. An indefinite *complete form* of the Furuta inequality and some exponential operator inequalities for J-selfadjoint matrices are also obtained. The parallelism between the inequalities in Hilbert spaces and the corresponding indefinite versions in Krein spaces is pointed out.

1. Introduction

Let M_n denote the algebra of $n \times n$ complex matrices. For a selfadjoint involution $J \in M_n$, that is, $J = J^*$ and $J^2 = I_n$, we consider $\mathbb{C}^n$ with a Krein space structure induced by the indefinite inner product $[x,y] = \langle Jx,y\rangle$, $x,y \in \mathbb{C}^n$, where $\langle\cdot,\cdot\rangle$ denotes the standard inner product in $\mathbb{C}^n$. For $A \in M_n$, the J-*adjoint* of A is the matrix $A^{[*]}$ defined by $[Ax,y] = [x,A^{[*]}y]$, $x,y \in \mathbb{C}^n$, or equivalently, $A^{[*]} = JA^*J$, and A is said to be J-*selfadjoint* if $A = A^{[*]}$. These matrices appear in several problems of relativistic quantum mechanics and quantum physics, and recently inequalities involving them deserved the attention of researchers [2, 4, 5, 15, 16]. The theory of inequalities for selfadjoint matrices has a long and rich history. In contrast with selfadjoint matrices, the eigenvalues of J-selfadjoint matrices may not be real [14], and so attention should focus on classes of these matrices with real spectra.

As usual, $A \geqslant 0$ means that the selfadjoint matrix A is positive semidefinite. For J-selfadjoint matrices $A,B \in M_n$, we define $A \geqslant^J B$ by $[Ax,x] \geqslant [Bx,x]$, $x \in \mathbb{C}^n$, which means that $J(A-B) \geqslant 0$. If $A \in M_n$ is J-selfadjoint and $I_n \geqslant^J A$, then all the eigenvalues of A are real, because $I_n - A$ is the product of the selfadjoint involution J and a positive semidefinite matrix. A matrix $A \in M_n$ is called a J-*contraction* if $[x,x] \geqslant [Ax,Ax]$, for all $x \in \mathbb{C}^n$, that is, $I_n \geqslant^J A^{[*]}A$, and in this case all the eigenvalues of $A^{[*]}A$ are nonnegative [3]. Analogously, $A \in M_n$ is a J-*expansion* if $A^{[*]}A \geqslant^J I_n$. We observe that the results throughout this note stated for J-contractions have analogous counterparts for J-expansions.

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Let f be a real valued function defined on an open (finite or infinite) interval $\mathscr{I} \subseteq \mathbb{R}$. Such a continuous function f is said to be *operator monotone* if $A \geqslant B$ implies $f(A) \geqslant f(B)$ for any $A,B \in M_n$ with spectra in $\mathscr{I}$, $n \in \mathbb{N}$. If f is analytic on $\mathscr{I}$ and A is a J-selfadjoint matrix with real eigenvalues on $\mathscr{I}$, then $f(A)$ is defined by the Dunford-Riesz integral

$$f(A) = \frac{1}{2\pi i}\int_{\Gamma} f(\xi)(\xi I_n - A)^{-1} d\xi,$$

where Γ is a closed rectifiable contour in the domain of analytic continuation of f, surrounding positively the spectrum of A, Further, $f(A)$ is J-selfadjoint.

Ando [2] obtained the following result concerning operator monotone functions.

THEOREM 1.1. *Let $A,B \in M_n$ be J-selfadjoint matrices with eigenvalues in an open real interval $\mathscr{I}$. Then $A \geqslant^J B$ implies $f(A) \geqslant^J f(B)$ for any operator monotone function f defined on $\mathscr{I}$.*

Let $0 \leqslant \alpha \leqslant 1$. The Löwner-Heinz inequality [11, 13] for selfadjoint operators on a Hilbert space states that $A \geqslant B \geqslant 0$ implies $A^\alpha \geqslant B^\alpha$ and it has a famous extension: the Furuta inequality [8, 10]. Since $f(t) = t^\alpha$ is an operator monotone function on $(0,+\infty)$, it follows by Theorem 1.1 that $A \geqslant^J B$ implies $A^\alpha \geqslant^J B^\alpha$, whenever $A,B \in M_n$ are J-selfadjoint matrices with positive eigenvalues. The same result holds if the eigenvalues of A,B are nonnegative under the additional assumption $I_n \geqslant^J A \geqslant^J B$. In this case, the J-selfadjoint powers A^α, B^α are well defined and $I_n \geqslant^J A^\alpha \geqslant^J B^\alpha$ [15]. This is the Löwner inequality of indefinite type, at first obtained by Ando [1] for $\alpha = \frac{1}{2}$. Motivated by these results, the Furuta inequality of indefinite type [4, 15] was established as follows:

THEOREM 1.2. *Let $A,B \in M_n$ be J-selfadjoint with nonnegative eigenvalues and $\mu I_n \geqslant^J A \geqslant^J B$ for some $\mu > 0$. For each $r \geqslant 0$, the following inequalities hold*

$$\left(A^{\frac{r}{2}} A^p A^{\frac{r}{2}}\right)^{\frac{1}{q}} \geqslant^J \left(A^{\frac{r}{2}} B^p A^{\frac{r}{2}}\right)^{\frac{1}{q}}, \tag{1}$$

$$\left(B^{\frac{r}{2}} A^p B^{\frac{r}{2}}\right)^{\frac{1}{q}} \geqslant^J \left(B^{\frac{r}{2}} B^p B^{\frac{r}{2}}\right)^{\frac{1}{q}}, \tag{2}$$

for all $p \geqslant 0$ and $q \geqslant 1$ with $(1+r)q \geqslant p+r$.

The Löwner inequality of indefinite type is recovered by Theorem 1.2 in the particular case $r = 0$. If $q = \frac{p+r}{1+r}$ and $p > 1$ in Theorem 1.2, we obtain the so-called *essential part* of the indefinite Furuta inequality, because the indefinite Löwner inequality implies the remaining cases $q > \frac{p+r}{1+r}$ and $0 \leqslant p \leqslant 1$. The optimal domain on p,q,r for the validity of the Furuta inequality of indefinite type is an open problem (for the definite case see [17]). For J-selfadjoint matrices $A,B \in M_n$ with positive eigenvalues, the *J-chaotic order* is defined by $\operatorname{Log}(A) \geqslant^J \operatorname{Log}(B)$, where Log denotes the principal branch of the logarithm function. Since $\operatorname{Log}(t)$ is an operator monotone function on $(0,+\infty)$, we conclude by Theorem 1.1 that the J-chaotic order $\operatorname{Log}(A) \geqslant^J \operatorname{Log}(B)$

is weaker than the usual J-order $A \geqslant^J B$. Sano [16] obtained the following useful characterization of the J-chaotic order.

THEOREM 1.3. *Let $A,B \in M_n$ be J-selfadjoint matrices with positive eigenvalues such that $I_n \geqslant^J A$ and $I_n \geqslant^J B$. Then* $\mathrm{Log}(A) \geqslant^J \mathrm{Log}(B)$ *if and only if* $A^r \geqslant^J \left(A^{\frac{r}{2}} B^p A^{\frac{r}{2}}\right)^{\frac{r}{p+r}}$ *for all $p > 0$ and $r > 0$.*

The aim of this note is to present a survey of indefinite versions of well-known inequalities in the context of Hilbert spaces. In Section 2, some operator functions associated with the Furuta inequality of indefinite type are considered and additional characterizations of the J-chaotic order are presented. In Section 3, an indefinite *complete form* of the Furuta inequality is obtained. In Section 4, exponential operator inequalities are derived, being some of them indefinite variants of results obtained by Ando [1] and Uchiyama [18]. The proofs combine techniques established in the Hilbert space setting and are included for the sake of completeness. The specificities inherent to Krein spaces are emphasized.

2. Operator functions related to the indefinite Furuta inequality

Let $A,B \in M_n$ be J-selfadjoint matrices, such that $A \geqslant^J B$. If A,B have nonnegative eigenvalues and A is invertible, then $B^{\frac{1}{2}} A^{-\frac{1}{2}}$ is a J-contraction, because $I_n \geqslant^J A^{-\frac{1}{2}} B A^{-\frac{1}{2}}$. Then all the eigenvalues of $A^{-\frac{1}{2}} B A^{-\frac{1}{2}}$ are nonnegative and the J-selfadjoint power $\left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right)^{\alpha}$ is well defined for $0 \leqslant \alpha \leqslant 1$. Under these assumptions, the α*-power mean* of A and B is defined by

$$A \sharp_\alpha B \;=\; A^{\frac{1}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right)^{\alpha} A^{\frac{1}{2}},$$

and $A \geqslant^J A \sharp_\alpha B$ holds. If $AB = BA$, then $A \sharp_\alpha B = A^{1-\alpha} B^{\alpha}$. We remark that $\sharp_\alpha$ is positively homogeneous, that is, $(\mu A) \sharp_\alpha (\mu B) = \mu\,(A \sharp_\alpha B)$ for any $\mu > 0$ and if B is also invertible, then $(A \sharp_\alpha B)^{-1} = A^{-1} \sharp_\alpha B^{-1}$. If A,B have positive eigenvalues, the *J-relative entropy of A and B* is defined by

$$S(A|B) = A^{\frac{1}{2}} \,\mathrm{Log}\left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right) A^{\frac{1}{2}}.$$

Since $I_n \geqslant^J A^{-\frac{1}{2}} B A^{-\frac{1}{2}}$, it is easy to see that $0 \geqslant^J S(A|B)$. Moreover, $S(A|B) = 0$ if and only if $A = B$. This concept introduced in [4] extends the concept of relative operator entropy for positive definite operators due to Fujii and Kamei [6].

The following lemmas will be used throughout this section.

LEMMA 2.1. [4] *Let $A,B \in M_n$ be J-selfadjoint matrices. Then $X^{[*]} A X \geqslant^J X^{[*]} B X$ for all $X \in M_n$ if and only if $A \geqslant^J B$.*

LEMMA 2.2. [15] *If $A \in M_n$ is a J-selfadjoint matrix with nonnegative eigenvalues and $I_n \geqslant^J A$, then A^{λ} is well defined and $I_n \geqslant^J A^{\lambda}$ for all $\lambda > 0$.*

From Theorem 1.1 and the fact that $f(t) = -\frac{1}{t}$ is an operator monotone function on $(0, +\infty)$, the following result holds (see [15] for another proof).

LEMMA 2.3. *If $A, B \in M_n$ are J-selfadjoint with positive eigenvalues and $A \geqslant^J B$, then $B^{-1} \geqslant^J A^{-1}$.*

LEMMA 2.4. [15] *Let $A, B \in M_n$ be J-selfadjoint matrices with positive eigenvalues and $I_n \geqslant^J A$, $I_n \geqslant^J B$. Then*

$$(ABA)^{\lambda} = AB^{\frac{1}{2}} \left(B^{\frac{1}{2}} A^2 B^{\frac{1}{2}} \right)^{\lambda - 1} B^{\frac{1}{2}} A, \qquad \lambda \in \mathbb{R}.$$

The next proposition provides additional properties of the α-power mean, namely the effect of interchanging A and B, and the J-*monotonicity* in the second variable.

PROPOSITION 2.1. *Let $A, B, C \in M_n$ be J-selfadjoint matrices with positive eigenvalues and $0 \leqslant \alpha \leqslant 1$.*

(i) *If $A \geqslant^J \mu I_n \geqslant^J B$ for some $\mu > 0$, then $A \natural_{\alpha} B = B \natural_{1-\alpha} A$.*

(ii) *If $A \geqslant^J B$ and $A \geqslant^J C$, then $B \geqslant^J C$ implies $A \natural_{\alpha} B \geqslant^J A \natural_{\alpha} C$.*

Proof. (i) Without loss of generality, let $\mu = 1$. Otherwise, replace A, B respectively by $\frac{1}{\mu} A$, $\frac{1}{\mu} B$. By Lemmas 2.2 and 2.3, we have $I_n \geqslant^J A^{-\frac{1}{2}}$. Applying Lemma 2.4 to the J-selfadjoint matrices $A^{-\frac{1}{2}}, B$, we find

$$\left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right)^{\alpha} = A^{-\frac{1}{2}} B^{\frac{1}{2}} \left(B^{\frac{1}{2}} A^{-1} B^{\frac{1}{2}} \right)^{\alpha - 1} B^{\frac{1}{2}} A^{-\frac{1}{2}} = A^{-\frac{1}{2}} \left(B \natural_{1-\alpha} A \right) A^{-\frac{1}{2}},$$

which is equivalent to $A \natural_{\alpha} B = B \natural_{1-\alpha} A$.

(ii) The result follows from Lemma 2.1 and the Löwner inequality of indefinite type. □

The *essential part* of the indefinite Furuta inequality can be formulated in terms of the α-power mean as follows. If $A, B \in M_n$ are J-selfadjoint with positive eigenvalues, such that $\mu I_n \geqslant^J A \geqslant^J B$ for some $\mu > 0$, then

$$A \geqslant^J A^{-r} \natural_{\frac{1+r}{p+r}} B^p \qquad \text{and} \qquad B^{-r} \natural_{\frac{1+r}{p+r}} A^p \geqslant^J B$$

for all $p \geqslant 1$ and $r \geqslant 0$. A real valued continuous function f defined on a real interval $\mathscr{I}$ is said to be J-*increasing* if $f(r) \geqslant^J f(s)$ whenever $r \geqslant s$. Analogously, f is said to be J-*decreasing* if $f(r) \geqslant^J f(s)$ whenever $r \leqslant s$. The characterizations of the J-chaotic order in the next theorem have been stated in the context of Hilbert spaces [9, 10].

THEOREM 2.1. *Let $A, B \in M_n$ be J-selfadjoint matrices with positive eigenvalues, such that $\mu I_n \geqslant^J A$, $\mu I_n \geqslant^J B$ for some $\mu > 0$. Then the following statements are mutually equivalent:*

(i) $\operatorname{Log}(A) \geqslant^J \operatorname{Log}(B)$;

(ii) $I_n \geqslant^J A^{-r} \natural_{\frac{r}{p+r}} B^p$ *for all* $p \geqslant 0$ *and* $r \geqslant 0$;

(iii) $B^{-r} \natural_{\frac{r}{p+r}} A^p \geqslant^J I_n$ *for all* $p \geqslant 0$ *and* $r \geqslant 0$;

(iv) *For any fixed* $t \geqslant 0$, $F_t(p,r) = A^{-r} \natural_{\frac{t+r}{p+r}} B^p$ *is a J-decreasing function of both* $p \geqslant t$ *and* $r \geqslant 0$;

(v) $B^t \geqslant^J A^{-r} \natural_{\frac{t+r}{p+r}} B^p$ *for all* $p \geqslant t \geqslant 0$ *and* $r \geqslant 0$;

(vi) *For any fixed* $t \geqslant 0$ *and* $r \geqslant 0$, $H_{t,r}(p) = \frac{t+r}{p+r} S(A^{-r}|B^p)$ *is a J-decreasing function of* $p \geqslant t$;

(vii) *For any fixed* $t \geqslant 0$, $G_t(p,r) = B^{-r} \natural_{\frac{t+r}{p+r}} A^p$ *is a J-increasing function of both* $p \geqslant t$ *and* $r \geqslant 0$;

(viii) $B^{-r} \natural_{\frac{t+r}{p+r}} A^p \geqslant^J A^t$ *for all* $p \geqslant t \geqslant 0$ *and* $r \geqslant 0$;

(ix) *For any fixed* $t \geqslant 0$ *and* $r \geqslant 0$, $K_{t,r}(p) = \frac{t+r}{p+r} S(B^{-r}|A^p)$ *is a J-increasing function of* $p \geqslant t$;

(x) $\left(A^{\frac{r}{2}} A^p A^{\frac{r}{2}}\right)^{\frac{1}{q}} \geqslant^J \left(A^{\frac{r}{2}} B^p A^{\frac{r}{2}}\right)^{\frac{1}{q}}$ *for all* $p \geqslant 0$, $r \geqslant 0$ *and* $q \geqslant 1$ *with* $rq \geqslant p + r$;

(xi) $\left(B^{\frac{r}{2}} A^p B^{\frac{r}{2}}\right)^{\frac{1}{q}} \geqslant^J \left(B^{\frac{r}{2}} B^p B^{\frac{r}{2}}\right)^{\frac{1}{q}}$ *for all* $p \geqslant 0$, $r \geqslant 0$ *and* $q \geqslant 1$ *with* $rq \geqslant p + r$.

Proof. Without loss of generality, consider $\mu = 1$. Otherwise, replace A, B respectively by $A_\mu = \frac{1}{\mu} A$, $B_\mu = \frac{1}{\mu} B$ and observe that $\operatorname{Log}(A) \geqslant^J \operatorname{Log}(B)$ if and only if $\operatorname{Log}(A_\mu) \geqslant^J \operatorname{Log}(B_\mu)$. Further, the following relations are satisfied:

$$
\begin{aligned}
F_t(p,r) &= \mu^t \left(A_\mu^{-r} \natural_{\frac{t+r}{p+r}} B_\mu^p\right), \\
G_t(p,r) &= \mu^t \left(B_\mu^{-r} \natural_{\frac{t+r}{p+r}} A_\mu^p\right), \\
H_{t,r}(p) &= \frac{1}{\mu^r} \frac{t+r}{p+r} S\left(A_\mu^{-r}|B_\mu^p\right) + \log\left(\mu^{t+r}\right) A^{-r}, \\
K_{t,r}(p) &= \frac{1}{\mu^r} \frac{t+r}{p+r} S\left(B_\mu^{-r}|A_\mu^p\right) + \log\left(\mu^{t+r}\right) B^{-r}.
\end{aligned}
$$

(i) $\Leftrightarrow$ (ii) It is a consequence of the application of Lemma 2.1 to a rewriting of Theorem 1.3.

(ii) $\Leftrightarrow$ (iii) From Lemmas 2.2 and 2.3, we have $A^{-p} \geqslant^J I_n \geqslant^J B^r$ for $p \geqslant 0$, $r \geqslant 0$. Interchanging the roles of r and p in (ii) and bearing in mind Proposition 2.1, we get $I_n \geqslant^J A^{-p} \natural_{\frac{p}{p+r}} B^r = B^r \natural_{\frac{r}{p+r}} A^{-p}$. By Lemma 2.3, this is equivalent to $B^{-r} \natural_{\frac{r}{p+r}} A^p \geqslant^J I_n$ for all $p \geqslant 0$ and $r \geqslant 0$.

(i) $\Rightarrow$ (iv) By Theorem 1.3, the assumption (i) implies $A^r \geqslant^J (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{r}{p+r}}$ for $r > 0$ and $p \geqslant 0$. By the Löwner inequality of indefinite type with $\alpha = \frac{u}{r}$ and Lemma 2.3, we get

$$\left(A^{\frac{r}{2}}B^pA^{\frac{r}{2}}\right)^{-\frac{u}{p+r}} \geqslant^J A^{-u} \tag{3}$$

for all $r \geqslant u \geqslant 0$ and $p \geqslant 0$. On the other hand, we proved that (i) $\Leftrightarrow$ (iii) so, interchanging in (iii) the roles of r and p, by Lemma 2.1 and applying the above technique with $\alpha = \frac{v}{p}$, we find

$$\left(B^{\frac{p}{2}}A^rB^{\frac{p}{2}}\right)^{\frac{v}{p+r}} \geqslant^J B^v \tag{4}$$

for all $p \geqslant v \geqslant 0$ and $r \geqslant 0$. We have $I_n \geqslant^J A^{\frac{r}{2}}$, $I_n \geqslant^J B^p$. Therefore, by Lemma 2.4 with $\lambda = \frac{p+v+r}{p+r}$ and Lemma 2.1, we successively have

$$\begin{aligned}
F_t(p,r) &= A^{-\frac{r}{2}}\left(\left(A^{\frac{r}{2}}B^pA^{\frac{r}{2}}\right)^{\frac{p+v+r}{p+r}}\right)^{\frac{t+r}{p+v+r}}A^{-\frac{r}{2}}\\
&= A^{-\frac{r}{2}}\left(A^{\frac{r}{2}}B^{\frac{p}{2}}\left(B^{\frac{p}{2}}A^rB^{\frac{p}{2}}\right)^{\frac{v}{p+r}}B^{\frac{p}{2}}A^{\frac{r}{2}}\right)^{\frac{t+r}{p+v+r}}A^{-\frac{r}{2}}\\
&= A^{-r}\sharp_{\frac{t+r}{p+v+r}}\left(B^{\frac{p}{2}}\left(B^{\frac{p}{2}}A^rB^{\frac{p}{2}}\right)^{\frac{v}{p+r}}B^{\frac{p}{2}}\right).
\end{aligned}$$

Taking into account (4), Lemma 2.1 and the J-monotonicity of $\sharp_{\frac{t+r}{p+v+r}}$ in the second variable, we get

$$F_t(p,r) \geqslant^J A^{-r}\sharp_{\frac{t+r}{p+v+r}}\left(B^{\frac{p}{2}}B^vB^{\frac{p}{2}}\right) = F_t(p+v,r),$$

that is, $F_t(p,r)$ is a J-decreasing function of $p \geqslant t$. Next, we show that $F_t(p,r)$ is a J-decreasing function of $r > 0$. Since $A^{-r} \geqslant^J I_n \geqslant^J B^p$ by Proposition 2.1 we obtain

$$F_t(p,r) = A^{-r}\sharp_{\frac{t+r}{p+r}}B^p = B^p\sharp_{\frac{p-t}{p+r}}A^{-r}.$$

Since $I_n \geqslant^J A^r$, $I_n \geqslant^J B^{\frac{p}{2}}$, by Lemma 2.4 with $\lambda = \frac{p+r+u}{p+r}$, in conjunction with Lemma 2.1, we obtain

$$\begin{aligned}
F_t(p,r) &= B^p\sharp_{\frac{p-t}{p+r}}A^{-r}\\
&= B^{\frac{p}{2}}\left(\left(B^{\frac{p}{2}}A^rB^{\frac{p}{2}}\right)^{\frac{p+r+u}{p+r}}\right)^{-\frac{p-t}{p+r+u}}B^{\frac{p}{2}}\\
&= B^{\frac{p}{2}}\left(B^{\frac{p}{2}}A^{\frac{r}{2}}\left(A^{\frac{r}{2}}B^pA^{\frac{r}{2}}\right)^{\frac{u}{p+r}}A^{\frac{r}{2}}B^{\frac{p}{2}}\right)^{-\frac{p-t}{p+r+u}}B^{\frac{p}{2}}\\
&= B^p\sharp_{\frac{p-t}{p+r+u}}\left(A^{-\frac{r}{2}}\left(A^{\frac{r}{2}}B^pA^{\frac{r}{2}}\right)^{-\frac{u}{p+r}}A^{-\frac{r}{2}}\right).
\end{aligned}$$

Taking into account (3), Lemma 2.1 and the J-monotonicity of $\sharp_{\frac{p-t}{p+r+u}}$ in the second variable, we have

$$F_t(p,r) \geqslant^J B^p\sharp_{\frac{p-t}{p+r+u}}\left(A^{-\frac{r}{2}}A^{-u}A^{-\frac{r}{2}}\right) = A^{-r-u}\sharp_{\frac{t+r+u}{p+r+u}}B^p = F_t(p,r+u),$$

that is, $F_t(p,r)$ is a J-decreasing function of $r > 0$.

(iv) $\Rightarrow$ (v) From (iv), we easily get $B^t = F_t(t,0) \geqslant^J F_t(p,r)$ for all $p \geqslant t \geqslant 0$ and $r \geqslant 0$.

(v) $\Rightarrow$ (ii) Put $t = 0$ in (v).

(iv) $\Rightarrow$ (vi) For any fixed $t \geqslant 0$ and $r \geqslant 0$, we observe that

$$H_{t,r}(p) = A^{-\frac{r}{2}} \operatorname{Log}\left(A^{\frac{r}{2}} F_t(p,r) A^{\frac{r}{2}}\right) A^{-\frac{r}{2}}.$$

We are assuming that $F_t(p,r)$ is a J-decreasing function of $p \geqslant t$. The same conclusion holds for $H_{t,r}(p)$ from Lemma 2.1 since the J-chaotic order is weaker then the J-order.

(vi) $\Rightarrow$ (i) If $t = 0$ in (vi), then $H_{0,r}(0) \geqslant^J H_{0,r}(1)$ for $r > 0$. Letting $r \to 0^+$, this yields (i).

(i) $\Rightarrow$ (vii) Analogously to the proof of the implication (i) $\Rightarrow$ (iv), we may conclude that

$$A^v \geqslant^J \left(A^{\frac{p}{2}} B^r A^{\frac{p}{2}}\right)^{\frac{v}{p+r}} \tag{5}$$

for $p \geqslant v \geqslant 0$ and $r \geqslant 0$, and

$$B^{-u} \geqslant^J \left(B^{\frac{r}{2}} A^p B^{\frac{r}{2}}\right)^{-\frac{u}{p+r}} \tag{6}$$

for $r \geqslant u \geqslant 0$ and $p \geqslant 0$. Changing the roles of A and B in the second part of the proof of (i) $\Rightarrow$ (iv), recalling (5) and (6), we conclude that $G_t(p,r)$ is a J-increasing function of both $p \geqslant t$ and $r \geqslant 0$.

(vii) $\Rightarrow$ (viii) From (vii), we easily get $G_t(p,r) \geqslant^J G_t(t,0) = A^t$ for all $p \geqslant t \geqslant 0$ and $r \geqslant 0$.

(viii) $\Rightarrow$ (iii) Put $t = 0$ in (viii).

(vii) $\Rightarrow$ (ix) The proof is analogous to the proof of the implication (viii) $\Rightarrow$ (vi), observing that

$$K_{t,r}(p) = A^{-\frac{r}{2}} \operatorname{Log}\left(A^{\frac{r}{2}} G_t(p,r) A^{\frac{r}{2}}\right) A^{-\frac{r}{2}}.$$

(ix) $\Rightarrow$ (i) If $t = 0$ in (ix), then $K_{0,r}(0) \leqslant^J K_{0,r}(1)$ for $r > 0$ and letting $r \to 0^+$ we get (i).

(i) $\Leftrightarrow$ (x) Was proved in [4, Theorem 2.2].

(i) $\Rightarrow$ (xi) By the implication (i) $\Rightarrow$ (iii) proved above and Lemma 2.1, we get $\left(B^{\frac{r}{2}} A^p B^{\frac{r}{2}}\right)^{\frac{r}{p+r}} \geqslant^J B^r$ and then by the indefinite Löwner inequality with $\alpha = \frac{p+r}{rq}$ we obtain (xi).

(xi) $\Rightarrow$ (i) Put $q = \frac{p+r}{r}$ with $r > 0$ in (xi), then use Lemma 2.1 and the implication (iii) $\Rightarrow$ (i). $\square$

The next corollary yields an indefinite version of Kamei's satellite to Furuta inequality [12].

COROLLARY 2.1. *If $A,B \in M_n$ are J-selfadjoint matrices with positive eigenvalues and $\mu I_n \geqslant^J A \geqslant^J B$ for some $\mu > 0$, then*

$$B^{-r}\sharp_{\frac{1+r}{p+r}}A^p \geqslant^J A \geqslant^J B \geqslant^J A^{-r}\sharp_{\frac{1+r}{p+r}}B^p$$

for all $p \geqslant 1$ and $r \geqslant 0$.

Proof. Since $A \geqslant^J B$ implies $\mathrm{Log}(A) \geqslant^J \mathrm{Log}(B)$, then the result readily follows from Theorem 2.1 (i) $\Rightarrow$ (v) and (i) $\Rightarrow$ (viii) in the particular case $t = 1$. $\square$

3. Complete form of the Furuta inequality of indefinite type

Theorem 3.1 provides an indefinite *complete form* of the Furuta inequality (cf. [19] in the case of Hilbert spaces). In fact, it will be shown in Remark 3.1 that this theorem, in the particular case $p_0 = 1$, implies Theorem 1.2, this justifying the used terminology. Firstly, we recall the following useful lemma from [15].

LEMMA 3.1. *Let $A \in M_n$ be a J-selfadjoint matrix with nonnegative eigenvalues, $I_n \geqslant^J A$ and let $A_m \in M_n$, $m \in \mathbb{N}$, be J-selfadjoint matrices with positive eigenvalues such that $A_m \to A$ as $m \to +\infty$. Then $A_m^\alpha \to A^\alpha$ for $0 \leqslant \alpha \leqslant 1$, as $m \to +\infty$.*

THEOREM 3.1. *Let $A,B \in M_n$ be J-selfadjoint with nonnegative eigenvalues such that $\mu I_n \geqslant^J A \geqslant^J B$ for some $\mu > 0$. If $r \geqslant 0$, $p > p_0 > 0$ and $s = \min\{p, 2p_0 + 1, 2p_0 + r\}$, then*

$$\left(A^{\frac{r}{2}}B^{p_0}A^{\frac{r}{2}}\right)^{\frac{s+r}{p_0+r}} \geqslant^J \left(A^{\frac{r}{2}}B^{p}A^{\frac{r}{2}}\right)^{\frac{s+r}{p+r}}, \tag{7}$$

$$\left(B^{\frac{r}{2}}A^{p}B^{\frac{r}{2}}\right)^{\frac{s+r}{p+r}} \geqslant^J \left(B^{\frac{r}{2}}A^{p_0}B^{\frac{r}{2}}\right)^{\frac{s+r}{p_0+r}}. \tag{8}$$

Proof. Without loss of generality, we may suppose that $\mu = 1$. Assume that A, B are invertible. Otherwise, replace A,B by $A + \frac{1}{m}I_n$, $B + \frac{1}{m}I_n$ and then use Lemma 3.1.

(I) If $p \leqslant 2p_0 + r$ and $p \leqslant 2p_0 + 1$, then $s = p$. By Lemma 2.2 we have $I_n \geqslant^J A^{\frac{r}{2}}$, $I_n \geqslant^J B^{p_0}$, and by Lemma 2.4 we get

$$\left(A^{\frac{r}{2}}B^{p_0}A^{\frac{r}{2}}\right)^{\frac{p+r}{p_0+r}} = A^{\frac{r}{2}}B^{\frac{p_0}{2}}\left(B^{\frac{p_0}{2}}A^{r}B^{\frac{p_0}{2}}\right)^{\frac{p-p_0}{p_0+r}}B^{\frac{p_0}{2}}A^{\frac{r}{2}}. \tag{9}$$

Now, replace r,p respectively by p_0, r and let $q = \frac{p_0+r}{p-p_0}$ in inequality (2) of Theorem 1.2. Note that $q \geqslant 1$, because $p \leqslant 2p_0 + r$ and that $(1+p_0)q \geqslant r + p_0$. Then $\left(B^{\frac{p_0}{2}}A^{r}B^{\frac{p_0}{2}}\right)^{\frac{p-p_0}{p_0+r}} \geqslant^J B^{p-p_0}$ and by Lemma 2.1 we find

$$A^{\frac{r}{2}}B^{\frac{p_0}{2}}\left(B^{\frac{p_0}{2}}A^{r}B^{\frac{p_0}{2}}\right)^{\frac{p-p_0}{p_0+r}}B^{\frac{p_0}{2}}A^{\frac{r}{2}} \geqslant^J A^{\frac{r}{2}}B^{p}A^{\frac{r}{2}}.$$

Bearing in mind (9), we obtain

$$\left(A^{\frac{r}{2}}B^{p_0}A^{\frac{r}{2}}\right)^{\frac{p+r}{p_0+r}} \geqslant^J \left(A^{\frac{r}{2}}B^{p}A^{\frac{r}{2}}\right)^{\frac{p+r}{p+r}}, \tag{10}$$

and so (7) follows because $s=p$. Similarly, (8) follows from inequality (1) in Theorem 1.2.

(II) If $p \geqslant 2p_0+r$ or $p \geqslant 2p_0+1$, then $s=2p_0+m_r$ with $m_r=\min\{1,r\}$. Let $n\in\mathbb{N}$ such that $p_n<p\leqslant p_{n+1}$, where $p_n=p_0+(2^n-1)(p_0+m_r)$. Note that $p_{n+1}=2p_n+m_r\geqslant p$, $n\in\mathbb{N}$. Replacing p by p_1 in (10), we have

$$\left(A^{\frac{r}{2}}B^{p_0}A^{\frac{r}{2}}\right)^{\frac{p_1+r}{p_0+r}} \geqslant^J \left(A^{\frac{r}{2}}B^{p_1}A^{\frac{r}{2}}\right)^{\frac{p_1+r}{p_1+r}}. \tag{11}$$

Similarly, since $p_2=2p_1+m_r\geqslant p_1$, by (10) we get

$$\left(A^{\frac{r}{2}}B^{p_1}A^{\frac{r}{2}}\right)^{\frac{p_2+r}{p_1+r}} \geqslant^J \left(A^{\frac{r}{2}}B^{p_2}A^{\frac{r}{2}}\right)^{\frac{p_2+r}{p_2+r}}.$$

Applying the Löwner inequality of indefinite type with $\alpha=\frac{p_1+r}{p_2+r}$ to the above inequality and combining the result so obtained with (11), we find

$$\left(A^{\frac{r}{2}}B^{p_0}A^{\frac{r}{2}}\right)^{\frac{p_1+r}{p_0+r}} \geqslant^J A^{\frac{r}{2}}B^{p_1}A^{\frac{r}{2}} \geqslant^J \left(A^{\frac{r}{2}}B^{p_2}A^{\frac{r}{2}}\right)^{\frac{p_1+r}{p_2+r}}.$$

Repeating the above procedure for $p>p_n$ and $p_{n+1}=2p_n+m_r\geqslant p$, $n\in\mathbb{N}$, by (10) we obtain

$$\left(A^{\frac{r}{2}}B^{p_n}A^{\frac{r}{2}}\right)^{\frac{p+r}{p_n+r}} \geqslant^J \left(A^{\frac{r}{2}}B^{p}A^{\frac{r}{2}}\right)^{\frac{p+r}{p+r}}.$$

Now, by the Löwner inequality of indefinite type with $\alpha=\frac{p_1+r}{p+r}$, we get

$$\begin{aligned}\left(A^{\frac{r}{2}}B^{p_0}A^{\frac{r}{2}}\right)^{\frac{p_1+r}{p_0+r}} &\geqslant^J A^{\frac{r}{2}}B^{p_1}A^{\frac{r}{2}} \geqslant^J \left(A^{\frac{r}{2}}B^{p_2}A^{\frac{r}{2}}\right)^{\frac{p_1+r}{p_2+r}}\\ &\geqslant^J \cdots \geqslant^J \left(A^{\frac{r}{2}}B^{p_n}A^{\frac{r}{2}}\right)^{\frac{p_1+r}{p_n+r}} \geqslant^J \left(A^{\frac{r}{2}}B^{p}A^{\frac{r}{2}}\right)^{\frac{p_1+r}{p+r}}\end{aligned}$$

for $p_n<p\leqslant p_{n+1}$, $n\in\mathbb{N}$, and (7) holds since $s=p_1$. The proof of (8) follows similarly. □

REMARK 3.1. For $p_0=1$, applying to (7) and (8) the Löwner inequality of indefinite type with $\alpha=\frac{1+r}{s+r}$ and $s=\min\{p,3,2+r\}$, we obtain

$$A^{\frac{r}{2}}BA^{\frac{r}{2}} \geqslant^J \left(A^{\frac{r}{2}}B^{p}A^{\frac{r}{2}}\right)^{\frac{1+r}{p+r}} \qquad \text{and} \qquad \left(B^{\frac{r}{2}}A^{p}B^{\frac{r}{2}}\right)^{\frac{1+r}{p+r}} \geqslant^J B^{\frac{r}{2}}AB^{\frac{r}{2}},$$

respectively, for all $r\geqslant 0$ and $p>1$. If $A\geqslant^J B$, then by Lemma 2.1 we get

$$A^{\frac{r}{2}}AA^{\frac{r}{2}} \geqslant^J A^{\frac{r}{2}}BA^{\frac{r}{2}} \geqslant^J \left(A^{\frac{r}{2}}B^{p}A^{\frac{r}{2}}\right)^{\frac{1+r}{p+r}} \quad \text{and} \quad \left(B^{\frac{r}{2}}A^{p}B^{\frac{r}{2}}\right)^{\frac{1+r}{p+r}} \geqslant^J B^{\frac{r}{2}}AB^{\frac{r}{2}} \geqslant^J B^{\frac{r}{2}}BB^{\frac{r}{2}}$$

for all $r \geqslant 0$ and $p > 1$. This is the *essential part* of the indefinite Furuta inequality. Thus, Theorem 3.1 implies Theorem 1.2. On the other hand, we proved Theorem 3.1 using Theorem 1.2. Henceforth, these two results are equivalent.

The following corollary is a variant of Theorem 3.1.

COROLLARY 3.1. *Let $A, B \in M_n$ be J-selfadjoint matrices with nonnegative eigenvalues and $\delta > 0$ fixed. If $\mu I_n \geqslant^J A^\delta \geqslant^J B^\delta$ for some $\mu > 0$, then (7) and (8) hold for $r \geqslant 0$, $p > p_0 > 0$ and $s = \min\{p, 2p_0 + \delta, 2p_0 + r\}$.*

Proof. For a fixed $\delta > 0$, let $r_\delta = r/\delta$, $p_\delta = p/\delta$, $p_{0\delta} = p_0/\delta$. Since $r_\delta \geqslant 0$, $p_\delta > p_{0\delta} > 0$ and $s_\delta = \min\{p_\delta, 2p_{0\delta} + 1, 2p_{0\delta} + r_\delta\}$, we can apply Theorem 3.1 to obtain

$$\left(A^{\delta \frac{r_\delta}{2}} B^{\delta p_{0\delta}} A^{\delta \frac{r_\delta}{2}}\right)^{\frac{s_\delta + r_\delta}{p_{0\delta} + r_\delta}} \geqslant^J \left(A^{\delta \frac{r_\delta}{2}} B^{\delta p_\delta} A^{\delta \frac{r_\delta}{2}}\right)^{\frac{s_\delta + r_\delta}{p_\delta + r_\delta}}$$

and (7) follows. Analogously, we prove (8). □

Next, we derive an indefinite *complete form* of Furuta's inequality for J-selfadjoint matrices obeying the J-chaotic order, which can be seen as the case $\delta \to 0^+$ of Corollary 3.1, because

$$\lim_{\delta \to 0^+} \frac{A^\delta - I_n}{\delta} = \mathrm{Log}(A).$$

THEOREM 3.2. *Let $A, B \in M_n$ be J-selfadjoint matrices with positive eigenvalues, such that $\mu I_n \geqslant^J A$, $\mu I_n \geqslant^J B$ for some $\mu > 0$. If $\mathrm{Log}(A) \geqslant^J \mathrm{Log}(B)$, then (7) and (8)* hold for $r \geqslant 0$, $p > p_0 > 0$ and $s = \min\{p, 2p_0\}$.

Proof. Without loss of generality we may consider $\mu = 1$. If $r \geqslant 0$ and $2p_0 \geqslant p$, then $s = p$. Replace in Theorem 2.1 (i) $\Rightarrow$ (x) r by p_0, p by r and let $q = \frac{p_0 + r}{p - p_0}$. Using Lemmas 2.2, 2.4 and 2.1 as in the proof of Theorem 3.1 (I), we obtain (7) with $s = p$. Analogously, from Theorem 2.1 (i) $\Rightarrow$ (xi) we get (8) with $s = p$. If $r \geqslant 0$ and $2p_0 < p$, then $s = 2p_0$. Let $n \in \mathbb{N}$ be such that $p_n < p \leqslant p_{n+1}$ with $p_n = 2^n p_0$. Then $p_1 = s$ and $p_{n+1} = 2p_n$, $n \in \mathbb{N}$. Following the steps of the proof of Theorem 3.1 (II), using the inequalities deduced in (I) and the Löwner-Heinz inequality of indefinite type, the result easily follows. □

4. Some exponential operator inequalities of indefinite type

In this section we present indefinite versions of some exponential operator inequalities obtained by Uchiyama [18] and Ando [1] for bounded selfadjoint operators on a Hilbert space. We remark that if $A \in M_n$ is a J-selfadjoint matrix, then $\left(\mathrm{e}^A\right)^{[*]} = J\mathrm{e}^{A^*}J = \mathrm{e}^{JA^*J} = \mathrm{e}^A$, and so e^A is also J-selfadjoint.

The following lemma is useful for the development of this section.

LEMMA 4.1. *If $A \in M_n$ is J-selfadjoint and $\mu I_n \geqslant^J A$ for some $\mu \in \mathbb{R}$, then* $e^{\mu} I_n \geqslant^J e^A$.

Proof. If $\mu I_n \geqslant^J A$ for some $\mu \in \mathbb{R}$, then the eigenvalues of the J-selfadjoint matrix A are real. Then there exists $\tau > 0$, such that $\mu + \tau > 0$ and the eigenvalues of the matrix $A_\tau = A + \tau I_n$ are positive. We obviously have $\bar{\mu} I_n \geqslant^J A_\tau$, where $\bar{\mu} = \mu + \tau$. Next, we prove by induction that $\bar{\mu}^k I_n \geqslant^J A_\tau^k$, $k \in \mathbb{N}$. We show that the inequality still holds when we replace k by $k+1$. Since the eigenvalues of A_τ are positive, the J-selfadjoint square root of A_τ is well defined. Replacing x by $A_\tau^{\frac{1}{2}} x$ in $\bar{\mu}^k [x,x] \geqslant [A_\tau^k x, x]$, we find

$$\bar{\mu}^k [A_\tau^{\frac{1}{2}} x, A_\tau^{\frac{1}{2}} x] \geqslant [A_\tau^{\frac{2k+1}{2}} x, A_\tau^{\frac{1}{2}} x]$$

and so $\bar{\mu}^k [A_\tau x, x] \geqslant [A_\tau^{k+1} x, x]$. Having in mind that $\bar{\mu} > 0$, we have $\bar{\mu}^{k+1}[x,x] \geqslant \bar{\mu}^k [A_\tau x, x]$ for any $x \in \mathbb{C}^n$. We have just proved that $\bar{\mu}^{k+1} I_n \geqslant^J A_\tau^{k+1}$. Then it may be easily shown that

$$\sum_{k=0}^{+\infty} \frac{\bar{\mu}^k}{k!} I_n \geqslant^J \sum_{k=0}^{+\infty} \frac{1}{k!} A_\tau^k,$$

that is, $e^\tau e^\mu I_n = e^{\bar{\mu}} I_n \geqslant^J e^{A_\tau} = e^\tau e^A$ and the desired result $e^\mu I_n \geqslant^J e^A$ is readily obtained. □

For J-selfadjoint matrices $A, B \in M_n$, let us consider

$$A \bigtriangledown_\lambda B = \lambda A + (1-\lambda) B, \qquad A \,!_\lambda\, B = \left(\lambda A^{-1} + (1-\lambda) B^{-1}\right)^{-1}$$

for $\lambda \geqslant 0$, with invertibility of A, B required in $A \,!_\lambda\, B$. If $\lambda = \frac{1}{2}$, these are the *arithmetic mean* and the *harmonic mean* of A and B, respectively.

THEOREM 4.1. *Let $A, B \in M_n$ be J-selfadjoint matrices, such that $\mu I_n \geqslant^J A \geqslant^J B$ for some $\mu \in \mathbb{R}$. For all $p \geqslant 0$, $r \geqslant s \geqslant 0$ and $\lambda \geqslant 0$ the following inequalities hold:*

(a) $e^{sA} \geqslant^J \left(e^{\frac{r}{2}A} e^{p(B \bigtriangledown_\lambda A)} e^{\frac{r}{2}A}\right)^{\frac{s}{p+r}}$;

(b) $e^{sA} \geqslant^J \left(e^{\frac{r}{2}A} e^{p(B \,!_\lambda A)} e^{\frac{r}{2}A}\right)^{\frac{s}{p+r}}$ *for invertible A, B.*

Proof. **(a)** From the hypothesis, it is easy to see that $\mu I_n \geqslant^J A \geqslant^J B \bigtriangledown_\lambda A$ for all $\lambda \geqslant 0$. Then all the eigenvalues of A and $B \bigtriangledown_\lambda A$ are real. Moreover, $e^\mu I_n \geqslant^J e^A$, $e^\mu I_n \geqslant^J e^{B \bigtriangledown_\lambda A}$ and the J-selfadjoint matrices e^A, $e^{B \bigtriangledown_\lambda A}$ have positive eigenvalues. If $s = 0$, the result is trivially satisfied. Now, let $p \geqslant 0$, $r \geqslant s > 0$ and $q = \frac{p+r}{s}$. Then $q \geqslant 1$ and $rq \geqslant p + r$. Thus (a) follows from Theorem 2.1 (i) $\Rightarrow$ (x), replacing A, B by $e^A, e^{B \bigtriangledown_\lambda A}$, respectively.

(b) For $\lambda \geqslant 0$ by Lemma 2.3 we have $\lambda B^{-1} \geqslant^J \lambda A^{-1}$ and it is easy to see that $(B!_\lambda A)^{-1} \geqslant^J A^{-1}$. Then $\mu I_n \geqslant^J A \geqslant^J B!_\lambda A$ and the proof follows analogously to (a). □

In the following corollary, we reformulate the previous result in order to obtain an indefinite version of an inequality obtained by Uchiyama [18].

COROLLARY 4.1. *Under the assumptions of Theorem* 4.1*, we have*

$$\mathrm{e}^{sA} \geqslant^J \left(\mathrm{e}^{\frac{r}{2}A}\mathrm{e}^{qA+pB}\mathrm{e}^{\frac{r}{2}A}\right)^{\frac{s}{p+q+r}} \tag{12}$$

for all $p \geqslant 0,\ p+q \geqslant 0$ *and* $r \geqslant s \geqslant 0$.

Proof. If $p \geqslant 0$, then $A \geqslant^J B$ implies $0 \geqslant^J p(B-A)$, and so $I_n \geqslant^J \mathrm{e}^{p(B-A)}$. If $r \geqslant s \geqslant 0$, by Lemma 2.1 and the Löwner inequality of indefinite type with $\alpha = \frac{s}{r} \in [0,1]$, we find

$$\mathrm{e}^{sA} \geqslant^J \left(\mathrm{e}^{\frac{r}{2}A}\mathrm{e}^{p(B-A)}\mathrm{e}^{\frac{r}{2}A}\right)^{\frac{s}{r}},$$

that is, (12) when $p+q=0$. If $p \geqslant 0$ and $p+q>0$, then for $\lambda = \frac{p}{p+q} \geqslant 0$ we get $B \nabla_\lambda A = \frac{1}{p+q}(qA+pB)$ and so (12) is obtained from Theorem 4.1 (a) with p replaced by $p+q$. □

The following result is a converse of Theorem 4.1 when $s=p=r$ and $\lambda = 1$ (cf. [7, 9]).

THEOREM 4.2. *Let* $A,B \in M_n$ *be* J*-selfadjoint matrices such that* $\mu I_n \geqslant^J A$, $\mu I_n \geqslant^J B$ *for some* $\mu \in \mathbb{R}$*. If*

$$\mathrm{e}^{tA} \geqslant^J \left(\mathrm{e}^{\frac{t}{2}A}\mathrm{e}^{tB}\mathrm{e}^{\frac{t}{2}A}\right)^{\frac{1}{2}} \tag{13}$$

for every $t>0$*, then* $A \geqslant^J B$.

Proof. As $X^{\frac{1}{2}} - I_n = \left(X^{\frac{1}{2}} + I_n\right)^{-1}(X-I_n)$, from (13) we get

$$\frac{\mathrm{e}^{tA}-I_n}{t} \geqslant^J \left(\left(\mathrm{e}^{\frac{t}{2}A}\mathrm{e}^{tB}\mathrm{e}^{\frac{t}{2}A}\right)^{\frac{1}{2}} + I_n\right)^{-1} \frac{\mathrm{e}^{\frac{t}{2}A}\mathrm{e}^{tB}\mathrm{e}^{\frac{t}{2}A}-I_n}{t} \tag{14}$$

for any $t>0$. Now, we observe that

$$\frac{\mathrm{e}^{\frac{t}{2}A}\mathrm{e}^{tB}\mathrm{e}^{\frac{t}{2}A}-I_n}{t} = \frac{\mathrm{e}^{\frac{t}{2}A}\left(\mathrm{e}^{tB}-I_n\right)\mathrm{e}^{\frac{t}{2}A}}{t} + \frac{\mathrm{e}^{tA}-I_n}{t}.$$

Since

$$\lim_{t\to 0^+} \frac{\mathrm{e}^{tA}-I_n}{t} = A,$$

letting $t \to 0^+$ in (14), we easily conclude that $A \geqslant^J \frac{1}{2}(B+A)$, and so $A \geqslant^J B$. □

Next we present a slight generalization of the previous theorem in the spirit of Uchiyama [18].

THEOREM 4.3. *Let $A,B \in M_n$ be J-selfadjoint matrices such that $\mu I_n \geqslant^J A$, $\mu I_n \geqslant^J B$ for some $\mu \in \mathbb{R}$. If there exist p,q,r,s with $p>0$, $p+q \geqslant 0$ and $r \geqslant s > 0$, such that either*

$$e^{stA} \geqslant^J \left(e^{\frac{rt}{2}A} e^{t(qA+pB)} e^{\frac{rt}{2}A}\right)^{\frac{s}{p+q+r}} \tag{15}$$

or

$$\left(e^{\frac{rt}{2}B} e^{t(qA+pB)} e^{\frac{rt}{2}B}\right)^{\frac{s}{p+q+r}} \geqslant^J e^{stB} \tag{16}$$

holds for every $t>0$, then $A \geqslant^J B$.

Proof. We assume that (15) holds. If $s=p+q+r$, then $p+q=0$ and $r=s$. From (15) by straightforward computations we get $I_n \geqslant^J e^{tp(B-A)}$ for every $t>0$. Letting $t \to 0^+$ in

$$\frac{e^{tp(B-A)} - I_n}{tp} \leqslant^J 0,$$

we easily conclude that $A \geqslant^J B$. If $s < p+q+r$, then,

$$\frac{e^{stA} - I_n}{t} \geqslant^J \frac{\left(e^{\frac{rt}{2}A} e^{t(qA+pB)} e^{\frac{rt}{2}A}\right)^{\frac{s}{p+q+r}} - I_n}{t}$$

for any $t>0$ and standard computations yield

$$sA + \frac{O(t^2)}{t} \geqslant^J \frac{s}{p+q+r}(rA+qA+pB) + \frac{O(t^2)}{t},$$

where $\left\|\frac{JO(t^2)}{t}\right\| \to 0$ as $t \to 0^+$. Henceforth, taking limits as $t \to 0^+$, we get $(p+q+r)A \geqslant^J (r+q)A + pB$ and the result follows. If (16) holds, the proof is analogous. □

Theorem 4.4 includes a partial reformulation of Theorem 2.1 for exponential operators and an indefinite version of a famous result due to Ando (cf [1]).

THEOREM 4.4. *Let $A,B \in M_n$ be J-selfadjoint matrices, such that $\mu I_n \geqslant^J A$, $\mu I_n \geqslant^J B$ for some $\mu \in \mathbb{R}$. Then the following statements are mutually equivalent:*

(i) $A \geqslant^J B$;

(ii) $I_n \geqslant^J e^{-rA} \sharp_{\frac{1}{2}} e^{rB}$ *for all* $r \geqslant 0$;

(iii) $f(r) = e^{-rA} \sharp_{\frac{1}{2}} e^{rB}$ *is a J-decreasing function of* $r \geqslant 0$;

(iv) *For any fixed* $t \geqslant 0$, $f_t(p,r) = \mathrm{e}^{-rA} \sharp_{\frac{t+r}{p+r}} \mathrm{e}^{pB}$ *is a* J*-decreasing function of both* $p \geqslant t$ *and* $r \geqslant 0$*;*

(v) *For any fixed* $t \geqslant 0$, $g_t(p,r) = \mathrm{e}^{-rB} \sharp_{\frac{t+r}{p+r}} \mathrm{e}^{pA}$ *is a* J*-increasing function of both* $p \geqslant t$ *and* $r \geqslant 0$.

Proof. According to the hypothesis, A,B have real eigenvalues and $\mathrm{e}^A, \mathrm{e}^B$ are J-selfadjoint matrices with positive eigenvalues, such that $\mathrm{e}^{\mu} I_n \geqslant^J \mathrm{e}^A$, $\mathrm{e}^{\mu} I_n \geqslant^J \mathrm{e}^B$.

The equivalences (i) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v) readily follow from the equivalences (i) $\Leftrightarrow$ (v) $\Leftrightarrow$ (viii) in Theorem 2.1, replacing A, B by e^A, e^B, respectively.

(i) $\Rightarrow$ (iii) Let $t = 0$ and $r = p$ in (i) $\Rightarrow$ (v) and the result holds.

(iii) $\Rightarrow$ (ii) Since $f(r)$ is a J-decreasing function of $r \geqslant 0$, we have $I_n = f(0) \geqslant^J f(r)$ for all $r \geqslant 0$.

(ii) $\Rightarrow$ (i) It is a consequence of Theorem 4.2 and Lemma 2.1. $\square$

COROLLARY 4.2. *Let* $A,B \in M_n$ *be* J*-selfadjoint matrices, such that* $\mu I_n \geqslant^J A \geqslant^J B$ *for some* $\mu \in \mathbb{R}$. *For all* $p \geqslant t \geqslant 0$, $r \geqslant 0$ *and* $0 \leqslant \alpha \leqslant 1$ *the following inequalities hold:*

(a) $\mathrm{e}^{-rB} \sharp_{\frac{t+r}{p+r}} \mathrm{e}^{p(A \nabla_\alpha B)} \geqslant^J \mathrm{e}^{t(A \nabla_\alpha B)} \geqslant^J \mathrm{e}^{-rA} \sharp_{\frac{t+r}{p+r}} \mathrm{e}^{p(A \nabla_\alpha B)}$;

(b) $\mathrm{e}^{-rB} \sharp_{\frac{t+r}{p+r}} \mathrm{e}^{p(A\,!_\alpha B)} \geqslant^J \mathrm{e}^{t(A\,!_\alpha B)} \geqslant^J \mathrm{e}^{-rA} \sharp_{\frac{t+r}{p+r}} \mathrm{e}^{p(A\,!_\alpha B)}$ *for invertible* A,B.

Proof. **(a)** For $0 \leqslant \alpha \leqslant 1$ we have $\mu I_n \geqslant^J A \geqslant^J A \nabla_\alpha B \geqslant^J B$. For $p \geqslant t \geqslant 0$ let us consider the functions on $r \geqslant 0$ defined by

$$\tilde{f}_{p,t}(r) = \mathrm{e}^{-rA} \sharp_{\frac{t+r}{p+r}} \mathrm{e}^{p(A \nabla_\alpha B)} \qquad \text{and} \qquad \tilde{g}_{p,t}(r) = \mathrm{e}^{-rB} \sharp_{\frac{t+r}{p+r}} \mathrm{e}^{p(A \nabla_\alpha B)}.$$

By Theorem 4.4 (i) $\Rightarrow$ (iv), $A \geqslant^J A \nabla_\alpha B$ implies that $\tilde{f}_{p,t}(r)$ is a J-decreasing function of $r \geqslant 0$. On the other hand, by Theorem (i) $\Rightarrow$ (v), $A \nabla_\alpha B \geqslant^J B$ implies that $\tilde{g}_{p,t}(r)$ is a J-increasing function of $r \geqslant 0$. Then,

$$\tilde{g}_{p,t}(r) \geqslant^J \tilde{g}_{p,t}(0) = \mathrm{e}^{t(A \nabla_\alpha B)} = \tilde{f}_{p,t}(0) \geqslant^J \tilde{f}_{p,t}(r),$$

that is, (a) holds for all $p \geqslant t \geqslant 0$, $r \geqslant 0$ and $0 \leqslant \alpha \leqslant 1$.

(b) The proof is similar to (a), bearing in mind that $A \geqslant^J A\,!_\alpha B \geqslant^J B$ holds for $0 \leqslant \alpha \leqslant 1$. $\square$

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