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REVERSED DETERMINANTAL INEQUALITIES FOR ACCRETIVE–DISSIPATIVE MATRICES

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Abstract. A matrix $A \in M_n(\mathbf{C})$ is said to be accretive-dissipative if, in its Toeplitz decomposition $A = B + iC$, $B = B^*$, $C = C^*$, both matrices B and C are positive definite. Let $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ be an accretive-dissipative matrix, k and l be the orders of A_{11} and A_{22}, respectively, and let $m = \min\{k,l\}$. It is proved

$$|\det A| \geqslant \frac{(4\kappa)^m}{(1+\kappa)^{2m}} |\det A_{11}||\det A_{22}|,$$

where κ is the maximum of the condition numbers of B and C.

1. Introduction and the main result

Let $M_n(\mathbf{C})$ be the space of complex matrices of size $n \times n$. A matrix $A \in M_n(\mathbf{C})$ is said to be accretive-dissipative if, in its Toeplitz decomposition (sometimes also called the Hermitian decomposition)

$$A = B + iC, \quad B = B^*, \quad C = C^*, \tag{1.1}$$

both matrices B and C are Hermitian positive definite. Comformably partition A, B, C as

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} \\ B_{12}^* & B_{22} \end{bmatrix} + i \begin{bmatrix} C_{11} & C_{12} \\ C_{12}^* & C_{22} \end{bmatrix} \tag{1.2}$$

such that all diagonal blocks are square. Say k and l ($k, l > 0$ and $k + l = n$) the order of A_{11} and A_{22}, respectively, and let $m = \min\{k,l\}$.

An accretive-dissipative matrix $A \in M_n(\mathbf{C})$ is said to be a Buckley matrix if, in representation (1.1), $B = I_n$ (the identity matrix of order n).

Let $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \in M_n(\mathbf{C})$. If A_{22} is invertible, then the Schur complement of A_{22} in A is denoted by $A/A_{22} := A_{11} - A_{12}A_{22}^{-1}A_{21}$. For nonsingular matrix A, its condition number is denoted by $\kappa(A) := \sqrt{\frac{\lambda_{\max}(A^*A)}{\lambda_{\min}(A^*A)}}$, the ratio of largest and smallest

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singular value of A. For Hermitian matrices $B, C \in M_n(\mathbf{C})$, we write $B > (\geqslant) C$ to mean that $B - C$ is Hermitian positive (semi)definite.

In [3], the following determinantal inequality of Fischer type (see, e.g. [4]) is obtained for accretive-dissipative matrices.

THEOREM 1.1. *Let* $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ *be an accretive-dissipative matrix,* k *and* l *be the orders of* A_{11} *and* A_{22}*, respectively, and let* $m = \min\{k, l\}$*. Then*

$$|\det A| \leqslant 3^m |\det A_{11}||\det A_{22}|. \tag{1.3}$$

The purpose of this paper is to show a reversed inequality of (1.3), our main result can be stated as:

THEOREM 1.2. *Under the same condition of Theorem 1.1, we have*

$$|\det A| \geqslant \frac{(4\kappa)^m}{(1+\kappa)^{2m}} |\det A_{11}||\det A_{22}|, \tag{1.4}$$

where κ *is the maximum of the condition numbers of* B *and* C *appearing in (1.1). The inequality (1.4) is sharp.*

If A is a Buckley matrix, then $\kappa = \kappa(C)$. The proof of Theorem 1.2 is given in the next section.

2. Auxiliary result and the proof

LEMMA 2.1. *Let* $B = \begin{bmatrix} B_{11} & B_{12} \\ B_{12}^* & B_{22} \end{bmatrix}$ *be Hermitian positive definite, then*

$$B/B_{22} \geqslant \frac{4\kappa(B)}{(1+\kappa(B))^2} B_{11}. \tag{2.1}$$

Proof. We know $\left(\frac{1-\kappa(B)}{1+\kappa(B)}\right)^2 B_{11} \geqslant B_{12} B_{22}^{-1} B_{12}^*$ (see, e.g. [5, (6)]), so

$$B_{11} - B_{12} B_{22}^{-1} B_{12}^* \geqslant \frac{4\kappa(B)}{(1+\kappa(B))^2} B_{11},$$

i.e., (2.1) holds. □

Formula (2.2) below for the Schur complement of a complex matrix and the Schur complement of its real and imaginary part should be of interest in its own right. The formula for difference of two Schur complements can be found in [1].

LEMMA 2.2. *Let* $A = B + iC$*,* $B = B^*, C = C^*$*, be partitioned as in (1.2). If* B_{22}, C_{22} *are invertible, then*

$$A/A_{22} = B/B_{22} + i(C/C_{22}) + X(B_{22}^{-1} - iC_{22}^{-1})^{-1} X^*, \tag{2.2}$$

where $X = B_{12} B_{22}^{-1} - C_{12} C_{22}^{-1}$.

Proof. Firstly, observe that A_{22} is invertible. Since B_{22}, C_{22} are invertible Hermitian matrices, so B_{22}^2 and C_{22}^2 are positive definite. For any nonzero vector x, we have

$$x^*A_{22}^*A_{22}x = x^*(B_{22} - iC_{22})(B_{22} + iC_{22})x = x^*B_{22}^2x + x^*C_{22}^2x > 0,$$

thus $A_{22}^*A_{22}$ is positive definite, in particular, A_{22} is invertible.

The proof of (2.2) is then by direct verification.

$$\begin{aligned}
&X(B_{22}^{-1} - iC_{22}^{-1})^{-1}X^* \\
&= (B_{12}B_{22}^{-1} - C_{12}C_{22}^{-1})iC_{22}(B_{22} + iC_{22})^{-1}B_{22}(B_{22}^{-1}B_{12}^* - C_{22}^{-1}C_{12}^*) \\
&= (B_{12}B_{22}^{-1}iC_{22} - iC_{12})(B_{22} + C_{22})^{-1}(B_{12}^* - B_{22}C_{22}^{-1}C_{12}^*) \\
&= (B_{12}B_{22}^{-1}iC_{22} + B_{12} - B_{12} - iC_{12})(B_{22} + iC_{22})^{-1}(B_{12}^* - B_{22}C_{22}^{-1}C_{12}^*) \\
&= (B_{12}B_{22}^{-1}iC_{22} + B_{12})(B_{22} + iC_{22})^{-1}(B_{12}^* - B_{22}C_{22}^{-1}C_{12}^*) \\
&\quad -(B_{12} + iC_{12})(B_{22} + iC_{22})^{-1}(B_{12}^* - B_{22}C_{22}^{-1}C_{12}^*) \\
&= B_{12}B_{22}^{-1}(B_{12}^* - B_{22}C_{22}^{-1}C_{12}^*) \\
&\quad -(B_{12} + iC_{12})(B_{22} + iC_{22})^{-1}(B_{12}^* + iC_{12}^* - iC_{12}^* - B_{22}C_{22}^{-1}C_{12}^*) \\
&= B_{12}B_{22}^{-1}B_{12}^* - B_{12}C_{22}^{-1}C_{12}^* - (B_{12} + iC_{12})(B_{22} + iC_{22})^{-1}(B_{12}^* + iC_{12}^*) \\
&\quad +(B_{12} + iC_{12})(B_{22} + iC_{22})^{-1}(iC_{12}^* + B_{22}C_{22}^{-1}C_{12}^*) \\
&= B_{12}B_{22}^{-1}B_{12}^* - B_{12}C_{22}^{-1}C_{12}^* - (B_{12} + iC_{12})(B_{22} + iC_{22})^{-1}(B_{12}^* + iC_{12}^*) \\
&\quad +(B_{12} + iC_{12})C_{22}^{-1}C_{12}^* \\
&= B_{12}B_{22}^{-1}B_{12}^* + iC_{12}C_{22}^{-1}C_{12}^* - (B_{12} + iC_{12})(B_{22} + iC_{22})^{-1}(B_{12}^* + iC_{12}^*)
\end{aligned}$$

The identity (2.2) becomes clear after expanding $A/A_{22}, B/B_{22}, i(C/C_{22})$. □

COROLLARY 2.3. *Let $A = B + iC$, $B = B^*, C = C^*$, be accretive-dissipative and be partitioned as in (1.2). If $A/A_{22} = R + iS$ is its Toeplitz decomposition, then*

$$R \geqslant B/B_{22} \text{ and } S \geqslant C/C_{22}. \tag{2.3}$$

Proof. It is known that A/A_{22} is accretive-dissipative [2, Property 6]. Moreover, $(B_{22}^{-1} - iC_{22}^{-1})^{-1}$ is also accretive-dissipative [2, Property 1], so is $X(B_{22}^{-1} - iC_{22}^{-1})^{-1}X^*$ [2, Property 3]). The conclusion follows due to (2.2). □

LEMMA 2.4. [3] *Let $A_1, A_2 \in M_n(\mathbf{C})$ and let $A_1 = B_1 + iC_1$, $A_2 = B_2 + iC_2$ be the Toeplitz decompositions of these matrices. If $B_1 \geqslant B_2, C_1 \geqslant C_2$, then*

$$|\det A_1| \geqslant |\det A_2|.$$

We are now in a position to give the proof of our main result.

Proof of Theorem 1.2. Without loss of generality, we assume $m = k$. Since $|\det A| = |\det A_{22}||\det(A/A_{22})|$, so it suffices to show

$$|\det(A/A_{22})| \geqslant \frac{(4\kappa)^m}{(1+\kappa)^{2m}}|\det A_{11}|, \tag{2.4}$$

where $\kappa = \max\{\kappa(B), \kappa(C)\}$. Note that $f(x) = \frac{4x}{(1+x)^2}$ is decreasing in $x \in [1,\infty)$. By Lemma 2.4, (2.3), and Lemma 2.1, we have

$$\begin{aligned}
|\det(A/A_{22})| &\geqslant |\det(B/B_{22} + iC/C_{22})| \\
&\geqslant \left|\det\left(\frac{4\kappa(B)}{(1+\kappa(B))^2}B_{11} + i\frac{4\kappa(C)}{(1+\kappa(C))^2}C_{11}\right)\right| \\
&\geqslant \left|\det\left(\frac{4\kappa}{(1+\kappa)^2}B_{11} + i\frac{4\kappa}{(1+\kappa)^2}C_{11}\right)\right| \\
&= \frac{(4\kappa)^m}{(1+\kappa)^{2m}}|\det(B_{11} + iC_{11})| \\
&= \frac{(4\kappa)^m}{(1+\kappa)^{2m}}|\det A_{11}|.
\end{aligned}$$

The inequality (1.4) is sharp. For example, let $A = \begin{bmatrix} I_k + iI_k & 0 \\ 0 & I_l + iI_l \end{bmatrix}$. Then the equality in (1.4) holds. This completes the proof. □

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