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EXTENSIONS OF THE HERMITE–HADAMARD INEQUALITY WITH APPLICATIONS

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Abstract. The main aim of this paper is to give improvements of various forms of the Hermite-Hadamard inequality, namely, that of Fejèr, Lupaş, Brenner-Alzer, Beesack-Pečarić. It is interesting that these improvements also imply the Hammer-Bullen inequality which deals with a comparison of the left-hand and the right-hand side of the Hermite-Hadamard inequality. These improvements are given in terms of positive linear functionals. Obtained results are used in constructing a new family of exponentially convex functions.

1. Introduction

Many important inequalities are established for the class of convex functions, but one of the most famous is so called Hermite-Hadamard inequality. This double inequality, which was first discovered by Hermite in 1881, is stated as follows (see for example [11, p. 137]): let f be a convex function on $[a,b] \subset \mathbb{R}$, where $a < b$. Then

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leqslant \frac{f(a)+f(b)}{2}. \tag{1.1}$$

This result was later incorrectly attributed to Hadamard who apparently was not aware of Hermite's discovery and today, when relating to (1.1), we use both names. Maybe it is interesting to mention that the term *convex* also stems from a result obtained by Hermite in 1881.

Note that the first inequality in (1.1) is stronger than the second one: if f is convex on $[a,b]$ then

$$\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x - f\left(\frac{a+b}{2}\right) \leqslant \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x. \tag{1.2}$$

A geometric proof of (1.2) was given in [4] and analytic one in [2] (see also [11, p. 140]). In the rest of the paper we will refer to (1.2) as the *Hammer-Bullen inequality*.

In 1906 Fejér, while studying trigonometric polynomials, obtained inequalities which generalize those of Hermite. He proved that if $w : [a,b] \to \mathbb{R}$ is a nonnegative integrable function such that the curve $y = w(x)$ is symmetric with respect to the

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straight line $x = (a+b)/2$ then for every convex function $f : [a,b] \to \mathbb{R}$ the following inequalities hold (see [11, p. 138]):

$$f\left(\frac{a+b}{2}\right)\int_a^b w(x)\mathrm{d}x \leqslant \int_a^b w(x)f(x)\mathrm{d}x \leqslant \frac{f(a)+f(b)}{2}\int_a^b w(x)\mathrm{d}x. \tag{1.3}$$

Obviously, for $w = \mathbf{1}$ the inequalities in (1.3) become the Hermite-Hadamard inequalities. Another generalization of the Hermite-Hadamard inequalities was given in [12] and [7] (or see [11, p. 143]).

THEOREM 1. *Let p,q be given positive numbers and $[a,b] \subseteq I$, $a < b$. Then the inequalities*

$$f\left(\frac{pa+qb}{p+q}\right) \leqslant \frac{1}{2y}\int_{T-y}^{T+y} f(x)\mathrm{d}x \leqslant \frac{pf(a)+qf(b)}{p+q} \tag{1.4}$$

hold for $T = (pa+qb)/(p+q)$, $y > 0$ and all continuous convex functions $f : I \to \mathbb{R}$ if

$$y \leqslant \frac{b-a}{p+q}\min\{p,q\}.$$

It can be easily verified that for $p = q = 1$ and $y = (b-a)/2$ the inequalities in (1.4) become the Hermite-Hadamard inequalities. Using the same technique as in the proof of (1.2) (see [10]) it can be proved that the first inequality in (1.4) is stronger than the second one, that is,

$$\frac{1}{2y}\int_{T-y}^{T+y} f(x)\mathrm{d}x - f\left(\frac{pa+qb}{p+q}\right) \leqslant \frac{pf(a)+qf(b)}{p+q} - \frac{1}{2y}\int_{T-y}^{T+y} f(x)\mathrm{d}x. \tag{1.5}$$

In [1] Brenner and Alzer proved the following generalization of the Hermite-Hadamard inequalities which is in fact a Fejér type variant of (1.4).

THEOREM 2. *Let p,q be given positive numbers and let $w : [a,b] \to \mathbb{R}_0^+$ be integrable and symmetric with respect to the line $x = (pa+qb)/(p+q) = T$ in the sense that $w(T+t) = w(T-t)$ for all $t \in \left[0, \frac{b-a}{p+q}\min\{p,q\}\right]$. If $f : [a,b] \to \mathbb{R}$ is a convex function then for all $y \in \mathbb{R}$ such that*

$$0 < y \leqslant \frac{b-a}{p+q}\min\{p,q\} \tag{1.6}$$

the following inequalities hold

$$f\left(\frac{pa+qb}{p+q}\right)\int_{T-y}^{T+y} w(x)\mathrm{d}x \leqslant \int_{T-y}^{T+y} w(x)f(x)\mathrm{d}x \leqslant \frac{pf(a)+qf(b)}{p+q}\int_{T-y}^{T+y} w(x)\mathrm{d}x. \tag{1.7}$$

Theorem 1 was generalized for positive linear functionals in [9], but before stating it let us first introduce some notation.

Let E be a nonempty set and L a *linear class* of functions $f : E \to \mathbb{R}$ having the properties:

(L1) $(\forall f,g\in L)(\forall a,b\in\mathbb{R})\ af+bg\in L$;

(L2) $\mathbf{1}\in L$ (that is if $(\forall t\in E)\,f(t)=1$ then $f\in L$).

In other words L is a subspace of the vector space $\mathbb{R}^E$ over $\mathbb{R}$ containing $\mathbf{1}$.
We consider *positive linear functionals* $A:L\to\mathbb{R}$, that is, we assume:

(A1) $(\forall f,g\in L)(\forall a,b\in\mathbb{R})\ \ A(af+bg)=aA(f)+bA(g)$ (linearity)

(A2) $(\forall f\in L)\,(f\geqslant 0\longrightarrow A(f)\geqslant 0)$ (positivity).

If additionally the condition $A(\mathbf{1})=1$ is satisfied, we say that A is a *positive normalized linear functional*.

THEOREM 3. *Let L satisfy L1, L2 on a nonempty set E and let A be a positive normalized linear functional. If $f:I\to\mathbb{R}$ is a continuous convex function and $[a,b]\subseteq I$, where $a<b$, then for all $g\in L$ such that $f(g)\in L$ the inequalities*

$$f\left(\frac{pa+qb}{p+q}\right)\leqslant A\left(f\left(g\right)\right)\leqslant\frac{pf\left(a\right)+qf\left(b\right)}{p+q}\tag{1.8}$$

hold, where p and q are any nonnegative real numbers such that

$$A\left(g\right)=\frac{pa+qb}{p+q}.\tag{1.9}$$

REMARK 1. It can be easily verified that Theorem 2 (and therefore Theorem 1) can be obtained as a special case of Theorem 3. Namely, for given positive numbers p and q, T and w as in Theorem 2 and y satisfying (1.6) such that $\overline{w}=\int_{T-y}^{T+y}w\left(x\right)\mathrm{d}x\neq 0$ we define $E=[a,b]$, $L=\mathscr{R}\left(E\right)$, $g=id_E$ and

$$A\left(f\right)=\frac{1}{\overline{w}}\int_{T-y}^{T+y}w\left(x\right)f(x)\mathrm{d}x.$$

Here $\mathscr{R}\left(E\right)$ denotes the subspace of all (bounded) R-integrable functions on $E=[a,b]$. Observe that A is a positive normalized linear functional and

$$A\left(g\right)=A\left(id_E\right)=\frac{1}{\overline{w}}\int_{T-y}^{T+y}w\left(x\right)x\mathrm{d}x=T=\frac{pa+qb}{p+q}.$$

By Theorem 3 we immediately obtain (1.7).

In Section 2 we give improvements of various forms of the Hermite-Hadamard inequality given in this Introduction. Using these improvements in Section 3, we construct new families of exponentially convex functions.

2. Main results

Throughout the rest of the paper with I we denote an interval in $\mathbb{R}$ and with $[a,b]$ an interval in $\mathbb{R}$ such that $-\infty < a < b < \infty$. We also need to equip our linear class L from Introduction with an additional property denoted by (L3):

(L3) $(\forall f,g \in L)\,(\min\{f,g\} \in L \wedge \max\{f,g\} \in L)$ (*lattice* property).

Obviously, $\left(\mathbb{R}^E, \leqslant\right)$ (with standard ordering) is a lattice. Also, it can be easily verified that a subspace $X \subseteq \mathbb{R}^E$ is lattice if and only if $x \in X$ implies $|x| \in X$. This is a simple consequence of the fact that for every $x \in X$ the functions $|x|$, x^- and x^+ can be defined by

$$|x|(t) = |x(t)|, \quad x^+(t) = \max\{0, x(t)\}, \quad x^-(t) = -\min\{0, x(t)\}, \qquad t \in E,$$

and

$$x^+ + x^- = |x|, \qquad x^+ - x^- = x,$$

$$\min\{x,y\} = \frac{1}{2}(x+y-|x-y|), \quad \max\{x,y\} = \frac{1}{2}(x+y+|x-y|).$$

Let us note here that $\mathscr{R}([a,b])$ from Remark 1 is a lattice since $f \in \mathscr{R}([a,b])$ implies $|f| \in \mathscr{R}([a,b])$.

A general form of the well known Jensen's inequality for convex functions (see [11, p. 45]) which involves positive normalized linear functionals is given in the following theorem.

THEOREM 4. *Let L satisfy L1, L2 on a nonempty set E and let A be a positive normalized linear functional. If ϕ is a continuous convex function on an interval I then for all $g \in L$ such that $\phi(g) \in L$ we have $A(g) \in I$ and*

$$\phi(A(g)) \leqslant A(\phi(g)). \tag{2.1}$$

Inequality (2.1) is called Jessen's inequality for convex functions.

We will also need the following lemma which is a special case of [8, p. 717, Theorem 1] for $n = 2$.

LEMMA 1. *Let ϕ be a convex function on D_ϕ, $x,y \in D_\phi$ and $p,q \in [0,1]$ such that $p+q=1$. Then*

$$\begin{aligned} &\min\{p,q\}\left[\phi(x)+\phi(y)-2\phi\left(\frac{x+y}{2}\right)\right] \\ &\leqslant p\phi(x)+q\phi(y)-\phi(px+qy) \\ &\leqslant \max\{p,q\}\left[\phi(x)+\phi(y)-2\phi\left(\frac{x+y}{2}\right)\right]. \end{aligned} \tag{2.2}$$

As our main result we give the following improvement of Theorem 3.

THEOREM 5. *Let L satisfy $(L1),(L2)$ and $(L3)$ on a nonempty set E and let A be a positive normalized linear functional. If $f: I \to \mathbb{R}$ is a continuous convex function and $[a,b] \subseteq I$ then for all $g \in L$ such that $g(E) \subseteq [a,b]$ and $f(g) \in L$ we have $A(g) \in [a,b]$ and*

$$f\left(\frac{pa+qb}{p+q}\right) \leqslant A(f(g)) \leqslant \frac{pf(a)+qf(b)}{p+q} - A(\tilde{g})\,\delta_f, \tag{2.3}$$

where p and q are any nonnegative real numbers such that

$$A(g) = \frac{pa+qb}{p+q} \tag{2.4}$$

and $\tilde{g}$, δ_f are defined by

$$\tilde{g} = \frac{1}{2}\mathbf{1} - \frac{\left|g - \frac{a+b}{2}\mathbf{1}\right|}{b-a}, \quad \delta_f = f(a)+f(b)-2f\left(\frac{a+b}{2}\right).$$

Proof. First observe that $g(E) \subseteq [a,b]$ implies

$$a = A(a\mathbf{1}) \leqslant A(g) \leqslant A(b\mathbf{1}) = b,$$

hence there exist a unique nonnegative real number $\lambda \in [0,1]$ such that $A(g) = \lambda a + (1-\lambda)b$. If p,q are nonnegative real numbers satisfying (2.4) then

$$\frac{p}{p+q} = \lambda, \quad \frac{q}{p+q} = 1-\lambda.$$

From Theorem 4 we have

$$f\left(\frac{pa+qb}{p+q}\right) = f(A(g)) \leqslant A(f(g)),$$

which is the first inequality in (2.3).

By Lemma 1 we have

$$\begin{aligned} f(g(x)) &= f\left(\frac{b-g(x)}{b-a}a + \frac{g(x)-a}{b-a}b\right) \\ &\leqslant \frac{b-g(x)}{b-a}f(a) + \frac{g(x)-a}{b-a}f(b) \\ &\quad - \min\left\{\frac{b-g(x)}{b-a}, \frac{g(x)-a}{b-a}\right\}\left[f(a)+f(b)-2f\left(\frac{a+b}{2}\right)\right]. \end{aligned}$$

Applying A to the above inequality we obtain

$$A(f(g)) \leqslant \frac{b-A(g)}{b-a}f(a) + \frac{A(g)-a}{b-a}f(b) - A(\tilde{g})\left[f(a)+f(b)-2f\left(\frac{a+b}{2}\right)\right]$$

where $\tilde{g}$ is defined on E by

$$\tilde{g}(x) = \min\left\{\frac{b-g(x)}{b-a}, \frac{g(x)-a}{b-a}\right\} = \frac{1}{2} - \frac{\left|g(x) - \frac{a+b}{2}\right|}{b-a}$$

and by $L3$ it belongs to L. By (2.4) we obtain

$$A(f(g)) \leqslant \frac{pf(a)+qf(b)}{p+q} - A(\tilde{g})\,\delta_f,$$

which is the second inequality in (2.3). □

REMARK 2. Theorem 5 is an improvement of Theorem 3 since under the required assumptions we have

$$A(\tilde{g})\,\delta_f = A\left(\frac{1}{2}\mathbf{1} - \frac{\left|g - \frac{a+b}{2}\mathbf{1}\right|}{b-a}\right)\left(f(a)+f(b)-2f\left(\frac{a+b}{2}\right)\right) \geqslant 0.$$

Furthermore (this will be important later)

$$0 \leqslant A\left(\frac{1}{2}\mathbf{1} - \frac{\left|g - \frac{a+b}{2}\mathbf{1}\right|}{b-a}\right) \leqslant \frac{1}{2}.$$

The following theorem is an improvement of Theorem 3 also.

THEOREM 6. *Let L satisfy $(L1), (L2)$ and $(L3)$ on a nonempty set E and let A be a positive normalized linear functional. If $f : I \to \mathbb{R}$ is a continuous convex function and $[a,b] \subseteq I$ then for all $g \in L$ such that*

$$g(E) \subseteq [a,b] \quad \textit{and} \quad f(g) \in L$$

and for all y such that

$$0 < y \leqslant \frac{b-a}{p+q}\min\{p,q\} \tag{2.5}$$

we have

$$f\left(\frac{pa+qb}{p+q}\right) \leqslant A(f(g)) \tag{2.6}$$
$$\leqslant \frac{pf(a)+qf(b)}{p+q} - 2A(\tilde{g})\left[\frac{pf(a)+qf(b)}{p+q} - f\left(\frac{pa+qb}{p+q}\right)\right],$$

where p and q are any nonnegative real numbers such that

$$A(g) = \frac{pa+qb}{p+q} \tag{2.7}$$

and $\tilde{g}$ is defined by

$$\tilde{g} = \frac{1}{2}\mathbf{1} - \frac{|g - A(g)\mathbf{1}|}{2y}.$$

Proof. First observe that from $g(E) \subseteq [a,b]$ follows $A(g) \in [a,b]$ and by (2.5) we have

$$a \leqslant A(g) - y < A(g) + y \leqslant b.$$

If we apply Theorem 5 on $a_1 = A(g) - y,\ b_1 = A(g) + y$ we have that

$$A(g) = \frac{A(g) - y + A(g) + y}{2} = \frac{a_1 + b_1}{2},$$

which implies that we can set $p = q = 1$ and by (2.3) we obtain

$$f(A(g)) \leqslant A(f(g))$$

and

$$\begin{aligned} A(f(g)) &\leqslant \frac{f(A(g)-y) + f(A(g)+y)}{2} \\ &\quad - A(\tilde{g})\left[f(A(g)-y) + f(A(g)+y) - 2f(A(g))\right] \\ &= (1 - 2A(\tilde{g}))\frac{f(A(g)-y) + (A(g)+y)}{2} + 2A(\tilde{g}) f(A(g)). \end{aligned}$$

Since f is convex on $[a,b]$ we know that

$$\begin{aligned} f(A(g)-y) &\leqslant \frac{b-(A(g)-y)}{b-a} f(a) + \frac{A(g)-y-a}{b-a} f(b), \\ f(A(g)+y) &\leqslant \frac{b-(A(g)+y)}{b-a} f(a) + \frac{A(g)+y-a}{b-a} f(b), \end{aligned}$$

hence

$$\frac{f(A(g)-y) + f(A(g)+y)}{2} \leqslant \frac{b-A(g)}{b-a} f(a) + \frac{A(g)-a}{b-a} f(b).$$

If p and q any nonnegative numbers such that (2.7) holds (observe that they are different from those we started with) we obtain

$$\frac{f(A(g)-y) + f(A(g)+y)}{2} \leqslant \frac{pf(a) + qf(b)}{p+q}.$$

Considering all this and the fact that $1 - 2A(\tilde{g}) \geqslant 0$ (see Remark 2) we deduce

$$\begin{aligned} A(f(g)) &\leqslant (1 - 2A(\tilde{g}))\frac{pf(a) + qf(b)}{p+q} + 2A(\tilde{g}) f(A(g)) \\ &= \frac{pf(a) + qf(b)}{p+q} - 2A(\tilde{g})\left[\frac{pf(a) + qf(b)}{p+q} - f\left(\frac{pa+qb}{p+q}\right)\right]. \qquad \square \end{aligned}$$

From (2.6) we can easily obtain a Hammer-Bullen type inequality for positive linear functionals.

COROLLARY 1. *Under the conditions of Theorem 6 the following inequality holds:*

$$(1-2A(\tilde{g}))\left[\frac{pf(a)+qf(b)}{p+q}-A(f(g))\right]\geqslant 2A(\tilde{g})\left[A(f(g))-f\left(\frac{pa+qb}{p+q}\right)\right].$$

In the following we show how these results can be used to obtain refinements of the inequalities given in Introduction as well as the related Hammer-Bullen type inequalities.

COROLLARY 2. *Let p,q be given positive numbers and let $w:[a,b]\to\mathbb{R}_0^+$ be an integrable function symmetric with respect to the line $x=(pa+qb)/(p+q)=T$ in the sense that*

$$\left(\forall t\in\left[0,\frac{b-a}{p+q}\min\{p,q\}\right]\right)w(T+t)=w(T-t).$$

If $f:[a,b]\to\mathbb{R}$ is a convex function then for all $y\in\mathbb{R}$ such that

$$0<y\leqslant\frac{b-a}{p+q}\min\{p,q\} \tag{2.8}$$

and

$$\overline{w}=\int_{T-y}^{T+y}w(x)\mathrm{d}x\neq 0$$

the following inequalities hold

$$f\left(\frac{pa+qb}{p+q}\right)\leqslant\frac{1}{\overline{w}}\int_{T-y}^{T+y}w(x)f(x)\mathrm{d}x\leqslant\frac{pf(a)+qf(b)}{p+q}-\Delta_w\delta_f, \tag{2.9}$$

where

$$\Delta_w=\frac{1}{2}-\frac{1}{\overline{w}}\int_{T-y}^{T+y}w(x)\frac{\left|x-\frac{a+b}{2}\right|}{b-a}\mathrm{d}x,$$
$$\delta_f=f(a)+f(b)-2f\left(\frac{a+b}{2}\right).$$

Proof. This is a special case of Theorem 5. First observe that for some given positive numbers p,q and $T=(pa+qb)/(p+q)$ the assumptions on y imply $a\leqslant T-y<T+y\leqslant b$, hence f is defined on $[T-y,T+y]$. If we choose E, L, A and g as in Remark 1 all the conditions of Theorem 5 will be satisfied and (2.3) accordingly becomes

$$f\left(\frac{pa+qb}{p+q}\right)\leqslant\frac{1}{\overline{w}}\int_{T-y}^{T+y}w(x)f(x)\mathrm{d}x\leqslant\frac{pf(a)+qf(b)}{p+q}-A(\tilde{g})\delta_f$$

where

$$\begin{aligned}A(\tilde{g})&=\frac{1}{\overline{w}}\int_{T-y}^{T+y}w(x)\tilde{g}(x)\mathrm{d}x \\ &=\frac{1}{2}-\frac{1}{\overline{w}}\int_{T-y}^{T+y}w(x)\frac{\left|x-\frac{a+b}{2}\right|}{b-a}\mathrm{d}x=\Delta_w.\end{aligned} \tag{2.10}$$

The condition that f has to be continuous on $[a,b]$, which is with an arbitrary A required in Theorem 5 for the same reasons as in Jessen's inequality, can be omitted in this special case. $\square$

REMARK 3. Let us emphasize here that under the conditions of Corollary 2 we have $\Delta_w \delta_f > 0$, hence (2.9) is a refinement of (1.7).

If we want to simplify Δ_w from the previous theorem we have to consider four cases:

1. $T \in (a,(3a+b)/4]$ and y satisfying (2.8) or $T \in ((3a+b)/4,(a+b)/2]$ and $0 < y \leqslant (a+b)/2 - T$.

   For such T and y we have $x-(a+b)/2 \leqslant 0$ for all $x \in [T-y,T+y]$ hence

$$\Delta_w = \frac{1}{2} + \frac{1}{\overline{W}} \int_{T-y}^{T+y} w(x) \frac{x-\frac{a+b}{2}}{b-a} \mathrm{d}x$$
$$= \frac{1}{2} + \frac{T}{b-a} - \frac{a+b}{2(b-a)} = \frac{T-a}{b-a}.$$

   Here we used the fact that symmetry of w yields

$$\frac{1}{\overline{W}} \int_{T-y}^{T+y} w(x)x\mathrm{d}x = T.$$

2. $T \in ((3a+b)/4,(a+b)/2]$ and $y > (a+b)/2 - T$ but still satisfying (2.8).

   For such T and y the function defined by $v = x-(a+b)/2$ changes sign on $[T-y,T+y]$ hence we leave Δ_w in the form (2.10).

3. $T \in ((a+b)/2,(a+3b)/4,]$ and $y > T-(a+b)/2$ but still satisfying (2.8).

   For such T and y the function v defined by $v = x-(a+b)/2$ changes sign on $[T-y,T+y]$ hence we again leave Δ_w in the form (2.10).

4. $T \in ((a+b)/2,(a+3b)/4,]$ and $0 < y \leqslant T-(b+a)/2$ or $T \in [(a+3b)/4,b)$ and y satisfying (2.8).

   For such T and y we have $x-(a+b)/2 \geqslant 0$ for all $x \in [T-y,T+y]$ hence in a similar way as in (ii) we obtain

$$\Delta_w = \frac{b-T}{b-a}.$$

As a special case of Corollary 2 we obtain the Hammer-Bullen inequality (1.2).

COROLLARY 3. *If $f:[a,b] \to \mathbb{R}$ is a convex function then the inequality*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \geqslant \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x - f\left(\frac{a+b}{2}\right) \tag{2.11}$$

holds.

Proof. This is a special case of Corollary 2 for $w = \mathbf{1}$, $p = q = 1$, $y = (b-a)/2$. In this case we have

$$\int_{T-y}^{T+y} w(x)\mathrm{d}x = \int_a^b \mathrm{d}x = b - a,$$

so from (2.9) follows

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \leqslant \frac{f(a)+f(b)}{2} - \Delta_w \delta_f. \tag{2.12}$$

A simple calculation gives $\Delta_w = 1/4$, hence

$$\begin{aligned} \frac{f(a)+f(b)}{2} - \Delta_w \delta_f &= \frac{f(a)+f(b)}{2} - \frac{1}{4}\left[f(a)+f(b)-2f\left(\frac{a+b}{2}\right)\right] \\ &= \frac{1}{2}f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{4}. \end{aligned} \tag{2.13}$$

From (2.12) and (2.13) we obtain

$$\begin{aligned} 2f\left(\frac{a+b}{2}\right) &\leqslant \frac{2}{b-a}\int_a^b f(x)\,\mathrm{d}x \\ &\leqslant f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2}, \end{aligned}$$

which implies

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \geqslant \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x - f\left(\frac{a+b}{2}\right). \qquad \square$$

In a similar way as a special case of Corollary 2 we obtain (1.5) but we skip the proof here.

COROLLARY 4. *Let p, q be given positive numbers and let $w : [a,b] \to \mathbb{R}_0^+$ be an integrable function symmetric with respect to the line $x = (pa+qb)/(p+q) = T$ in the sense that*

$$(\forall t \in [0, \min\{T-a, b-T\}])\, w(T+t) = w(T-t).$$

If $f : [a,b] \to \mathbb{R}$ is a convex function then for all y such that

$$0 < y \leqslant \frac{b-a}{p+q}\min\{p,q\} \text{ and } \overline{w} = \int_{T-y}^{T+y} w(x)\mathrm{d}x \neq 0$$

the following inequalities hold

$$\begin{aligned} f\left(\frac{pa+qb}{p+q}\right) &\leqslant \frac{1}{\overline{w}}\int_{T-y}^{T+y} w(x)f(x)\mathrm{d}x \\ &\leqslant \frac{pf(a)+qf(b)}{p+q} - \Delta_w\left(\frac{pf(a)+qf(b)}{p+q} - f\left(\frac{pa+qb}{p+q}\right)\right), \end{aligned} \tag{2.14}$$

where

$$\Delta_w = 1 - \frac{1}{y\overline{w}} \left[\int_T^{T+y} w(x)\,x\mathrm{d}x - \int_{T-y}^{T} w(x)\,x\mathrm{d}x \right].$$

Proof. This is a special case of Theorem 6 for E, L, A and g as in Remark 1. In this case (2.6) becomes

$$\begin{aligned} f\left(\frac{pa+qb}{p+q}\right) &\leqslant \frac{1}{\overline{w}} \int_{T-y}^{T+y} w(x)\,f(x)\mathrm{d}x \\ &\leqslant \frac{pf(a)+qf(b)}{p+q} - 2A(\tilde{g}) \left[\frac{pf(a)+qf(b)}{p+q} - f\left(\frac{pa+qb}{p+q}\right) \right], \end{aligned}$$

where

$$\begin{aligned} A(\tilde{g}) &= \frac{1}{2} - \frac{1}{2y\overline{w}} \int_{T-y}^{T+y} w(x)\,|x-T|\,\mathrm{d}x \\ &= \frac{1}{2} - \frac{1}{2y\overline{w}} \left[T\int_{T-y}^{T} w(x)\,\mathrm{d}x - T\int_T^{T+y} w(x)\,\mathrm{d}x - \int_{T-y}^{T} w(x)\,x\mathrm{d}x + \int_T^{T+y} w(x)\,x\mathrm{d}x \right] \\ &= \frac{1}{2} - \frac{1}{2y\overline{w}} \left[\int_T^{T+y} w(x)\,x\mathrm{d}x - \int_{T-y}^{T} w(x)\,x\mathrm{d}x \right] = \frac{1}{2}\Delta_w. \qquad \square \end{aligned}$$

REMARK 4. A Hammer-Bullen type inequality easily follows from (2.14): under the conditions of Corollary 4 the following inequality holds

$$\begin{aligned} &(1-\Delta_w) \left[\frac{pf(a)+qf(b)}{p+q} - \frac{1}{\overline{w}} \int_{T-y}^{T+y} w(x)\,f(x)\mathrm{d}x \right] \\ &\quad \geqslant \Delta_w \left[\frac{1}{\overline{w}} \int_{T-y}^{T+y} w(x)\,f(x)\mathrm{d}x - f\left(\frac{pa+qb}{p+q}\right) \right]. \end{aligned}$$

In the next corollary we give a refinement of the discrete analogue of the Hermite-Hadamard inequalities (see [11, p. 145]).

COROLLARY 5. *Let $x_1 < x_2 < \cdots < x_n$ be equidistant points in I. Then for every convex function $f : I \to \mathbb{R}$ the following inequalities are valid:*

$$\begin{aligned} f\left(\frac{x_1+x_n}{2}\right) &\leqslant \frac{1}{n}\sum_{i=1}^{n} f(x_i) \\ &\leqslant \frac{f(x_1)+f(x_n)}{2} - \Delta_n \left(\frac{f(x_1)+f(x_n)}{2} - f\left(\frac{x_1+x_n}{2}\right) \right), \end{aligned}$$

where

$$\Delta_n = \begin{cases} 1 - \frac{k+1}{2k+1}, & n = 2k+1 \\ 1 - \frac{k}{2k-1}, & n = 2k \end{cases}, \quad k \in \mathbb{N}_0.$$

Proof. This is a special case of Theorem 5 for $E=[a,b]=[x_1,x_n]$, $L=\mathbb{R}^E$, $g=id_E$ and A defined by

$$A(f)=\frac{1}{n}\sum_{i=1}^{n} f(x_i).$$

Since

$$(\forall i\in\{1,\dots,n-1\})\ \ x_{i+1}-x_i=h,$$

we have

$$\begin{aligned} A(g)=A(id_E)&=\frac{1}{n}\sum_{i=1}^{n} x_i=\frac{nx_1+\frac{(n-1)(h+(n-1)h)}{2}}{n}\\ &=\frac{2x_1+(n-1)h}{2}=\frac{x_1+x_n}{2}, \end{aligned}$$

that is, we can choose $p=q=1$ and (2.3) becomes

$$f\left(\frac{x_1+x_n}{2}\right)\leqslant \frac{1}{n}\sum_{i=1}^{n} f(x_i)\leqslant \frac{f(x_1)+f(x_n)}{2}-A(\tilde{g})\,\delta_f,$$

where

$$\delta_f=2\left(\frac{f(x_1)+f(x_n)}{2}-f\left(\frac{x_1+x_n}{2}\right)\right)$$

and

$$\begin{aligned} A(\tilde{g})&=\frac{1}{2}-\frac{1}{n(x_n-x_1)}\sum_{i=1}^{n}\left|x_i-\frac{x_1+x_n}{2}\right|\\ &=\frac{1}{2}-\frac{1}{n(n-1)h}\sum_{i=1}^{n}\left|x_1+(i-1)h-\frac{2x_1+(n-1)h}{2}\right|\\ &=\frac{1}{2}-\frac{1}{2n(n-1)}\sum_{i=1}^{n}|2i-n-1|. \end{aligned}$$

Considering the parity of n we obtain

$$\begin{aligned} A(\tilde{g})&=\begin{cases} \frac{1}{2}-\frac{1}{2k(2k+1)}\sum_{i=1}^{k}2i, & n=2k+1\\ \frac{1}{2}-\frac{1}{2k(2k-1)}\sum_{i=1}^{k}(2i-1), & n=2k \end{cases}\\ &=\begin{cases} \frac{1}{2}\left(1-\frac{k+1}{2k+1}\right), & n=2k+1\\ \frac{1}{2}\left(1-\frac{k}{2k-1}\right), & n=2k \end{cases}=\frac{1}{2}\Delta_n. \end{aligned}$$

Observe that for $n=1$ and $n=2$ we have $\Delta_n=0$. □

To give our next result we need to add yet another property to the linear class L.

Let $\mathscr{A}$ be an algebra of subsets of E and let L be a class of functions $f:E\to\mathbb{R}$ having the properties $(L1)$, $(L2)$, $(L3)$ and

$(L4)$ $(\forall f \in L)(\forall E_1 \in \mathscr{A})\ fC_{E_1} \in L;$

where C_{E_1} is the characteristic function of E_1, that is,

$$C_{E_1}(t) = \begin{cases} 1, & t \in E_1 \\ 0, & t \in E \setminus E_1 \end{cases}.$$

It can be easily seen that for every $E_1 \in \mathscr{A}$ the following assertions hold true:

(i) $C_{E_1} \in L$.

(ii) If A is a positive linear functional on L such that $A(C_{E_1}) > 0$ and $g \in L$ then A_1 defined by

$$A_1(g) = \frac{A(gC_{E_1})}{A(C_{E_1})}$$

is a positive normalized linear functional.

(iii) If A is a positive linear functional on L and $g \in L$ then

$$A(C_{E_1}) + A\left(C_{E \setminus E_1}\right) = 1$$

and

$$A(gC_{E_1}) + A\left(gC_{E \setminus E_1}\right) = A(g).$$

THEOREM 7. *Let L satisfy $(L1)-(L4)$ on a nonempty set E and let $f : I \to \mathbb{R}$ be a continuous convex function while $g, h \in L$ are such that $f(g), f(h) \in L$. Let A, B be two positive normalized linear functionals on L such that $A(h) = B(g)$. If $E_1 \in \mathscr{A}$ satisfies $A(C_{E_1}) > 0$, $A\left(C_{E \setminus E_1}\right) > 0$ and*

$$(\forall t \in E)\quad a \leqslant g(t) \leqslant b,$$

where

$$a = \min\left\{\frac{A(hC_{E_1})}{A(C_{E_1})}, \frac{A\left(hC_{E \setminus E_1}\right)}{A\left(C_{E \setminus E_1}\right)}\right\},$$
$$b = \max\left\{\frac{A(hC_{E_1})}{A(C_{E_1})}, \frac{A\left(hC_{E \setminus E_1}\right)}{A\left(C_{E \setminus E_1}\right)}\right\}$$

then

$$f(A(h)) \leqslant B(f(g)) \leqslant A(f(h)) - B(\tilde{g})\,\delta_f, \tag{2.15}$$

where $\tilde{g}$ and δ_f are defined as in Theorem 5. In the limiting case $a = b$, (2.15) becomes

$$f(A(h)) = B(f(g)) \leqslant A(f(h)).$$

Proof. By Jessen's inequality (see (ii) above) we have

$$f\left(\frac{A(hC_{E_1})}{A(C_{E_1})}\right) \leqslant \frac{A(f(h)C_{E_1})}{A(C_{E_1})}$$

and

$$f\left(\frac{A\left(hC_{E\backslash E_1}\right)}{A\left(C_{E\backslash E_1}\right)}\right) \leqslant \frac{A\left(f(h)C_{E\backslash E_1}\right)}{A\left(C_{E\backslash E_1}\right)}.$$

Without loss of generality we may assume

$$a = \min\left\{\frac{A(hC_{E_1})}{A(C_{E_1})}, \frac{A\left(hC_{E\backslash E_1}\right)}{A\left(C_{E\backslash E_1}\right)}\right\} = \frac{A(hC_{E_1})}{A(C_{E_1})},$$

$$b = \max\left\{\frac{A(hC_{E_1})}{A(C_{E_1})}, \frac{A\left(hC_{E\backslash E_1}\right)}{A\left(C_{E\backslash E_1}\right)}\right\} = \frac{A\left(hC_{E\backslash E_1}\right)}{A\left(C_{E\backslash E_1}\right)}.$$

If $a < b$ and

$$p = A(C_{E_1}), \quad q = A\left(C_{E\backslash E_1}\right),$$

we have

$$p + q = A(C_E) = A(\mathbf{1}) = 1,$$

$$B(g) = A(h) = A(hC_{E_1}) + A\left(hC_{E\backslash E_1}\right) = pa + qb,$$

and applying Theorem 5 to B and g by (2.3) we obtain

$$\begin{aligned} f(A(h)) = f(B(g)) &\leqslant B(f(g)) \leqslant pf(a) + qf(b) - B(\tilde{g})\,\delta_f \\ &= A(C_{E_1})\, f\left(\frac{A(hC_{E_1})}{A(C_{E_1})}\right) + A\left(C_{E\backslash E_1}\right) f\left(\frac{A\left(hC_{E\backslash E_1}\right)}{A\left(C_{E\backslash E_1}\right)}\right) - B(\tilde{g})\,\delta_f \\ &\leqslant A(f(h)C_{E_1}) + A\left(f(h)C_{E\backslash E_1}\right) - B(\tilde{g})\,\delta_f \\ &= A(f(h)) - B(\tilde{g})\,\delta_f. \end{aligned}$$

If $a = b$ it follows that g is a constant function and the limiting case follows immediately. □

Theorem 7 is an improvement of [11, Theorem 5.14] and at the same time it gives a refinement of Jessen's inequality (2.1). We also give the following improvement of [11, Theorem 5.14].

THEOREM 8. *Suppose that the assumptions of Theorem 7 hold. If $a < b$, then for all y such that*

$$0 < y \leqslant \min\{B(g) - a, b - B(g)\} \tag{2.16}$$

the following inequalities are valid:

$$\begin{aligned} f(A(h)) &\leqslant B(f(g)) \\ &\leqslant A(f(h)) - 2B(\tilde{g})\left[A(C_{E_1})\, f(a) + A\left(C_{E\backslash E_1}\right) f(b) - f(B(g))\right], \end{aligned}$$

where

$$\tilde{g} = \frac{1}{2}\mathbf{1} - \frac{|g - B(g)\mathbf{1}|}{2y}.$$

Proof. This proof is almost identical to the proof of Theorem 7 except that we use Theorem 6 instead of Theorem 5, hence for $a < b$ and y satisfying (2.16) using (2.6) we obtain

$$\begin{aligned} f(A(h)) &\leqslant B(f(g)) \\ &\leqslant A(f(h)) - 2B(\tilde{g})\left[A(C_{E_1})f(a) + A\left(C_{E\setminus E_1}\right)f(b) - f(B(g))\right]. \end{aligned}$$

□

3. $n-$exponential convexity and exponential convexity of Hammer-Bullen differences, applications to Stolarsky type means

Motivated by Theorems 5 and 6, we define two functionals $\Phi_i : L_g \to \mathbb{R}$, $i = 1,2$, by

$$\Phi_1(f) = \frac{pf(a) + qf(b)}{p+q} - A(f(g)) - A(\tilde{g})\delta_f \tag{3.1}$$

where $A, g, \tilde{g}, p$ and q are as in Theorem 5, $L_g = \{f : I \to \mathbb{R} : f(g) \in L\}$, $[a,b] \subseteq I$ and

$$\Phi_2(f) = \frac{pf(a) + qf(b)}{p+q} - A(f(g)) - 2A(\tilde{g})\left[\frac{pf(a) + qf(b)}{p+q} - f\left(\frac{pa+qb}{p+q}\right)\right] \tag{3.2}$$

where $A, g, \tilde{g}, p$ and q are as in Theorem 6, L_g as in the above and $[a,b] \subseteq I$. Obviously, Φ_1 and Φ_2 are linear.

If f is additionally continuous and convex then Theorems 5 and 6 imply $\Phi_i(f) \geqslant 0, i = 1,2$.

In the following with f_0 we denote the function defined by $f_0(x) = x^2$ on whatever domain we need.

Now, we give mean value theorems for the functionals Φ_i, $i = 1,2$.

THEOREM 9. *Let L satisfy $(L1),(L2)$ and $(L3)$ on a nonempty set E and let A be a positive normalized linear functional on L. Let $g \in L$ be such that $f_0 \in L_g$, $g(E) \in [a,b], [a,b] \subseteq I$ and let $f \in C^2(I)$ be such that $f \in L_g$. If Φ_1 and Φ_2 are linear functionals defined as in (3.1) and (3.2) then there exist $\xi_i \in [a,b]$ such that*

$$\Phi_i(f) = \frac{f''(\xi_i)}{2}\Phi_i(f_0), \quad i = 1,2.$$

Proof. We give a proof for the functional Φ_1. Since $f \in C^2(I)$ there exist real numbers $m = \min_{x\in[a,b]} f''(x)$ and $M = \max_{x\in[a,b]} f''(x)$. It is easy to show that the functions f_1, f_2 defined by

$$f_1(x) = \frac{M}{2}x^2 - f(x), \quad f_2(x) = f(x) - \frac{m}{2}x^2$$

are continuous and convex, therefore $\Phi_1(f_1) \geqslant 0, \Phi_1(f_2) \geqslant 0$. This implies

$$\frac{m}{2}\Phi_1(f_0) \leqslant \Phi_1(f) \leqslant \frac{M}{2}\Phi_1(f_0).$$

If $\Phi_1(f_0) = 0$, there is nothing to prove. Suppose $\Phi_1(f_0) > 0$. We have

$$m \leqslant \frac{2\Phi_1(f)}{\Phi_1(x^2)} \leqslant M.$$

Hence, there exists $\xi_1 \in [a,b]$ such that

$$\Phi_1(f) = \frac{f''(\xi_1)}{2}\Phi_1(f_0). \qquad \square$$

THEOREM 10. *Let L satisfy $(L1),(L2)$ and $(L3)$ on a non-empty set E and let A be a positive normalized linear functional on L. Let $g \in L$ be such that $f_0 \in L_g$, $g(E) \in [a,b], [a,b] \subseteq I$ and $f_1, f_2 \in C^2(I)$ such that $f_1, f_2 \in L_g$. If Φ_1 and Φ_2 are linear functionals defined as in (3.1) and (3.2) then there exist $\xi_i \in [a,b]$ such that*

$$\frac{\Phi_i(f_1)}{\Phi_i(f_2)} = \frac{f_1''(\xi_i)}{f_2''(\xi_i)}, \quad i = 1,2$$

provided that the denominators are non-zero.

Proof. We give a proof for the functional Φ_1. Define $f_3 \in C^2([a,b])$ by

$$f_3 = c_1 f_1 - c_2 f_2, \text{ where } c_1 = \Phi_1(f_2), \ c_2 = \Phi_1(f_1).$$

Using Theorem 9 we get that there exists $\xi_1 \in [a,b]$ such that

$$\left(c_1 \frac{f_1''(\xi_1)}{2} - c_2 \frac{f_2''(\xi_1)}{2}\right)\Phi_1(f_0) = 0.$$

Since $\Phi_1(f_0) \neq 0$, (otherwise we have a contradiction with $\Phi_1(f_2) \neq 0$, by Theorem 9), we obtain

$$\frac{\Phi_1(f_1)}{\Phi_1(f_2)} = \frac{f_1''(\xi_1)}{f_2''(\xi_1)}. \qquad \square$$

Next we introduce some function properties which are going to be explored here and immediately after that we give some characterizations of these properties.

DEFINITION 1. A function $\psi\colon I \to \mathbb{R}$ is $n-$exponentially convex in the Jensen sense on I if

$$\sum_{i,j=1}^{n} \xi_i \xi_j \psi\left(\frac{x_i + x_j}{2}\right) \geqslant 0$$

holds for all choices $\xi_i \in \mathbb{R}$ and $x_i \in I$, $i = 1,\dots,n$.

A function $\psi\colon I \to \mathbb{R}$ is $n-$exponentially convex if it is $n-$exponentially convex in the Jensen sense and continuous on I.

REMARK 5. It is clear from the definition that $1-$exponentially convex functions in the Jensen sense are in fact non-negative functions. Also, $n-$exponentially convex functions in the Jensen sense are $k-$exponentially convex in the Jensen sense for every $k \in \mathbb{N}$, $k \leqslant n$.

By definition of positive semi-definite matrices and some basic linear algebra we have the following proposition.

PROPOSITION 1. *If ψ is an n-exponentially convex in the Jensen sense, then the matrix* $\left[\psi\left(\frac{x_i+x_j}{2}\right)\right]_{i,j=1}^{k}$ *is a positive semi-definite matrix for all $k \in \mathbb{N}, k \leqslant n$. Particularly,* $\det\left[\psi\left(\frac{x_i+x_j}{2}\right)\right]_{i,j=1}^{k} \geqslant 0$ *for all $k \in \mathbb{N}$, $k \leqslant n$.*

DEFINITION 2. A function $\psi\colon I \to \mathbb{R}$ is exponentially convex in the Jensen sense on I if it is $n-$exponentially convex in the Jensen sense for all $n \in \mathbb{R}$.

A function $\psi\colon I \to \mathbb{R}$ is exponentially convex if it is exponentially convex in the Jensen sense and continuous.

REMARK 6. It is known (and easy to show) that $\psi : I \to \mathbb{R}^+$ is log-convex in the Jensen sense if and only if

$$\alpha^2\psi(x) + 2\alpha\beta\psi\left(\frac{x+y}{2}\right) + \beta^2\psi(y) \geqslant 0$$

holds for every $\alpha, \beta \in \mathbb{R}$ and $x, y \in I$. It follows that a positive function is log-convex in the Jensen-sense if and only if it is $2-$exponentially convex in the Jensen sense. Also, using basic convexity theory, it follows that a positive function is log-convex if and only if it is $2-$exponentially convex.

We will also need the following result (see for example [11]).

PROPOSITION 2. *If Ψ is a convex function on I and if $x_1 \leqslant y_1, x_2 \leqslant y_2, x_1 \neq x_2, y_1 \neq y_2$ then the following inequality is valid*

$$\frac{\Psi(x_2)-\Psi(x_1)}{x_2-x_1} \leqslant \frac{\Psi(y_2)-\Psi(y_1)}{y_2-y_1}. \tag{3.3}$$

If Ψ is concave on I the inequality reverses.

When dealing with functions with different degree of smoothness divided differences are found to be very useful.

DEFINITION 3. The second order devided difference of a function $f\colon I \to \mathbb{R}$ at mutually different points $y_0, y_1, y_2 \in I$ is defined recursively by

$$
\begin{aligned}
[y_i; f] &= f(y_i), \quad i = 0,1,2 \\
[y_i, y_{i+1}; f] &= \frac{f(y_{i+1}) - f(y_i)}{y_{i+1} - y_i}, \quad i = 0,1 \\
[y_0, y_1, y_2; f] &= \frac{[y_1, y_2; f] - [y_0, y_1; f]}{y_2 - y_0}.
\end{aligned}
\tag{3.4}
$$

REMARK 7. The value $[y_0, y_1, y_2; f]$ is independent of the order of the points y_0, y_1 and y_2. This definition may be extended to include the case in which some or all the points coincide. Namely, taking the limit $y_1 \to y_0$ in (3.4), we get

$$
\lim_{y_1 \to y_0} [y_0, y_1, y_2; f] = [y_0, y_0, y_2; f] = \frac{f(y_2) - f(y_0) - f'(y_0)(y_2 - y_0)}{(y_2 - y_0)^2}, y_2 \neq y_0
$$

provided f' exists, and furthermore, taking the limits $y_i \to y_0, i = 1,2$ in (3.4), we get

$$
\lim_{y_2 \to y_0} \lim_{y_1 \to y_0} [y_0, y_1, y_2; f] = [y_0, y_0, y_0; f] = \frac{f''(y_0)}{2}
$$

provided that f'' exists.

We use an idea from [5] to give an elegant method of producing an $n-$ exponentially convex functions and exponentially convex functions applying the functionals Φ_1 and Φ_2 on a given family with the same property.

THEOREM 11. *Let $\Phi_i, \quad i = 1,2,$ be linear functionals defined as in (3.1) and (3.2). Let $\Upsilon = \{f_s : s \in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions defined on an open interval I such that $\Upsilon \subseteq L_g$ and that the function $s \mapsto [y_0, y_1, y_2; f_s]$ is $n-$exponentially convex in the Jensen sense on J for every three mutually different points $y_0, y_1, y_2 \in I$. Then $s \mapsto \Phi_i(f_s)$ is an $n-$exponentially convex function in the Jensen sense on J. If the function $s \mapsto \Phi_i(f_s)$ is continuous on J then it is $n-$exponentially convex on J.*

Proof. For $\xi_i \in \mathbb{R}$, $i = 1, \ldots, n$ and $s_i \in J$, $i = 1, \ldots, n$, we define the function $h : I \to \mathbb{R}$ by

$$
h(y) = \sum_{i,j=1}^{n} \xi_i \xi_j f_{\frac{s_i + s_j}{2}}(y).
$$

Using the assumption that the function $s \mapsto [y_0, y_1, y_2; f_s]$ is $n-$exponentially convex in the Jensen sense we obtain

$$
[y_0, y_1, y_2; h] = \sum_{i,j=1}^{n} \xi_i \xi_j [y_0, y_1, y_2; f_{\frac{s_i + s_j}{2}}] \geqslant 0,
$$

which in turn implies that h is a convex (and continuous) function on I, therefore $\Phi_i(h) \geqslant 0, \quad i = 1,2$. Hence

$$\sum_{i,j=1}^{n} \xi_i \xi_j \Phi_i(f_{\frac{s_i+s_j}{2}}) \geqslant 0.$$

We conclude that the function $s \mapsto \Phi_i(f_s)$ is $n-$exponentially convex on J in the Jensen sense. If the function $s \mapsto \Phi_i(f_s)$ is also continuous on J, then $s \mapsto \Phi_i(f_s)$ is $n-$exponentially convex by definition. □

The following corollary is an immediate consequence of the above theorem.

COROLLARY 6. *Let $\Phi_i, \quad i = 1,2,$ be linear functionals defined as in (3.1) and (3.2). Let $\Upsilon = \{f_s : s \in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions defined on an open interval I such that $\Upsilon \subseteq L_g$ and that the function $s \mapsto [y_0, y_1, y_2; f_s]$ is exponentially convex in the Jensen sense on J for every three mutually different points $y_0, y_1, y_2 \in I$. Then $s \mapsto \Phi_i(f_s)$ is an exponentially convex function in the Jensen sense on J. If the function $s \mapsto \Phi_i(f_s)$ is continuous on J then it is exponentially convex on J.*

COROLLARY 7. *Let $\Phi_i, \quad i = 1,2,$ be linear functionals defined as in (3.1) and (3.2). Let $\Omega = \{f_s : s \in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions defined on an open interval I such that $\Omega \subseteq L_g$ and that the function $s \mapsto [y_0, y_1, y_2; f_s]$ is $2-$exponentially convex in the Jensen sense on J for every three mutually different points $y_0, y_1, y_2 \in I$. Then the following statements hold:*

(i) *If the function $s \mapsto \Phi_i(f_s)$ is continuous on J then it is $2-$exponentially convex function on J. If $s \mapsto \Phi_i(f_s)$ is additionally strictly positive than it is also log-convex on J.*

(ii) *If the function $s \mapsto \Phi_i(f_s)$ is strictly positive and differentiable on J then for every $s, q, u, v \in J$, such that $s \leqslant u$ and $q \leqslant v$, we have*

$$\mu_{s,q}(\Phi_i, \Omega) \leqslant \mu_{u,v}(\Phi_i, \Omega), \;\; i = 1,2, \tag{3.5}$$

where

$$\mu_{s,q}(\Phi_i, \Omega) = \begin{cases} \left(\frac{\Phi_i(f_s)}{\Phi_i(f_q)}\right)^{\frac{1}{s-q}} & , s \neq q, \\ \exp\left(\frac{\frac{d}{ds}\Phi_i(f_s)}{\Phi_i(f_s)}\right) & , s = q. \end{cases} \tag{3.6}$$

for $f_s, f_q \in \Omega$.

Proof. (i) This is an immediate consequence of Theorem 11 and Remark 6.

(ii) Since by (i) the function $s \mapsto \Phi_i(f_s)$ is log-convex on J, that is, the function $s \mapsto \log \Phi_i(f_s)$ is convex on J. Applying Proposition 2 we get

$$\frac{\log \Phi_i(f_s) - \log \Phi_i(f_q)}{s-q} \leqslant \frac{\log \Phi_i(f_u) - \log \Phi_i(f_v)}{u-v} \tag{3.7}$$

for $s \leqslant u, q \leqslant v, s \neq q, u \neq v$, and therefrom conclude that

$$\mu_{s,q}(\Phi_i,\Omega) \leqslant \mu_{u,v}(\Phi_i,\Omega), \quad i=1,2.$$

Cases $s=q$ and $u=v$ follow from (3.7) as limit cases. □

REMARK 8. Note that the results from Theorem 11, Corollary 6, Corollary 7 still hold when two of the points $y_0, y_1, y_2 \in I$ coincide, say $y_1 = y_0$, for a family of differentiable functions f_s such that the function $s \mapsto [y_0, y_1, y_2; f_s]$ is $n-$exponentially convex in the Jensen sense (exponentially convex in the Jensen sense, log-convex in the Jensen sense), and furthermore, they still hold when all three points coincide for a family of twice differentiable functions with the same property. The proofs are obtained by recalling Remark 7 and suitable characterization of convexity.

Now, we present several families of functions which fulfil the conditions of Theorem 11, Corollary 6 and Corollary 7 (and Remark 8). This enable us to construct a large families of functions which are exponentially convex. For a discussion related to this problem see [3].

In the rest of the section we consider only Φ_1 and Φ_2 defined as in (3.1) and (3.2) with A which is continuous and g such that compositions with any function from the chosen familly Ω_i as well as with other functions which appear as arguments of Φ_1 and Φ_2 remain in L.

EXAMPLE 1. Consider a family of functions

$$\Omega_1 = \{g_s : \mathbb{R} \to [0,\infty) : s \in \mathbb{R}\}$$

defined by

$$g_s(x) = \begin{cases} \frac{1}{s^2} e^{sx}, & s \neq 0, \\ \frac{1}{2} x^2, & s = 0. \end{cases}$$

We have $\frac{d^2 g_s}{dx^2}(x) = e^{sx} > 0$ which shows that g_s is convex on $\mathbb{R}$ for every $s \in \mathbb{R}$ and $s \mapsto \frac{d^2 g_s}{dx^2}(x)$ is exponentially convex by definition. Using analogous arguing as in the proof of Theorem 11 we also have that $s \mapsto [y_0, y_1, y_2; g_s]$ is exponentially convex (and so exponentially convex in the Jensen sense). Using Theorem 6 we conclude that $s \mapsto \Phi_i(g_s)$, $i=1,2$, are exponentially convex in the Jensen sense. It is easy to verify that these mappings are continuous (although mapping $s \mapsto g_s$ is not continuous for $s=0$), so they are exponentially convex.

For this family of functions, $\mu_{s,q}(\Phi_i,\Omega_1)$, $i=1,2$, from (3.6) become

$$\mu_{s,q}(\Phi_i,\Omega_1) = \begin{cases} \left(\frac{\Phi_i(g_s)}{\Phi_i(g_q)}\right)^{\frac{1}{s-q}}, & s \neq q, \\ \exp\left(\frac{\Phi_i(id \cdot g_s)}{\Phi_i(g_s)} - \frac{2}{s}\right), & s = q \neq 0, \\ \exp\left(\frac{\Phi_i(id \cdot g_0)}{3\Phi_i(g_0)}\right), & s = q = 0, \end{cases}$$

and using (3.5) they are monotonous functions in parameters s and q.

Using Theorem 10 it follows that for $i = 1,2$

$$M_{s,q}(\Phi_i,\Omega_1) = \log \mu_{s,q}(\Phi_i,\Omega_1)$$

satisfy $a \leqslant M_{s,q}(\Phi_i,\Omega_1) \leqslant b$, which shows that $M_{s,q}(\Phi_i,\Omega_1)$ are means (of a function g). Notice that by (3.5) they are monotonous.

EXAMPLE 2. Consider a family of functions

$$\Omega_2 = \{f_s : (0,\infty) \to \mathbb{R} : s \in \mathbb{R}\}$$

defined by

$$f_s(x) = \begin{cases} \frac{x^s}{s(s-1)}, \ s \neq 0,1, \\ -\log x, \ s = 0, \\ x \log x, \ s = 1. \end{cases}$$

Here, $\frac{d^2 f_s}{dx^2}(x) = x^{s-2} = e^{(s-2)\ln x} > 0$ which shows that f_s is convex for $x > 0$ and $s \mapsto \frac{d^2 f_s}{dx^2}(x)$ is exponentially convex by definition. Arguing as in Example 1 we get that the mappings $s \mapsto \Phi_i(g_s), \quad i = 1,2$ are exponentially convex. Functions (3.6) in this case are equal to:

$$\mu_{s,q}(\Phi_i,\Omega_2) = \begin{cases} \left(\frac{\Phi_i(f_s)}{\Phi_i(f_q)}\right)^{\frac{1}{s-q}}, & s \neq q, \\ \exp\left(\frac{1-2s}{s(s-1)} - \frac{\Phi_i(f_s f_0)}{\Phi_i(f_s)}\right), & s = q \neq 0,1, \\ \exp\left(1 - \frac{\Phi_i(f_0^2)}{2\Phi_i(f_0)}\right), & s = q = 0, \\ \exp\left(-1 - \frac{\Phi_i(f_0 f_1)}{2\Phi_i(f_1)}\right), & s = q = 1. \end{cases}$$

If Φ_i is positive, then Theorem 10 applied for $f = f_s \in \Omega_2$ and $g = f_q \in \Omega_2$ yields that there exists $\xi \in [a,b]$ such that

$$\xi^{s-q} = \frac{\Phi_i(f_s)}{\Phi_i(f_q)}.$$

Since the function $\xi \mapsto \xi^{s-q}$ is invertible for $s \neq q$, we then have

$$a \leqslant \left(\frac{\Phi_i(f_s)}{\Phi_i(f_q)}\right)^{\frac{1}{s-q}} \leqslant b, \tag{3.8}$$

which together with the fact that $\mu_{s,q}(\Phi_i,\Omega_2)$ is continuous, symmetric and monotonous (by (3.5)), shows that $\mu_{s,q}(\Phi_i,\Omega_2)$ is a mean (of a function h).

EXAMPLE 3. Consider a family of functions

$$\Omega_3 = \{h_s : (0,\infty) \to (0,\infty) : s \in (0,\infty)\}$$

defined by

$$h_s(x) = \begin{cases} \frac{s^{-x}}{\ln^2 s}, & s \neq 1, \\ \frac{x^2}{2}, & s = 1. \end{cases}$$

Since $s \mapsto \frac{d^2 h_s}{dx^2}(x) = s^{-x}$ is the Laplace transform of a non-negative function (see [13]) it is exponentially convex. Obviously h_s are convex functions for every $s > 0$.

For this family of functions, $\mu_{s,q}(\Phi_i, \Omega_3)$, from (3.6) becomes

$$\mu_{s,q}(\Phi_i, \Omega_3) = \begin{cases} \left(\frac{\Phi_i(h_s)}{\Phi_i(h_q)}\right)^{\frac{1}{s-q}}, & s \neq q, \\ \exp\left(-\frac{\Phi_i(id \cdot h_s)}{s\Phi_i(h_s)} - \frac{2}{s \ln s}\right), & s = q \neq 1, \\ \exp\left(-\frac{2\Phi_i(id \cdot h_1)}{3\Phi_i(h_1)}\right), & s = q = 1, \end{cases}$$

and it is monotonous in parameters s and q by (3.5).

Using Theorem 10, it follows that

$$M_{s,q}(\Phi_i, \Omega_3) = -L(s,q) \log \mu_{s,q}(\Phi_i, \Omega_3),$$

satisfies $a \leqslant M_{s,q}(\Phi_i, \Omega_3) \leqslant b$, which shows that $M_{s,q}(\Phi_i, \Omega_3)$ is a mean (of a function h). $L(s,q)$ is the logarithmic mean defined by $L(s,q) = \frac{s-q}{\log s - \log q}$, $s \neq q$, $L(s,s) = s$.

EXAMPLE 4. Consider a family of functions

$$\Omega_4 = \{k_s : (0,\infty) \to (0,\infty) : s \in (0,\infty)\}$$

defined by

$$k_s(x) = \frac{e^{-x\sqrt{s}}}{s}$$

Since $s \mapsto \frac{d^2 k_s}{dx^2}(x) = e^{-x\sqrt{s}}$ is the Laplace transform of a non-negative function (see [13]) it is exponentially convex. Obviously k_s are convex functions for every $s > 0$.

For this family of functions, $\Phi_i, \Omega_4)$ from (3.6) becomes

$$\mu_{s,q}(\Phi_i, \Omega_4) = \begin{cases} \left(\frac{\Phi_i(k_s)}{\Phi_i(k_q)}\right)^{\frac{1}{s-q}}, & s \neq q, \\ \exp\left(-\frac{\Phi_i(id \cdot k_s)}{2\sqrt{s}\Phi_i(k_s)} - \frac{1}{s}\right), & s = q, \end{cases}$$

and it is monotonous function in parameters s and q by (3.5).

Using Theorem 10, it follows that

$$M_{s,q}(\Phi_i, \Omega_4) = -\left(\sqrt{s} + \sqrt{q}\right) \log \mu_{s,q}(\Phi_i, \Omega_4)$$

satisfies $a \leqslant M_{s,q}(\Phi_i, \Omega_4) \leqslant b$, which shows that $M_{s,q}(\Phi_i, \Omega_4)$ is a mean (of a function h).

REFERENCES

[1] J. L. BRENNER, H. ALZER, *Integral inequalities for concave functions with applications to special functions*, Proc. Roy. Soc. Edinburgh Sect. A **118**, 1–2 (1991), 173–192.

[2] P. S. BULLEN, *Error estimates for some elementary quadrature rules*, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. **602–633** (1978), 97–103.

[3] W. EHM, MARC G. GENTON, T. GNEITING, *Stationary covariances associated with exponentially convex functions*, Bernoulli **9**, 4 (2003), 607–615.

[4] P. C. HAMMER, *The midpoint method of numerical integration*, Math. Mag. **31** (1957/1958), 193–195.

[5] J. JAKŠETIĆ, JOSIP E. PEČARIĆ, *Exponential convexity method*, submitted.

[6] B. JESSEN, *Bemaerkinger om konvekse Funktioner og Uligheder imellem Middelvaerdier I*, Mat. Tidsskrift **B** (1931), 17–29.

[7] A. LUPAŞ, *A generalization of Hadamard's inequalities for convex functions*, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. **544–576** (1976), 115–121.

[8] D. S. MITRINOVIĆ, J. E. PEČARIĆ, A. M. FINK, *Classical and new inequalities in analysis*, Mathematics and its Applications (East European Series), 61. Kluwer Academic Publishers Group, Dordrecht, 1993. xviii+740 pp. ISBN: 0-7923-2064-6.

[9] J. PEČARIĆ, P. BEESACK, *On Jessen's inequality for convex functions* II, J. Math. Anal. Appl. **118**, 1 (1986), 125–144.

[10] J. E. PEČARIĆ, *Notes on convex functions*, General inequalities, 6 (Oberwolfach, 1990), 449–454, Internat. Ser. Numer. Math., **103**, Birkhäuser, Basel, 1992.

[11] JOSIP E. PEČARIĆ, FRANK PROSCHAN, Y. L. TONG, *Convex functions, partial orderings, and statistical applications*, Mathematics in Science and Engineering, 187. Academic Press, Inc., Boston, MA, 1992. xiv+467 pp. ISBN: 0-12-549250-2

[12] P. M. VASIĆ, I. B. LACKOVIĆ, *Some complements to the paper: "On an inequality for convex functions"*, (Univ.Beograd. Publ. Elektrotech. Fak. Ser. Mat. Fiz. No. **461–497** (1974), 63–66), Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. **544–576** (1976), 59–62.

[13] D. V. WIDDER, *The Laplace Transform*, Princeton Mathematical Series, v. 6. Princeton University Press, Princeton, N. J., 1941.

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