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SOME INEQUALITIES FOR THE GENERALIZED DISTORTION FUNCTIONS

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Abstract. In this paper, we establish some functional inequalities for the generalized Hersch-Pfluger distortion function $\varphi_K(a,r)$, and prove a submultiplicative property for the generalized Agard distortion function $\eta_K(a,x)$.

1. Introduction

For real numbers a, b and c with $c \neq 0,-1,-2,\cdots$, the Gaussian hypergeometric function is defined by

$$F(a,b;c;x) = {}_2F_1(a,b;c;x) = \sum_{n=0}^{\infty} \frac{(a,n)(b,n)}{(c,n)} \frac{x^n}{n!}, \text{ for } |x| < 1. \tag{1.1}$$

Here, $(a,0) = 1$ for $a \neq 0$ and (a,n) denotes the shifted factorial function

$$(a,n) = a(a+1)(a+2)(a+3)\cdots(a+n-1)$$

for $n = 1,2,\cdots$. For a survey of these functions, see [1, 8].

For $r \in (0,1)$, $a \in (0,1)$ and $r' = \sqrt{1-r^2}$, the generalized elliptic integrals of the first and second kind [4, 19] are defined by

$$\begin{cases} \mathcal{K}_a = \mathcal{K}_a(r) = \frac{\pi}{2} F(a,1-a;1;r^2), \\ \mathcal{K}_a{}' = \mathcal{K}_a{}'(r) = \mathcal{K}_a(r'), \\ \mathcal{K}_a(0) = \pi/2, \ \mathcal{K}_a(1) = \infty \end{cases} \tag{1.2}$$

and

$$\begin{cases} \mathcal{E}_a = \mathcal{E}_a(r) = \frac{\pi}{2} F(a-1,1-a;1;r^2), \\ \mathcal{E}_a{}' = \mathcal{E}_a'(r) = \mathcal{E}_a(r'), \\ \mathcal{E}_a(0) = \pi/2, \ \mathcal{E}_a(1) = \sin(\pi a)/[2(1-a)], \end{cases} \tag{1.3}$$

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respectively. In particular, when $a = 1/2$, the functions $\mathscr{K}_a(r)$ and $\mathscr{E}_a(r)$ reduce to $\mathscr{K}(r)$ and $\mathscr{E}(r)$, respectively, which are the complete elliptic integrals of the first and second kind [1, 3, 7–10, 12, 13, 19]. By symmetry of (1.2), we assume that $a \in (0, 1/2]$ in the sequel.

Ramanujan's generalized modular equation [11] with signature $1/a$ and degree p is given by

$$\frac{F(a, 1-a; 1; 1-s^2)}{F(a, 1-a; 1; s^2)} = p\frac{F(a, 1-a; 1; 1-r^2)}{F(a, 1-a; 1; r^2)}, \tag{1.4}$$

where $a \in (0, 1/2]$, $r \in (0, 1)$ and $p > 0$. To rewrite (1.4) in a slightly shorter form, we use the decreasing homeomorphism $\mu_a : (0, 1) \to (0, \infty)$ defined by

$$\mu_a(r) = \frac{\pi}{2\sin \pi a}\frac{\mathscr{K}_a{}'(r)}{\mathscr{K}_a(r)},$$

for $a \in (0, 1/2]$. We can now write (1.4) as

$$\mu_a(s) = p\mu_a(r), \quad 0 < r < 1. \tag{1.5}$$

The solution of (1.5) is given by

$$s = \varphi_K(a, r) = \mu_a{}^{-1}(\mu_a(r)/K), \quad K = 1/p. \tag{1.6}$$

We call $\varphi_K(a, r)$ the generalized modular function with signature $1/a$ and degree $p = 1/K$ or generalized Hersch-Pflguer distortion function.

For $x, K \in (0, \infty)$, the generalized Agard distortion function is defined by

$$\eta_K(a, x) = \left[\frac{\varphi_K(a, r)}{\varphi_{1/K}(a, r')}\right]^2, r = \sqrt{\frac{x}{x+1}}. \tag{1.7}$$

If $a = 1/2$, then the functions defined in (1.6) and (1.7) reduce to the Hersch-Pfluger distortion function $\varphi_K(r)$ and Agard distortion function $\eta_K(x)$, respectively, which play a crucial role in quasiconformal mappings, quasiregular mappings, quasisymmetric functions and some related fields [2, 7, 14, 20, 21]. In particular, some remarkable properties and inequalities for them can be found in the literature [5, 6, 15–18, 22, 23].

For $a \in (0, 1/2]$, the so-called Ramanujan constant $R(a)$ is defined by

$$R(a) = -2\gamma - \psi(a) - \psi(1-a)$$

with $R(1/2) = \log 16$, where $\gamma = 0.577215\cdots$ is the Euler-Mascheroni constant, and ψ is the classical psi function.

The main purpose of this paper is to establish some interesting functional inequalities for the generalized distortion function $\varphi_K(a, r)$, and to prove a submultiplicative property for the generalized Agard distortion function $\eta_K(a, x)$. Our main results are the following Theorems 1.1–1.3.

THEOREM 1.1. *For $r \in (0,1)$ and $K \in (0,\infty)$, define the function F on $[2,\infty)$ by*

$$F(x) = 2[1-\varphi_{1/K}(a,r)^x]/[x\varphi_K(a,\sqrt{2(1-r^x)/x})^2].$$

Then $F(x)=1$ if $K=1$, and $F(x)$ is strictly decreasing from $[2,\infty)$ onto $(0,1]$ if $K\in(1,\infty)$, and strictly increasing from $[2,\infty)$ onto $[1,\infty)$ if $K\in(0,1)$. In particular, if $r\in(0,1)$ and $K,b\in[1,\infty)$, then

$$\varphi_K(a,r)^{2b} + b\varphi_{1/K}\left(a,\sqrt{(1-r^{2b})/b}\right)^2 \leqslant 1, \tag{1.8}$$

$$\varphi_{1/K}(a,r)^{2b} + b\varphi_K\left(a,\sqrt{(1-r^{2b})/b}\right)^2 \geqslant 1, \tag{1.9}$$

with equality in either case if and only if $b=1$ or $K=1$.

THEOREM 1.2. *Let $K,p\in(0,\infty)$ and $r\in(0,1)$, then*

(1) $g(r)=\varphi_K(a,r)^p/\varphi_K(a,r^p)$ is strictly decreasing from $(0,1)$ onto $(1,e^{R(a)(1-1/K)(p-1)/2})$ if $p,K\in(1,\infty)$ or $p,K\in(0,1)$, and strictly increasing from $(0,1)$ onto $(e^{R(a)(1-1/K)(p-1)/2},1)$ if $1/p,K\in(0,1)$ or $1/p,K\in(1,\infty)$. In particular, one has

$$\begin{cases} \varphi_K(a,r)^p \in (\varphi_K(a,r^p), e^{R(a)(1-1/K)(p-1)/2}\varphi_K(a,r^p)), \\ \text{if } p,K\in(1,\infty) \text{ or } K,p\in(0,1); \\ \varphi_K(a,r)^p = \varphi_K(a,r^p), \\ \text{if } K=1 \text{ or } p=1; \\ \varphi_K(a,r)^p \in (e^{R(a)(1-1/K)(p-1)/2}\varphi_K(a,r^p), \varphi_K(a,r^p)), \\ \text{if } 1/p,K\in(0,1) \text{ or } 1/p,K\in(1,\infty). \end{cases} \tag{1.10}$$

(2) $G(K)=\varphi_K(a,r)^p/\varphi_{K^p}(a,r)$ is strictly decreasing if $1/p,K\in(1,\infty)$ or $1/p,K\in(0,1)$, and strictly increasing if $p,K\in(1,\infty)$ or $p,K\in(0,1)$. In particular, one has

$$\begin{cases} \varphi_K(a,r)^p \in (r^{p-1}\varphi_{K^p}(a,r), r^{p/K}), \text{ if } 1/p,K\in(0,1); \\ \varphi_K(a,r)^p \in (\varphi_{K^p}(a,r), r^{p-1}\varphi_{K^p}(a,r)), \text{ if } 1/p,K\in(1,\infty); \\ \varphi_K(a,r)^p = \varphi_{K^p}(a,r), \text{ if } p=1; \\ \varphi_K(a,r)^p = r^{p-1}\varphi_{K^p}(a,r), \text{ if } K=1; \\ \varphi_K(a,r)^p \in (r^{p-1}\varphi_{K^p}(a,r), \varphi_{K^p}(a,r)), \text{ if } K,p\in(1,\infty); \\ \varphi_K(a,r)^p \in (r^{p/K}e^{pR(a)(1-1/K)/2}, r^{p-1}\varphi_{K^p}(a,r)), \text{ if } K,p\in(0,1). \end{cases} \tag{1.11}$$

(3) Inequality

$$\varphi_K(a,r^{1/p})^p \leqslant \varphi_{K^p}(a,r) \tag{1.12}$$

holds for $p,K\in[1,\infty)$ or $p,K\in(0,1)$, with equality if and only if $K=1$ or $p=1$. The inequality is reversed if $1/p,K\in(1,\infty)$ or $1/p,K\in(0,1)$.

THEOREM 1.3. *If* $K \geqslant 1$ *and* $x,y \in (0,\infty)$, *then*

$$[\min\{x^K, x^{1/K}\}\min\{y^K, y^{1/K}\}\eta_K(a,x)\eta_K(a,y)]^{1/2} \leqslant \eta_K(a,xy) \leqslant e^{R(a)(1/K-1)}\eta_K(a,x)\eta_K(a,y). \tag{1.13}$$

The first equality holds if and only if $K=1$ *or* $x=y=1$, *and the second equality holds if and only if* $K=1$. *The coefficient* $e^{R(a)(1/K-1)}$ *of the upper bound cannot be replaced by a smaller one depending only on* K.

2. Properties of $\varphi_K(a,r)$

In this section, we introduce several monotonicity properties of some functions defined in terms of $\varphi_K(a,r)$, and prove Theorems 1.1 and 1.2.

First, let us recall the following formulas:

$$\lim_{r\to o}\mu_a(r)+\log r = \frac{R(a)}{2}, \tag{2.1}$$

$$\varphi_K(a,r)^2+\varphi_{1/K}(a,r')^2 = 1, \tag{2.2}$$

$$\frac{\partial\varphi_K(a,r)}{\partial r} = \frac{1}{K}\frac{ss'^2\mathcal{K}_a(s)^2}{rr'^2\mathcal{K}_a(r)^2} = K\frac{ss'^2\mathcal{K}_a'(s)^2}{rr'^2\mathcal{K}_a'(r)^2}, \tag{2.3}$$

$$\frac{\partial\varphi_K(a,r)}{\partial K} = \frac{2}{\pi K\sin(\pi a)}ss'^2\mathcal{K}_a(s)\mathcal{K}_a'(s). \tag{2.4}$$

where $s=\varphi_K(a,r)$, $r\in(0,1)$ and $K\in(0,\infty)$.

The following Lemma 2.1 can be found in [4, Theorem 6.7].

LEMMA 2.1. *For each* $a\in(0,1/2]$ *and* $K\in(1,\infty)$, *let* f, g *be defined on* $(0,1]$ *by*

$$f(r)=r^{-1/K}\varphi_K(a,r) \quad \text{and} \quad g(r)=r^{-K}\varphi_{1/K}(a,r).$$

Then f *is strictly decreasing and* g *is strictly increasing, with* $f((0,1])=[1,e^{R(a)(1-1/K)/2})$ *and* $g((0,1])=(e^{R(a)(1-K)/2},1]$.

The following Lemma 2.2 can be found in [4, Lemma 6.2(1), (2), (4) and (5), and Lemma 5.4(1)].

LEMMA 2.2. *For each* $a\in(0,1/2]$, $K\in(1,\infty)$, $r\in(0,1)$, *let* $s=\varphi_K(a,r)$ *and* $t=\varphi_{1/K}(a,r)$. *Then the function*

(1) $h_1(r)=\mathcal{K}_a(s)/\mathcal{K}_a(r)$ *is strictly increasing from* $(0,1)$ *onto* $(1,K)$;

(2) $h_2(r)=s'\mathcal{K}_a(s)^2/[r'\mathcal{K}_a(r)^2]$ *is strictly decreasing from* $(0,1)$ *onto* $(0,1)$;

(3) $h_3(r)=\mathcal{K}_a(t)/\mathcal{K}_a(r)$ *is strictly decreasing from* $(0,1)$ *onto* $(1/K,1)$;

(4) $h_4(r)=t'\mathcal{K}_a(t)^2/[r'\mathcal{K}_a(r)^2]$ *is strictly increasing from* $(0,1)$ *onto* $(1,\infty)$;

(5) $h_5(r) = r'^c \mathscr{K}_a(r)$ *is strictly decreasing if and only if* $c \geqslant 2a(1-a)$*, in which case* $r'^c \mathscr{K}_a(r)$ *is strictly decreasing from* $(0,1)$ *onto* $(0,\pi/2)$*.*

Proof of theorem 1.1. Clearly, $F(2) = 1$. Then making use of Lemma 2.1 we get

$$\begin{aligned}\lim_{x\to\infty} F(x) = &\lim_{x\to\infty} \frac{2}{x\varphi_K(a,\sqrt{2(1-r^x)/x})^2}\\ =&2\lim_{x\to\infty}\left[\frac{(\sqrt{2(1-r^x)/x})^{1/K}}{\varphi_K(a,\sqrt{2(1-r^x)/x})}\right]^2\cdot\left(\sqrt{\frac{x}{2(1-r^x)}}\right)^{2/K}\cdot\frac{1}{x}\\ =&2^{1-1/K}\cdot e^{R(a)(1/K-1)}\cdot\lim_{x\to\infty}\frac{x^{1/K-1}}{(1-r^x)^{1/K}}\\ =&\begin{cases}0, & \text{if } K\in(1,\infty),\\ \infty, & \text{if } K\in(0,1).\end{cases}\end{aligned} \tag{2.5}$$

Let $t = \sqrt{2(1-r^x)/x}, s = \varphi_K(a,t)$ and $u = \varphi_{1/K}(a,r)$. Then by logarithmic differentiation and (2.3), we have

$$\frac{xF'(x)}{F(x)} = \frac{s'^2\mathscr{K}_a(s)\mathscr{K}_a'(s)}{t'^2\mathscr{K}_a(t)\mathscr{K}_a'(t)}[F_1(r^x)+1]-[F_1(u^x)+1], \tag{2.6}$$

where $F_1(x) = (x\log x)/(1-x)$, which is strictly decreasing from $(0,1)$ onto $(-1,0)$. By Lemma 2.2(5), the function $s'^2\mathscr{K}_a(s)\mathscr{K}_a'(s)$ is strictly decreasing in K on $(0,\infty)$. Hence, from (2.6) we have

$$\begin{cases}\dfrac{xF'(x)}{F(x)} > F_1(r^x)-F_1(u^x), \text{ if } K\in(0,1),\\[2ex] \dfrac{xF'(x)}{F(x)} < F_1(r^x)-F_1(u^x), \text{ if } K\in(1,\infty).\end{cases} \tag{2.7}$$

Since

$$\begin{cases} u>r \Rightarrow u^x > r^x \Rightarrow F_1(r^x) > F_1(u^x), \text{ if } K\in(0,1),\\ u<r \Rightarrow u^x < r^x \Rightarrow F_1(r^x) < F_1(u^x), \text{ if } K\in(1,\infty),\end{cases} \tag{2.8}$$

from (2.7) and (2.8) one has

$$\begin{cases}\dfrac{xF'(x)}{F(x)} > 0, \text{ if } K\in(0,1),\\[2ex] \dfrac{xF'(x)}{F(x)} < 0, \text{ if } K\in(1,\infty).\end{cases} \tag{2.9}$$

Therefore, the monotonicity of F follows from (2.9). Moreover, taking $x = 2b$ in Theorem 1.1, inequalities (1.8) and (1.9) are clear. □

REMARK 2.3. If $b = 1$, then inequalities (1.8) and (1.9) reduce to equation (2.2).

Proof of theorem 1.2. For part (1), clearly $g(1^-)=1$, and making use of Lemma 2.1 we get $g(0^+)=e^{R(a)(1-1/K)(p-1)/2}$. Let $x=r^p$, $s=\varphi_K(a,r)$ and $u=\varphi_K(a,x)$, then $g(r)=s^p/u$ and

$$\frac{Kr}{p}\frac{g'(r)}{g(r)}=\frac{s'^2\mathscr{K}_a(s)^2}{r'^2\mathscr{K}_a(r)^2}-\frac{u'^2\mathscr{K}_a(u)^2}{x'^2\mathscr{K}_a(x)^2}=g_1(r)-g_1(x), \tag{2.10}$$

where $g_1(r)=s'^2\mathscr{K}_a(s)^2/[r'^2\mathscr{K}_a(r)^2]$. From Lemma 2.2 (1)–(4) we know that $g_1(r)$ is strictly decreasing if $K>1$, and strictly increasing if $0<K<1$.

Next, we divide the proof of part (1) into two cases.

Case 1.1. $p,K\in(1,\infty)$ or $p,K\in(0,1)$. Then $g_1(r)<g_1(x)$, and $g(r)$ is strictly decreasing in $(0,1)$ by (2.10).

Case 1.2. $1/p,K\in(1,\infty)$ or $1/p,K\in(0,1)$. Then $g_1(r)>g_1(x)$, and $g(r)$ is strictly increasing in $(0,1)$ follows from (2.10).

For part (2), clearly $G(1)-r^{p-1}=G(\infty)-1=0$. Let $T=K^p$, $s=\varphi_K(a,r)$ and $v=\varphi_T(a,r)$, then $G(K)=s^p/v$. By (2.1) we have

$$\begin{aligned}&\lim_{K\to 0}\log G(K)\\&=p\lim_{K\to 0}[\mu_a(s)+\log s]-\lim_{K\to 0}[\mu_a(v)+\log v]+\lim_{K\to 0}[\mu_a(v)-p\mu_a(s)]\\&=\frac{R(a)(p-1)}{2}+\lim_{K\to 0}\left[\frac{1-pK^{p-1}}{K^p}\right]\mu_a(r)\\&=\begin{cases}\infty, & 1/p,K\in(0,1),\\ -\infty, & p,K\in(0,1).\end{cases}\end{aligned}$$

Logarithmic differentiation of $G(K)$ gives

$$\frac{\pi K\sin(\pi a)}{2p}\frac{G'(K)}{G(K)}=G_1(s)-G_1(v), \tag{2.11}$$

where $G_1(r)=r'^2\mathscr{K}_a(r)\mathscr{K}_a'(r)$, which is decreasing follows from Lemma 2.2(5).

Next we divide the proof of part (2) into two cases.

Case 2.1. $1/p,K\in(1,\infty)$ or $1/p,K\in(0,1)$. Then $s>v$ and $G_1(s)<G_1(v)$. Thus from (2.11) we know that $G(K)$ is strictly decreasing.

Case 2.2. $p,K\in(1,\infty)$ or $p,K\in(0,1)$. Then $s<v$ and $G_1(s)>G_1(v)$. Thus from (2.11) we know that $G(K)$ is strictly increasing.

Therefore, (2.11) follows from the monotonicity and limiting values of $G(K)$ together with Lemma 2.1.

For part (3), define $f(r,K)=\varphi_K(a,r^{1/p})^p/\varphi_{K^p}(a,r)(p\neq 1)$ for $(r,K)\in D=(0,1)\times(0,\infty)$. Let $t=r^{1/p}$, $w=\varphi_K(a,t)$ and $v=\varphi_{K^p}(a,r)$, then equations (2.3) and (2.4) lead to

$$\frac{r}{f(r,K)}\frac{\partial f(r,K)}{\partial r}=\frac{w'^2\mathscr{K}_a(w)\mathscr{K}_a'(w)}{t'^2\mathscr{K}_a(t)\mathscr{K}_a'(t)}-\frac{v'^2\mathscr{K}_a(v)\mathscr{K}_a'(v)}{r'^2\mathscr{K}_a(r)\mathscr{K}_a'(r)} \tag{2.12}$$

and

$$\frac{\pi K \sin(\pi a)}{2pf(r,K)}\frac{\partial f(r,K)}{\partial K} = w'^2 \mathscr{K}_a(w)\mathscr{K}_a'(w) - v'^2 \mathscr{K}_a(v)\mathscr{K}_a'(v), \tag{2.13}$$

respectively. Let $\partial f(r,K)/\partial r = \partial f(r,K)/\partial K = 0$, then from (2.12) and (2.13) we conclude that $r = t$ and $w = v$, which implies that $p = 1$ and leads to a contradiction with $p \neq 1$. Hence, f has no extreme points in D, and

$$\begin{cases} \sup\limits_{(r,t)\in D} f(r,K) = \sup\limits_{(r,t)\in \partial D} f(r,K), \\ \inf\limits_{(r,t)\in D} f(r,K) = \inf\limits_{(r,t)\in \partial D} f(r,K). \end{cases} \tag{2.14}$$

Next we divide the proof of part (3) into three cases.

Case 3.1. $K, p \in (1,\infty)$ or $K, p \in (0,1)$. Then

$$f(r,K) = \frac{w^p}{v} = \left(\frac{w}{t^{1/K}}\right)^p \cdot \frac{r^{1/K^p}}{v} \cdot r^{[1-K^{(1-p)}]/K}, \tag{2.15}$$

$$f(0^+,K) = f(1,K) - 1 = f(r,1) - 1 = \lim_{K\to\infty} f(r,K) - 1 = 0 \tag{2.16}$$

and

$$\begin{aligned} f(r,0^+) = &\lim_{K\to 0} \exp\left\{ p[\mu_a(w) + \log w] - [\mu_a(v) + \log v] + \frac{K^{1-p}\mu_a(r) - p\mu_a(t)}{K} \right\} \\ =&0. \end{aligned} \tag{2.17}$$

It follows from (2.15)–(2.17) that $f(r,K) \leqslant 1$ for all $(r,K) \in \overline{D}$ with $K, p \in (1,\infty)$ or $K, p \in (0,1)$, where $\overline{D}$ is the closure of D. Thus inequality (1.12) and its equality case follow.

Case 3.2. $1/p, K \in (0,1)$. Then from Case 3.1 one has

$$r = \varphi_{1/K}(a, \varphi_K(a, r^{1/p}))^p \leqslant \varphi_{1/K^p}(a, \varphi_K(a, r^{1/p})^p),$$

$$\varphi_{K^p}(a,r) \leqslant \varphi_K(a, r^{1/p})^p. \tag{2.18}$$

Case 3.3. $1/p, K \in (1,\infty)$. Then (2.15) and Lemma 2.1 yield

$$f(0^+,K) = \infty, \quad f(1,K) = f(r,1) = \lim_{K\to\infty} f(r,K) = 1. \tag{2.19}$$

Equations (2.14) and (2.19) lead to the conclusion that $f(r,K) \geqslant 1$ for all $(r,K) \in \overline{D}$ and inequality (2.18) holds again. $\square$

3. Properties of $\eta_K(a,x)$

In this section, we study some properties of $\eta_K(a,x)$, and complete the proof of Theorem 1.3.

LEMMA 3.1. *If $K>1$, then the function $J_1(x)=\eta_K(a,x)/x^K$ is strictly decreasing from $(0,\infty)$ onto $(e^{R(a)(K-1)},\infty)$, and the function $J_2(x)=\eta_K(a,x)/x^{1/K}$ is strictly increasing from $(0,\infty)$ onto $(e^{R(a)(1-1/K)},\infty)$.*

Proof. Let $r=\sqrt{x/(1+x)}$ and $s=\varphi_K(a,r)$, then

$$J_1(x)=H_1(r)=\left(\frac{s}{s'}\right)^2\left(\frac{r'}{r}\right)^{2K}$$

and

$$J_2(x)=H_2(r)=\left(\frac{s}{s'}\right)^2\left(\frac{r'}{r}\right)^{2/K}.$$

Then logarithmic differentiations give

$$\frac{H_1{}'(r)}{H_1(r)}=\frac{2K}{rr'^2}\left(\frac{\mathscr{K}_a(s)^2}{K^2\mathscr{K}_a(r)^2}-1\right),$$

$$\frac{H_2{}'(r)}{H_2(r)}=\frac{2}{rr'^2K}\left(\frac{\mathscr{K}_a(s)^2}{\mathscr{K}_a(r)^2}-1\right).$$

Therefore, the monotonicity properties of $H_1(r)$ and $H_2(r)$ follow from Lemma 2.2(1), and the limiting value follows from Lemma 2.1. □

LEMMA 3.2. *As a function of r, $g_K(r)=(\varphi_K(a,r)/\varphi_{1/K}(a,r'))(r'/r)^K$ is strictly decreasing from $(0,1)$ onto $(e^{R(a)(K-1)/2},\infty)$ for $K\in(1,\infty)$, and strictly increasing from $(0,1)$ onto $(0,e^{R(a)(K-1)/2})$ for $K\in(0,1)$. In particular, for $r\in(0,1)$ and $K\in[1,\infty)$,*

$$\varphi_K(a,r)\geqslant\max\{h_1(r,K),h_2(r,K)\},\tag{3.1}$$

and

$$\varphi_{1/K}(a,r)\leqslant\min\{h_1(r,1/K),h_2(r,1/K)\},\tag{3.2}$$

where

$$h_1(r,K)=e^{R(a)(K-1)/2}r^K/(r'^{2K}+e^{(K-1)R(a)}r^{2K})^{1/2},$$

$$h_2(r,K)=e^{R(a)(1-1/K)/2}r^{1/K}/(r'^{2/K}+e^{(1-1/K)R(a)}r^{2/K})^{1/2},$$

with the equalities holding if and only if $K=1$.

Proof. Let $t=(r/r')^2$. Then

$$g_K(r)=(\eta_K(a,t)/t^K)^{1/2},$$

and

$$g_{1/K}(r) = [(1/t)^{1/K}/\eta_K(a,1/t)]^{1/2},$$

and hence the monotonicity of g_K follows from Lemma 3.1.

By the monotonicity of g_K,

$$g_K(r) = (s/s')(r'/r)^K \geqslant e^{R(a)(K-1)/2},$$

where $s = \varphi_K(a,r)$, and $K \in [1,\infty)$, and it follows that

$$s \geqslant \frac{r^K e^{R(a)(K-1)/2}}{[r'^{2K} + r^{2K} e^{(K-1)R(a)}]^{1/2}} = h_1(r,K), \quad K \geqslant 1.$$

Likewise

$$g_{1/K}(r') = \left(\frac{s'}{s}\right)\left(\frac{r}{r'}\right)^{1/K} \leqslant e^{R(a)(1/K-1)/2}, \quad K \geqslant 1,$$

implies that

$$s \geqslant \frac{r^{1/K} e^{R(a)(1-1/K)/2}}{[r'^{2/K} + r^{2/K} e^{R(a)(1-1/K)}]^{1/2}} = h_2(r,K).$$

Hence (3.1) holds.

The proof of (3.2) is similar.

The equality case is clear. □

COROLLARY 3.3. *If $r \in (0,1)$ and $K \in (1,\infty)$, then*

$$\varphi_{1/K}(a,r) < \min\{r^K, e^{R(a)(1-K)/2}\varphi_K(a,r')(r/r')^K, \\ e^{R(a)(1/K-1)/2}\varphi_K(a,r')(r/r')^{1/K}\}.$$

Proof. The first bound follows from Lemma 2.1, and the second and third bounds follow from Lemma 3.2. □

Proof of theorem 1.3. Firstly, by Lemma 3.1 we get

$$\frac{\eta_K(a,xy)}{x^K \eta_K(a,y)} = \frac{\eta_K(a,xy)}{(xy)^K}\frac{y^K}{\eta_K(a,y)} \begin{cases} \geqslant 1, \text{ if } x \leqslant 1, \\ \leqslant 1, \text{ if } x \geqslant 1 \end{cases}$$

and

$$\frac{\eta_K(a,xy)}{x^{1/K} \eta_K(a,y)} = \frac{\eta_K(a,xy)}{(xy)^{1/K}}\frac{y^{1/K}}{\eta_K(a,y)} \begin{cases} \leqslant 1, \text{ if } x \leqslant 1, \\ \geqslant 1, \text{ if } x \geqslant 1, \end{cases}$$

with equalities if and only if $K = 1$ or $x = 1$. Hence

$$\min\{x^K, x^{1/K}\}\eta_K(a,y) \leqslant \eta_K(a,xy) \leqslant \max\{x^K, x^{1/K}\}\eta_K(a,y) \tag{3.3}$$

for $K \in [1,\infty)$ and $x,y \in (0,\infty)$, with equality in each case if and only if $K=1$ or $x=1$. By symmetry,

$$\min\{y^K, y^{1/K}\}\eta_K(a,x) \leqslant \eta_K(a,xy) \leqslant \max\{y^K, y^{1/K}\}\eta_K(a,x) \tag{3.4}$$

for $K \in [1,\infty)$ and $x,y \in (0,\infty)$, with equality in each case if and only if $K=1$ or $y=1$. Hence the first inequality in (1.13) and its equality case follow from (3.3) and (3.4).

To prove the second inequality in (1.13), set $D=(0,\infty)\times(0,\infty)$, and define the function f on D by

$$f(x,y) = \eta_K(a,xy)/\{\eta_K(a,x)\eta_K(a,y)\}.$$

Our goal is to show that

$$\sup_{(x,y)\in D} f(x,y) = e^{R(a)(1/K-1)}.$$

Let $r=\sqrt{xy/(1+xy)}$, $s=\sqrt{x/(1+x)}$, $t=\sqrt{y/(1+y)}$, $u=\varphi_K(a,r)$, $v=\varphi_K(a,s)$ and $w=\varphi_K(a,t)$. Then

$$\eta_K(a,x)=(v/v')^2, \quad \eta_K(a,y)=(w/w')^2, \quad \eta_K(a,xy)=(u/u')^2,$$

$$\frac{\partial r}{\partial x}=\frac{rr'^2}{2x}, \quad \frac{\partial r}{\partial y}=\frac{rr'^2}{2y}, \tag{3.5}$$

$$\frac{ds}{dx}=\frac{ss'^2}{2x}, \quad \frac{dt}{dy}=\frac{tt'^2}{2y}, \tag{3.6}$$

$$f(x,y)=\left(\frac{u}{u'}\frac{v'}{v}\frac{w'}{w}\right)^2. \tag{3.7}$$

Clearly, $f \in C^\infty(D)$.

Now we divide the proof into four steps.

Step (i). We first find $\sup_{(x,y)\in\partial D} f(x,y)$.

It follows from (3.7) that

$$f(x,y)=\left(\frac{u}{r^{1/K}}\frac{s^{1/K}}{v}\frac{v'w'}{u'w}\right)^2 \cdot \left[\frac{y(1+x)}{1+xy}\right]^{1/K}.$$

Hence, by Lemma 2.1,

$$f(0^+,y)=y^{1/K}(w'/w)^2=y^{1/K}/\eta_K(a,y). \tag{3.8}$$

Similarly,

$$f(x,0^+)=x^{1/K}/\eta_K(a,x). \tag{3.9}$$

Clearly, $f(x,y)$ can be rewritten as

$$f(x,y)=\left(\frac{u}{r^{1/K}}\cdot\frac{s^{1/K}}{v}\cdot\frac{t^{1/K}}{w}\cdot\frac{v'w'}{u'}\right)^2\left[\frac{(1+x)(1+y)}{1+xy}\right]^{1/K},$$

from which it follows that

$$\lim_{\substack{x\to 0^+\\ y\to 0^+}} f(x,y) = e^{R(a)(1/K-1)} \tag{3.10}$$

by Lemma 2.1.

From (3.8)–(3.10) together with Lemmas 3.1 and 3.2 we get

$$\sup_{(x,y)\in\partial D} f(x,y) = e^{R(a)(1/K-1)} \tag{3.11}$$

and

$$\lim_{y\to\infty} f(0^+,y) = \lim_{x\to\infty} f(x,0^+) = 0. \tag{3.12}$$

Step (ii). We want to find $\lim_{x^2+y^2\to\infty} f(x,y)$.

For this purpose, we rewrite $f(x,y)$ as

$$f(x,y) = \left(\frac{u}{vw}\right)^2 \left(\frac{r'^K}{u'}\frac{v'}{s'^K}\frac{w'}{t'^K}\right)^2 \left[\frac{1+xy}{(1+x)(1+y)}\right]^K,$$

from which and Lemma 2.1 we get

$$f(x,\infty) = \lim_{y\to\infty} f(x,y) = x^K/\eta_K(a,x), \tag{3.13}$$

$$f(\infty,y) = \lim_{x\to\infty} f(x,y) = y^K/\eta_K(a,y), \tag{3.14}$$

$$\lim_{\substack{x\to\infty\\ y\to\infty}} f(x,y) = e^{R(a)(1-K)}. \tag{3.15}$$

Step (iii). We want to estimate $\sup_{(x,y)\in C_M} f(x,y)$ for sufficiently large M, where $C_M = \{(x,y)\in D; x^2+y^2=M^2\}$, $0<M<\infty$.

By Lemma 3.2, it follows from (3.13)–(3.15) that

$$\lim_{M\to\infty} \sup_{(x,y)\in C_M} f(x,y) = f(\infty,\infty) = e^{R(a)(1-K)}. \tag{3.16}$$

It follows from [4, (3.6)] that $R(a)\geqslant \log 16$ for all $a\in(0,1/2]$. Thus, for each $K\in(1,\infty)$ one can take $\varepsilon>0$ so small that

$$e^{R(a)(1-K)} + \varepsilon < e^{R(a)(1/K-1)}. \tag{3.17}$$

For such ε, by (3.16), there must be an $M_0>\sqrt{2}$ such that

$$e^{R(a)(1-K)} - \varepsilon < \sup_{(x,y)\in C_M} f(x,y) < e^{R(a)(1-K)} + \varepsilon \tag{3.18}$$

for all $M\geqslant M_0$.

Step (iv). We now find $\sup_{(x,y)\in D(M)} f(x,y)$, where

$$D(M)=\{(x,y)\in D; x^2+y^2<M^2\}, \quad M\geqslant M_0.$$

By (2.3), (3.5) and (3.6), we have

$$\frac{Kx}{f(x,y)}\frac{df(x,y)}{dx}=\frac{\mathscr{K}_a(u)^2}{\mathscr{K}_a(r)^2}-\frac{\mathscr{K}_a(v)^2}{\mathscr{K}_a(s)^2} \tag{3.19}$$

and

$$\frac{Ky}{f(x,y)}\frac{df(x,y)}{dy}=\frac{\mathscr{K}_a(u)^2}{\mathscr{K}_a(r)^2}-\frac{\mathscr{K}_a(w)^2}{\mathscr{K}_a(t)^2} \tag{3.20}$$

for $(x,y)\in D$. Then from (3.19) and (3.20) together with Lemma 2.2(1) we know that if $(x_0,y_0)\in D(M)$ is an extreme point of f, then $r_0=s_0=t_0$, where $r_0=\sqrt{x_0y_0/(1+x_0y_0)}$, $s_0=\sqrt{x_0/(1+x_0)}$, and $t_0=\sqrt{y_0/(1+y_0)}$. It follows that $x_0=y_0=1$, and

$$f(x_0,y_0)=1/\eta_K(a,1). \tag{3.21}$$

By Lemma 3.2, we clearly see that

$$\eta_K(a,1)\geqslant e^{R(a)(K-1)}, \quad 1\leqslant K\leqslant\infty,$$

with equality if and only if $K=1$. Hence it follows from (3.21) that

$$f(x_0,y_0)\leqslant e^{R(a)(1-K)}\leqslant e^{R(a)(1/K-1)}. \tag{3.22}$$

Therefore, by (3.11), (3.17), (3.18) and (3.22),

$$\begin{aligned}\max_{(x,y)\in\bar{D}(M)} f(x,y)=&\max\left\{e^{R(a)(1/K-1)}, \sup_{(x,y)\in C_M} f(x,y), f(x_0,y_0)\right\}\\ =&e^{R(a)(1/K-1)}, \end{aligned} \tag{3.23}$$

where $\overline{D}(M)$ is the closure of $D(M)$.

Now, the second inequality in (1.13), its equality case and the sharpness of the constant $e^{R(a)(1/K-1)}$ follow from (3.11), (3.17), (3.18) and (3.23). Moreover, f takes its supremum $e^{R(a)(1/K-1)}$ only at the origin $(0,0)$. □

COROLLARY 3.4. *If $K\geqslant 1, x,y\in(0,\infty)$, then*

$$\begin{aligned}&e^{R(a)(1-1/K)}\eta_{1/K}(a,x)\eta_{1/K}(a,y)\leqslant\eta_{1/K}(a,xy)\\ &\leqslant\left[\max\{x^K,x^{1/K}\}\max\{y^K,y^{1/K}\}\eta_{1/K}(a,x)\eta_{1/K}(a,y)\right]^{1/2}.\end{aligned}$$

The first (second) equality holds if and only if $K=1$ $(K=1$ or $x=y=1)$. The coefficient $e^{R(a)(1-1/K)}$ of the lower bound is the best possible.

Proof. The result follows immediately from Theorem 1.3,

$$\eta_K(a,t)\eta_{1/K}(a,1/t)=1$$

and

$$\min\{t^{-K}, t^{-1/K}\}\max\{t^K, t^{1/K}\} = 1. \quad \square$$

REMARK 3.5. If $a = 1/2$, then Theorem 1.3, Lemmas 3.1 and 3.2, Corollary 3.3 and Corollary 3.4 reduce to Theorem 1.6, Theorem 3.49, Lemma 2.21, Corollary 2.26 and Corollary 3.47 in [18], respectively.

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