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A HARDY INEQUALITY FOR THE GRUSHIN TYPE OPERATORS

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Abstract. We prove a Hardy inequality related to Carnot-Carathéodory distance for the Grushin type operators like $\Delta_x + |x|^2\partial_t^2$. The procedure is based on a representation formula for such operators.

1. Introduction

The Hardy inequality in $\mathbb{R}^N$ states that, for all $f \in C_0^\infty(\mathbb{R}^N)$ and $N \geqslant 3$,

$$\int_{\mathbb{R}^N} |\nabla f|^2 dx \geqslant \frac{(N-2)^2}{4} \int_{\mathbb{R}^N} \frac{f^2}{|x|^2} dx. \tag{1.1}$$

Inequality (1.1) has been generalized to the degenerate elliptic differential operators by several authors for many years. For details, we refer to [2, 3, 4, 5, 10].

The aim of this note is to prove a Hardy type inequalities for the Grushin type operators like $\Delta_x + |x|^2\partial_t^2$, where non-isotropic gauge is replaced by the Carnot-Carathéodory distance d_{cc}. We refer to [3, 4] for the Hardy inequality related to non-isotropic gauge. We note that it has been proved by D'Ambrosio (see [3], Theorem 3.3) that the following Hardy inequality holds for all $f \in C_0^\infty(\mathbb{R}^{n+1})$

$$\int_{\mathbb{R}^{n+1}} |\nabla_L f|^2 \geqslant \left(\frac{n-2}{2}\right)^2 \int_{\mathbb{R}^{n+1}} \frac{|f|^2}{N(x,t)^2},$$

where $N(x,t) = \sqrt[4]{|x|^4 + 4t^2}$ and $\nabla_L = (\nabla_x, |x|\partial_t)$ is the gradient associated with the operators $\Delta_x + |x|^2\partial_t^2$. Notice that $N(x,t)$ and d_{cc} are equivalent, there exists a constant C, such that for all $f \in C_0^\infty(\mathbb{R}^{n+1})$,

$$\int_{\mathbb{R}^{n+1}} |\nabla_L f|^2 \geqslant C \int_{\mathbb{R}^{n+1}} \frac{|f|^2}{d_{cc}^2}.$$

In fact, one may choose the constant

$$C = \left(\frac{n-2}{2}\right)^2 \min_{(x,t)\in\mathbb{R}^{n+1}\setminus\{\mathbf{0}\}} \frac{d_{cc}(x,t)}{N(x,t)}.$$

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However, such constant C is not sharp. In this note, we shall give a new proof of Hardy inequalities related to d_{cc}. The corresponding constants is the same as the Hardy inequalities related to $N(x,t)$ (see [3], Theorem 3.1) and it seems that it is sharp though we fail to prove it (see Remark 3.3).

Recall that for the Grushin type operators, the sub-Riemannian metric is given by the vectors

$$X_1 = \frac{\partial}{\partial x_1}, \cdots, X_n = \frac{\partial}{\partial x_n}, \ T_1 = x_1 \frac{\partial}{\partial t}, \cdots, T_n = x_n \frac{\partial}{\partial t}.$$

By Chow's conditions, the Carnot-Carathéodory distance $d_{cc}(u,v)$ between any two points $u, v \in \mathbb{R}^{n+1}$ is finite. We denote by $d_{cc}(u) = d_{cc}(o,u)$, where $o = (\mathbf{0},0)$ is the origin.

Define on $\mathbb{R}^{n+1}$ the dilation δ_λ as

$$\delta_\lambda u = \delta_\lambda(x,t) := (\lambda x, \lambda^2 t), \ u = (x,t) \in \mathbb{R}^{n+1}.$$

For simplicity, we will write it as $\lambda u = (\lambda x, \lambda^2 t)$. The Jacobian determinant of δ_λ is λ^Q, where $Q = n+2$ is the homogeneous dimension. The Carnot-Carathéodory distance d_{cc} satisfies

$$d_{cc}(\lambda(x,t)) = \lambda d_{cc}(x,t), \ \lambda > 0.$$

To this end we have:

THEOREM 1.1. *Let $1 < p < Q - \alpha$. There holds, for all $f \in C_0^\infty(\mathbb{R}^{n+1})$,*

$$\int_{\mathbb{R}^{n+1}} \frac{|\nabla_L f|^p}{d_{cc}^\alpha} \geqslant \left(\frac{Q-p-\alpha}{p}\right)^p \int_{\mathbb{R}^{n+1}} \frac{|f|^p}{d_{cc}^{p+\alpha}}. \tag{1.2}$$

2. Geodesics for the Grushin type operators

In this section we shall give a parametrization of $\mathbb{R}^{n+1}$ using the geodesics. We refer to [7] for the analogous parametrization of the Heisenberg group.

Recall that the Grushin operator is given by

$$\Delta_L = \Delta_x + |x|^2 \frac{\partial^2}{\partial t^2}.$$

The associated Hamiltonian function $H(x,y,\xi,\theta)$ is of the form

$$H(x,t,\xi,\eta) = \frac{1}{2}(|\xi|^2 + |x|^2\eta^2).$$

We note all the geodesics are solutions of the Hamiltonian system (cf. [6, 11])

$$\begin{cases} \dot{x}(s) = \dfrac{\partial H}{\partial \xi} = \xi(s), \\ \dot{\xi}(s) = -\dfrac{\partial H}{\partial x} = -x\eta^2(s), \\ \dot{t}(s) = \dfrac{\partial H}{\partial \eta} = |x|^2\eta, \\ \dot{\eta}(s) = -\dfrac{\partial H}{\partial t} = 0, \ \textit{i.e.} \ \eta(s) = \eta(0), \end{cases} \tag{2.1}$$

Taking the initial date $(x(0),t(0)) = (\mathbf{0},0)$ and $(\xi(0),\eta(0)) = (A,\phi)$, one can find the solutions

$$\begin{cases} x(s) = A\dfrac{\sin\phi s}{\phi}, \\ t(s) = |A|^2\dfrac{2\phi s - \sin 2\phi s}{4\phi^2}, \end{cases}$$

where the time s is exactly the Carnot-Carathéodory distance. Letting $\phi \to 0$, one get the Euclidean geodesics

$$(x(s),t(s)) = (As,0)$$

and hence the correct normalization is $|A|^2 = 1$.

Set

$$\Omega = \{(A,\phi,\rho) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} : -\pi \leqslant \phi\rho \leqslant \pi,\ \rho \geqslant 0,\ |A|^2 = 1\} \subset \mathbb{R}^{n+2}.$$

and define $\Phi : \Omega \to \mathbb{R}^{n+1}$ by $\Phi(A,\phi,\rho) = (x(A,\phi,\rho),t(A,\phi,\rho))$, where

$$\begin{cases} x(A,\phi,\rho) = A\dfrac{\sin\phi\rho}{\phi}, \\ t(A,\phi,\rho) = \dfrac{2\phi\rho - \sin 2\phi\rho}{4\phi^2}. \end{cases} \tag{2.2}$$

We note if one fixes $\rho > 0$, equations (2.2) with $|A| = 1$ and $-\frac{\pi}{\rho} \leqslant \phi \leqslant \frac{\pi}{\rho}$ parameterize ∂B_ρ, where B_ρ is the Carnot-Carathéodory ball centered at the origin and of radius $\rho > 0$.

On the other hand, the Carnot-Carathéodory distance d_{cc} satisfies (see [11], Theorem 2.3)

$$d_{cc}(x,t) = d_{cc}((\mathbf{0},0),(x,t)) = \frac{\theta}{\sin\theta}|x|$$

for $x \neq \mathbf{0}$, where $\theta = \mu^{-1}\left(\frac{2t}{|x|^2}\right)$, μ^{-1} is the inverse function of μ and

$$\mu(\theta) = \frac{\theta}{\sin^2\theta} - \cot\theta = \frac{2\theta - \sin 2\theta}{2\sin^2\theta} : (-\pi,\pi) \to \mathbb{R}.$$

is a diffeomorphism of the interval $(-\pi,\pi)$ onto $\mathbb{R}$. When $x = \mathbf{0}$, d_{cc} satisfies $d_{cc}^2(\mathbf{0},t) = 2\pi|t|$ (see [11], Theorem 2.2).

By (2.2), we have

$$\mu(\theta) = \frac{2t}{|x|^2} = \frac{2\phi\rho - \sin 2\phi\rho}{2\sin^2\phi\rho} = \mu(\phi\rho).$$

Therefore,

$$\theta = \phi\rho \tag{2.3}$$

since μ is a diffeomorphism.

We finally recall the polar coordinates associated with d_{cc}. Given any $(0,0)\neq u=(x,t)\in\mathbb{R}^{n+1}$, set $x^*=\frac{x}{d_{cc}(u)}$, $t^*=\frac{t}{d_{cc}^2(u)}$ and $u^*=(x^*,t^*)$. It has been proved in [9] the following coarea formula

$$\int_{\mathbb{R}^{n+1}} f(u)|\nabla_L d_{cc}(u)|du=\int_{-\infty}^{+\infty}\int_{\{d_{cc}(u)=\lambda\}} f(u)dP(E_\lambda)d\lambda,$$

where $E_\lambda=\{u\in\mathbb{R}^{n+1}:d_{cc}(u)>\lambda\}$ and $P(E_\lambda)$ is the perimeter-measure. Notice that $|\nabla_L d_{cc}(u)|=1$ *a.e.* (cf. [9]) and it is easy to check $P(E_\lambda)=\lambda^{Q-1}P(E_1)$ through the dilation (see [8], Proposition 2.2 for the case of $n=1$.), we have the following polar coordinates

$$\int_{\mathbb{R}^{n+1}} f(u)du=\int_0^{+\infty}\int_\Sigma f(\lambda u^*)\lambda^{Q-1}d\sigma d\lambda,$$

when $f\in L^1(\mathbb{R}^{n+1})$, where Σ is the unit sphere associated with d_{cc}, i.e., $\Sigma=\{u\in\mathbb{R}^{n+1}:d_{cc}(u)=1\}$

3. The proof

To proved the main result, we first need the following representation formula. The idea is due to Cohn and Lu ([6], Theorem 1.2).

LEMMA 3.1. *Let* $\mathbb{Z}=\{\mathbf{0}\}\times\mathbb{R}\subset\mathbb{R}^{n+1}$. *Set* $\mathbb{Z}\cap\Sigma=\{(\mathbf{0},z_0),(\mathbf{0},z_1)\}$. *For each* $\delta>0$, *define* $\Sigma_\delta=\{(x,t)\in\Sigma|d_{cc}((x,t),(0,z_0))\geqslant\delta,d_{cc}((x,t),(0,z_1))\geqslant\delta\}$. *Then for all* $f\in C(\mathbb{R}^{n+1})\cap C^1(\mathbb{R}^{n+1}\setminus\mathbb{Z})$ *and* $R_2>R_1>0$, *there holds*

$$\int_{\Sigma_\delta} f(R_2u^*)d\sigma-\int_{\Sigma_\delta} f(R_1u^*)d\sigma=\int_{R_1}^{R_2}\int_{\Sigma_\delta}\langle\nabla_L f,\nabla_L d_{cc}\rangle d\sigma d\rho. \tag{3.1}$$

Proof. Notice that for any $u^*\in\Sigma_\delta$,

$$f(R_2u^*)-f(R_1u^*)=\int_{R_1}^{R_2}\frac{d}{d\rho}f(\rho u^*)d\rho$$
$$=\int_{R_1}^{R_2}\left(\sum_{i=1}^n\frac{\partial f(u)}{\partial x_i}\cdot\frac{\partial x_i}{\partial\rho}+\frac{\partial f(u)}{\partial t}\cdot\frac{\partial t}{\partial\rho}\right)d\rho,$$

where $u=(x,t)=\rho u^*$. Integrating over Σ_δ with respect to $d\sigma$, we have

$$\int_{\Sigma_\delta} f(R_2u^*)d\sigma-\int_{\Sigma_\delta} f(R_1u^*)d\sigma=\int_{\Sigma_\delta}\int_{R_1}^{R_2}\left(\sum_{i=1}^n\frac{\partial f(u)}{\partial x_i}\cdot\frac{\partial x_i}{\partial\rho}+\frac{\partial f(u)}{\partial t}\cdot\frac{\partial t}{\partial\rho}\right)d\rho d\sigma.$$

On the other hand, using equation (2.2), we have

$$\begin{aligned}\sum_{i=1}^n\frac{\partial f(u)}{\partial x_i}\cdot\frac{\partial x_i}{\partial\rho}+\frac{\partial f(u)}{\partial t}\cdot\frac{\partial t}{\partial\rho}=&\langle A\cos\phi\rho,\nabla_x f\rangle+\frac{\sin^2\phi\rho}{\phi}\frac{\partial f(u)}{\partial t}\\ =&\langle A\cos\phi\rho,\nabla_x f\rangle+\sin\phi\rho\cdot|x|\frac{\partial f(u)}{\partial t}.\end{aligned}\tag{3.2}$$

To finish the proof, it is enough to show

$$\nabla_x d_{cc}(u) = A\cos\phi\rho;\ \ |x|\partial_t d_{cc}(u) = \sin\phi\rho.$$

in $\mathbb{R}^n\setminus\{\mathbf{0}\}\times\mathbb{R}$. This will be proved in the following lemma 3.2. The proof of Lemma 3.1 is therefore completed. □

LEMMA 3.2. *There holds, for $x\neq\mathbf{0}$,*

$$\nabla_x d_{cc}(u) = A\cos\phi\rho;\ \ |x|\partial_t d_{cc}(u) = \sin\phi\rho.$$

Proof. Recall that if $x\neq\mathbf{0}$, then

$$d_{cc}(u) = d_{cc}(x,t) = \frac{\theta}{\sin\theta}|x|,$$

where $\theta = \mu^{-1}(2t/|x|^2)$. A simple calculation shows,

$$\mu'(\theta) = \frac{2\sin\theta - 2\theta\cos\theta}{\sin^3\theta};$$
$$\nabla_x\theta = \frac{1}{\mu'(\theta)}\cdot\frac{-4tx}{|x|^4};\quad \frac{\partial\theta}{\partial t} = \frac{1}{\mu'(\theta)}\cdot\frac{-2}{|x|^2}.$$

Therefore, if $x\neq\mathbf{0}$, then

$$\begin{aligned}\nabla_x d_{cc}(u) &= \nabla_x\left(\frac{\theta}{\sin\theta}|x|\right) = \frac{x}{|x|}\cdot\frac{\theta}{\sin\theta} + |x|\cdot\frac{\sin\theta-\theta\cos\theta}{\sin^2\theta}\cdot\nabla_x\theta\\ &= \frac{x}{|x|}\cdot\frac{\theta}{\sin\theta} - |x|\sin\theta\cdot\frac{-2tx}{|x|^4} = \frac{x}{|x|}\cdot\frac{\theta}{\sin\theta} - \frac{x}{|x|}\cdot\sin\theta\cdot\mu(\theta)\\ &= A\frac{\theta}{\sin\theta} - A\sin\theta\cdot\left(\frac{\theta}{\sin^2\theta} - \cot\theta\right)\\ &= A\cos\theta.\end{aligned}$$

On the other hand,

$$\begin{aligned}|x|\partial_t d_{cc}(u) &= |x|\frac{\partial d_{cc}(u)}{\partial t} = |x|\frac{\partial}{\partial t}\left(\frac{\theta}{\sin\theta}|x|\right)\\ &= |x|^2\cdot\frac{\sin\theta-\theta\cos\theta}{\sin^2\theta}\cdot\frac{\partial\theta}{\partial t}\\ &= |x|^2\cdot\frac{\sin\theta-\theta\cos\theta}{\sin^2\theta}\cdot\frac{1}{\mu'(\theta)}\cdot\frac{-2}{|x|^2}\\ &= \sin\theta.\end{aligned}$$

Therefore we obtain, by (2.3),

$$\nabla_x d_{cc}(u) = A\cos\theta = A\cos\phi\rho;\ \ |x|\partial_t d_{cc}(u) = \sin\theta = \sin\phi\rho.$$

This completes the proof of Lemma 3.2. $\square$

Proof of Theorem 1.1. Let $\varepsilon > 0$. Then $0 \leqslant f_\varepsilon := (|f|^2+\varepsilon^2)^{p/2} - \varepsilon^p \in C_0^\infty(\mathbb{R}^{n+1})$. In fact, f_ε has the same support as f. Since $d_{cc}(x,t) \in C(\mathbb{R}^{n+1}) \cap C^1(\mathbb{R}^{n+1} \setminus \mathbb{Z})$, we can put $f_\varepsilon d_{cc}^{Q-p-\alpha}(u)$ in Lemma 3.1 and get, for all $\delta > 0$,

$$
\begin{aligned}
&(Q-p-\alpha)\int_{R_1}^{R_2}\int_{\Sigma_\delta} f_\varepsilon d_{cc}^{Q-p-\alpha-1} d\sigma d\rho - R_2^{Q-p-\alpha} \\
&\times \int_{\Sigma_\delta} f_\varepsilon(R_2 u^*) d\sigma + R_1^{Q-p-\alpha}\int_{\Sigma_\delta} f_\varepsilon(R_1 u^*) d\sigma \\
&= -p\int_{R_1}^{R_2}\int_{\Sigma_\delta} (|f|^2+\varepsilon^2)^{(p-2)/2} f \langle \nabla_L f, \nabla_L d_{cc}\rangle d_{cc}^{Q-p-\alpha} d\sigma d\rho \\
&\leqslant p\int_{R_1}^{R_2}\int_{\Sigma_\delta} (|f|^2+\varepsilon^2)^{(p-2)/2} |f| \cdot |\nabla_L f| d_{cc}^{Q-p-\alpha} d\sigma d\rho \\
&\leqslant p\int_{R_1}^{R_2}\int_{\Sigma_\delta} (|f|^2+\varepsilon^2)^{(p-1)/2} |\nabla_L f| d_{cc}^{Q-p-\alpha} d\sigma d\rho.
\end{aligned}
$$

Let $\delta \to 0$, we obtain, since $\Sigma_\delta \to \Sigma$,

$$
\begin{aligned}
&(Q-p-\alpha)\int_{R_1}^{R_2}\int_{\Sigma} f_\varepsilon d_{cc}^{Q-p-\alpha-1} d\sigma d\rho - R_2^{Q-p-\alpha} \\
&\times \int_{\Sigma} f_\varepsilon(R_2 u^*) d\sigma + R_1^{Q-p-\alpha}\int_{\Sigma} f_\varepsilon(R_1 u^*) d\sigma \\
&\leqslant p\int_{R_1}^{R_2}\int_{\Sigma} (|f|^2+\varepsilon^2)^{(p-1)/2} |\nabla_L f| d_{cc}^{Q-p-\alpha} d\sigma d\rho.
\end{aligned}
$$

Letting $R_2 \to \infty$ and $R_1 \to 0+$ yields, since $Q-p-\alpha > 0$ and $f, f_\varepsilon \in C_0^\infty(\mathbb{R}^{n+1})$,

$$
(Q-p-\alpha)\int_0^\infty\int_\Sigma f_\varepsilon d_{cc}^{Q-p-\alpha-1} d\sigma d\rho \leqslant p\int_0^\infty\int_\Sigma (|f|^2+\varepsilon^2)^{(p-1)/2} |\nabla_L f| d_{cc}^{Q-p-\alpha} d\sigma d\rho.
$$

Rewriting the expression into a solid integral using the polar coordinates, we obtain

$$
(Q-p-\alpha)\int_{\mathbb{R}^{n+1}} \frac{f_\varepsilon}{d_{cc}^{p+\alpha}} \leqslant p\int_{\mathbb{R}^{n+1}} \frac{(|f|^2+\varepsilon^2)^{(p-1)/2}\cdot |\nabla_L f|}{d_{cc}^{p+\alpha-1}}
$$

By dominated convergence, letting $\varepsilon \to 0+$, we have,

$$
(Q-p-\alpha)\int_{\mathbb{R}^{n+1}} \frac{|f|^p}{d_{cc}^{p+\alpha}} \leqslant p\int_{\mathbb{R}^{n+1}} \frac{|f|^{p+\alpha-1}\cdot |\nabla_L f|}{d_{cc}^{p+\alpha-1}}.
$$

By Hölder's inequality:

$$
(Q-p-\alpha)\int_{\mathbb{R}^{n+1}} \frac{|f|^p}{d_{cc}^{p+\alpha}} \leqslant p\left(\int_{\mathbb{R}^{n+1}} \frac{|f|^p}{d_{cc}^{p+\alpha}}\right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^{n+1}} \frac{|\nabla_L f|^p}{d_{cc}^{\alpha}}\right)^{\frac{1}{p}}.
$$

Canceling and raising both sides to the power p, we get (1.2). $\square$

REMARK 3.3. It seems that the constant in Theorem 1.1 is sharp. To see this, one can follow [3] and consider the function

$$f_\varepsilon(x,t) = \begin{cases} 1, & d_{cc} \leqslant 1; \\ d_{cc}^{-(Q-p-\alpha)/p-\varepsilon}, & d_{cc} > 1. \end{cases}$$

It is easy to check that

$$\left(\frac{Q-p-\alpha}{p}\right)^p = \lim_{\varepsilon\to 0} \frac{\displaystyle\int_{\mathbb{R}^{n+1}} \frac{|\nabla_L f_\varepsilon|^p}{d_{cc}^{\alpha}}}{\displaystyle\int_{\mathbb{R}^{n+1}} \frac{|f_\varepsilon|^p}{d_{cc}^{p+\alpha}}}.$$

Since we do not know whether the function f_ε belong to the closure of $C_0^\infty(\mathbb{R}^{n+1})$, we fail to prove the sharpness of the constant in Theorem 1.1. However, the constant is sharp if the function belongs to a space which is larger than $C_0^\infty(\mathbb{R}^{n+1})$ (see [12] for the case of Heisenberg group).

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