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AN INTEGRAL–TYPE OPERATOR FROM BLOCH SPACES TO $\mathcal{Q}_p$ SPACES IN THE UNIT BALL

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Abstract. Let $H(B)$ denote the space of all holomorphic functions on the unit ball B of $\mathbb{C}^n$. Let $\alpha > 0$, $f \in H(B)$ with homogeneous expansion $f = \sum_{k=0}^{\infty} f_k$. The fractional derivative $\mathscr{D}^\alpha f$ is defined as

$$\mathscr{D}^\alpha f(z) = \sum_{k=0}^{\infty} (k+1)^\alpha f_k(z).$$

Let φ be a holomorphic self-map of B and $g \in H(B)$ such that $g(0) = 0$. In this paper we consider the following integral-type operator

$$\mathscr{D}^\alpha_{\varphi,g} f(z) = \int_0^1 \mathscr{D}^\alpha f(\varphi(tz))g(tz)\frac{dt}{t}, \quad f \in H(B).$$

The boundedness of the operator $\mathscr{D}^\alpha_{\varphi,g}$ from the Bloch space to the spaces $\mathcal{Q}_p$ and $\mathcal{Q}_{p,0}$ are investigated. In particular, the boundedness and compactness of the operator $\mathscr{D}^1_{\varphi,g}$ on the Bloch spaces are completely characterized.

1. Introduction

Let B denote the unit ball and ∂B the unit sphere in $\mathbb{C}^n$. Let $H(B)$ be the space of all holomorphic functions on B. Let $\Re f$ stands for the radial derivative of f, that is, $\Re f(z) = \sum_{j=1}^n z_j \frac{\partial f}{\partial z_j}(z)$, $z = (z_1, z_2, \cdots, z_n) \in B$. Let $\varphi_a(z)$ be the holomorphic involution exchanging 0 and a. For $f \in C^1(B)$, the invariant gradient $\widetilde{\nabla} f$ is defined by

$$(\widetilde{\nabla} f)(z) = \nabla(f \circ \varphi_z)(0),$$

where $\nabla f(z) = (\partial f/\partial z_1, \ldots, \partial f/\partial z_n)$ is the complex gradient of f.

A function $f \in H(B)$ is said to belong to the Bloch space, denoted by $\mathscr{B} = \mathscr{B}(B)$, if

$$b(f) = \sup_{z\in B}(1 - |z|^2)\,|\Re f(z)| < \infty.$$

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It is well-known that $\mathscr{B}$ is a Banach space with the norm $\|f\|_{\mathscr{B}} = |f(0)| + b(f)$. Let $\mathscr{B}_0$, called the little Bloch space, denote the subspace of $\mathscr{B}$ consisting of those $f \in \mathscr{B}$ for which

$$\lim_{|z|\to 1}(1-|z|^2)|\Re f(z)| = 0.$$

Recall that the Bergman space A^1 is the space of all $f \in H(B)$ such that

$$\int_B |f(z)|dv(z) < \infty,$$

where $dv(z)$ is the normalized volume measure on B. From [23], we know that $(\mathscr{B}_0)^* = A^1$ and $(A^1)^* = \mathscr{B}$.

Let $f \in H(B)$ with homogeneous expansion $f = \sum_{k=0}^{\infty} f_k$ and $\alpha > 0$. The fractional derivative $\mathscr{D}^\alpha f$ is defined as follows:

$$\mathscr{D}^\alpha f(z) = \sum_{k=0}^{\infty}(k+1)^\alpha f_k(z).$$

Note that $\Re f = \sum_{k=0}^{\infty} k f_k$ if f has homogeneous expansion $f = \sum_{k=0}^{\infty} f_k$. Hence $\mathscr{D}^1 f = \Re f + f$. For simplicity of notation, we denote $\mathscr{D}^1$ by $\mathscr{D}$. From [1] we known that $f \in \mathscr{B}$ if and only if

$$\sup_{z\in B}(1-|z|^2)^\alpha |\mathscr{D}^\alpha f(z)| < \infty.$$

$f \in \mathscr{B}_0$ if and only if $\lim_{|z|\to 1}(1-|z|^2)^\alpha |\mathscr{D}^\alpha f(z)| = 0$. Moreover,

$$\|f\|_{\mathscr{B}} \asymp \sup_{z\in B}(1-|z|^2)^\alpha |\mathscr{D}^\alpha f(z)|. \tag{1}$$

See [23] and the references therein for more characterizations of the Bloch space in the unit ball.

In recent years a special class of Möbius invariant function space in the unit disk, the so-called $\mathscr{Q}_p$ space, has attracted a lot of attention. See [20, 21] for a summary of recent research about the $\mathscr{Q}_p$ space in the unit disk. The $\mathscr{Q}_p$ space was generalized to the unit ball in [11]. For $0 < p < \infty$, recall that an $f \in H(B)$ is said to belong to the space $\mathscr{Q}_p$ if (see [11])

$$\sup_{a\in B}\int_B |\widetilde{\nabla} f(z)|^2 G^p(z,a)d\lambda(z) < \infty.$$

Here $d\lambda(z) = (1-|z|^2)^{-n-1}dv(z)$, $G(z,a) = g(\varphi_a(z))$,

$$g(z) = \frac{n+1}{2n}\int_{|z|}^1 (1-t^2)^{n-1}t^{-2n+1}dt.$$

Let $\mathscr{Q}_{p,0}$ denote the subspace of $\mathscr{Q}_p$ for which

$$\lim_{|a|\to 1}\int_B |\widetilde{\nabla} f(z)|^2 G^p(z,a)d\lambda(z) = 0.$$

By [11], we know that $\mathcal{Q}_p = \mathcal{B}$ (the Bloch space) when $1 < p < \frac{n}{n-1}$; $\mathcal{Q}_1 = BMOA$; and $\mathcal{Q}_p$ contains only the constant functions when $0 < p \leqslant \frac{n-1}{n}$ or $p \geqslant \frac{n}{n-1}$.

For $\xi \in \partial B$, $\delta > 0$, let

$$S(\xi,\delta) = \{z \in B : |1 - \langle z,\xi\rangle| < \delta\}.$$

A positive Borel measure μ on B ia called a $\beta-$Carleson measure if there exists a constant $C > 0$ such that $\mu(S(\xi,\delta)) \leqslant C\delta^{n\beta}$ for all $\xi \in \partial B$ and $\delta > 0$. If

$$\lim_{\delta \to 0} \frac{\mu(S(\xi,\delta))}{\delta^{n\beta}} = 0,$$

uniformly for $\xi \in \partial B$, we call μ a vanishing $\beta-$Carleson measure.

Let φ be a holomorphic self-map of the unit ball. Define a linear operator C_φ on $H(B)$, called the composition operator, by

$$(C_\varphi f)(z) = (f \circ \varphi)(z), f \in H(B).$$

It is interesting to provide a function theoretic characterization when φ induce a bounded or compact composition operator on various spaces. The book [2] contains much information on this topic.

Let φ be a holomorphic self-map of B and $g \in H(B)$ such that $g(0) = 0$. The generalized composition operator

$$C_\varphi^g f(z) = \int_0^1 \Re f(\varphi(tz))g(tz)\frac{dt}{t}, \quad f \in H(B), \tag{2}$$

was recently introduced in [14] and [24] respectively, motivated by [9]. See, for example, [4, 6, 7, 8, 10, 12, 13, 14, 15, 16, 17, 18, 19, 22, 24, 25, 26] for the study of the operator C_φ^g. Note that the generalized composition operator is induced by radial derivative. Now we introduce the integral-type operator induced by fractional derivative as follows:

$$\mathcal{D}_{\varphi,g}^\alpha f(z) = \int_0^1 \mathcal{D}^\alpha f(\varphi(tz))g(tz)\frac{dt}{t}, \quad f \in H(B). \tag{3}$$

In this paper we study the boundedness of the integral-type operator $\mathcal{D}_{\varphi,g}^\alpha$ from the Bloch space to the spaces $\mathcal{Q}_p$ and $\mathcal{Q}_{p,0}$. In particular, the boundedness and compactness of the operator $\mathcal{D}_{\varphi,g}^1$ on the Bloch spaces are completely characterized. To the best of our knowledge, the operator $\mathcal{D}_{\varphi,g}^\alpha$ is studied in the present paper for the first time.

Throughout this paper, constants are denoted by C, they are positive and may differ from one occurrence to the other. The notation $A \asymp B$ means that there is a positive constant C such that $C^{-1}B \leqslant A \leqslant CB$.

2. Main results and proofs

In this section we will give our main results and proofs. First we state several auxiliary results which we will use in the proofs of main results. The proof of the next lemma was essentially proved, for example, in [8] or [14].

LEMMA 1. *Let φ be a holomorphic self-map of B, $g \in H(B)$ such that $g(0) = 0$. Then*

$$\Re[\mathscr{D}^{\alpha}_{\varphi,g}(f)](z) = \mathscr{D}^{\alpha} f(\varphi(z))g(z), \quad f \in H(B).$$

The following lemma can be found in [1].

LEMMA 2. *There exists a positive integer $M = M(n)$ with the following property: there exist functions $f_i \in \mathscr{B}(1 \leqslant i \leqslant M)$ such that*

$$\sum_{i=1}^{M} |\mathscr{D}^{\alpha} f_i(z)| \geqslant \frac{1}{(1-|z|)^{\alpha}}, \quad z \in B.$$

LEMMA 3. [8] *A closed set K in $\mathscr{B}$ is compact if and only if it is bounded and satisfies*

$$\lim_{|z|\to 1} \sup_{f\in K} (1-|z|^2)|\Re f(z)| = 0.$$

The following criterion for compactness follows from standard arguments similar to those outlined in Proposition 3.11 of [2].

LEMMA 4. *Let φ be a holomorphic self-map of B, $g \in H(B)$ such that $g(0) = 0$. Then $\mathscr{D}^{\alpha}_{\varphi,g} : \mathscr{B}(\text{or } \mathscr{B}_0) \to \mathscr{B}$ is compact if and only if $\mathscr{D}^{\alpha}_{\varphi,g} : \mathscr{B}(\text{or } \mathscr{B}_0) \to \mathscr{B}$ is bounded and for any bounded sequence $(f_k)_{k\in\mathbb{N}}$ in $\mathscr{B}(\text{or } \mathscr{B}_0)$ which converges to zero uniformly on compact subset of B as $k \to \infty$, we have $\|\mathscr{D}^{\alpha}_{\varphi,g} f_k\|_{\mathscr{B}} \to 0$ as $k \to \infty$.*

Now, we are in a position to formulate and prove the main results of this paper.

THEOREM 1. *Let φ be a holomorphic self-map of B, $\frac{n-1}{n} < p < \frac{n}{n-1}$, $g \in H(B)$ such that $g(0) = 0$. Then $\mathscr{D}^{\alpha}_{\varphi,g} : \mathscr{B} \to \mathscr{Q}_p$ is bounded if and only if*

$$\sup_{a\in B} \int_B \frac{(1-|z|^2)^2 |g(z)|^2}{(1-|\varphi(z)|^2)^{2\alpha}} (1-|\varphi_a(z)|^2)^{np} d\lambda(z) < \infty. \tag{4}$$

Proof. Assume that (4) holds. From [5] we know that $f \in \mathscr{Q}_p$ if and only if

$$\sup_{a\in B} \int_B |\Re f(z)|^2 (1-|z|^2)^2 (1-|\varphi_a(z)|^2)^{np} d\lambda(z) < \infty. \tag{5}$$

For any $f \in \mathscr{B}$, using (1) and (5) we get

$$\begin{aligned} &\sup_{a\in B} \int_B |\Re(\mathscr{D}^{\alpha}_{\varphi,g} f)(z)|^2 (1-|z|^2)^2 (1-|\varphi_a(z)|^2)^{np} d\lambda(z) \\ &= \sup_{a\in B} \int_B |g(z)|^2 |\mathscr{D}^{\alpha} f(\varphi(z))|^2 (1-|z|^2)^2 (1-|\varphi_a(z)|^2)^{np} d\lambda(z) \\ &\leqslant C\|f\|^2_{\mathscr{B}} \sup_{a\in B} \int_B \frac{(1-|z|^2)^2 |g(z)|^2}{(1-|\varphi(z)|^2)^{2\alpha}} (1-|\varphi_a(z)|^2)^{np} d\lambda(z) < \infty, \end{aligned} \tag{6}$$

i.e. $\mathscr{D}^{\alpha}_{\varphi,g} f \in \mathscr{Q}_p$, hence $\mathscr{D}^{\alpha}_{\varphi,g} : \mathscr{B} \to \mathscr{Q}_p$ is bounded.

Conversely, assume that $\mathscr{D}^{\alpha}_{\varphi,g}:\mathscr{B}\to\mathscr{Q}_p$ is bounded. By Lemma 2, there exists a positive integer M and there exist functions $f_i\in\mathscr{B}(1\leqslant i\leqslant M)$ such that

$$\sum_{i=1}^{M}|\mathscr{D}^{\alpha}f_i(z)|\geqslant\frac{1}{(1-|z|)^{\alpha}},\quad z\in B. \tag{7}$$

By the assumption, we have $\mathscr{D}^{\alpha}_{\varphi,g}f_i\in\mathscr{Q}_p(1\leqslant i\leqslant M)$, i.e.

$$\sup_{a\in B}\int_B|\Re(\mathscr{D}^{\alpha}_{\varphi,g}f_i)(z)|^2(1-|z|^2)^2(1-|\varphi_a(z)|^2)^{np}d\lambda(z)<\infty,\ \ i=1,\cdots,M. \tag{8}$$

Since

$$\frac{1}{(1-|\varphi(z)|)^{2\alpha}}\leqslant\Big(\sum_{i=1}^{M}|\mathscr{D}^{\alpha}f_i(\varphi(z))|\Big)^2\leqslant C\sum_{i=1}^{M}|\mathscr{D}^{\alpha}f_i(\varphi(z))|^2,\quad z\in B, \tag{9}$$

we obtain

$$\begin{aligned}&\sup_{a\in B}\int_B\frac{(1-|z|^2)^2|g(z)|^2}{(1-|\varphi(z)|)^{2\alpha}}(1-|\varphi_a(z)|^2)^{np}d\lambda(z)\\ &\leqslant C\sup_{a\in B}\int_B\sum_{i=1}^{M}|\mathscr{D}^{\alpha}f_i(\varphi(z))|^2|g(z)|^2(1-|z|^2)^2(1-|\varphi_a(z)|^2)^{np}d\lambda(z)\\ &=C\sum_{i=1}^{M}\sup_{a\in B}\int_B|\Re(\mathscr{D}^{\alpha}_{\varphi,g}f_i)(z)|^2(1-|z|^2)^2(1-|\varphi_a(z)|^2)^{np}d\lambda(z)<\infty. \end{aligned} \tag{10}$$

Then (4) follows from (10). $\square$

REMARK 1. From [5] we know that $f\in\mathscr{Q}_p$ if and only if

$$|\Re f(z)|^2(1-|z|^2)^{np-n+1}dv(z) \tag{11}$$

is a $p-$Carleson measure. Using this characterization of $\mathscr{Q}_p$ and similarly to the above proof we see that $\mathscr{D}^{\alpha}_{\varphi,g}:\mathscr{B}\to\mathscr{Q}_p$ is bounded if and only if

$$\sup\frac{1}{\delta^{np}}\int_{S(\xi,\delta)}\frac{(1-|z|^2)^{np-n+1}|g(z)|^2}{(1-|\varphi(z)|^2)^{2\alpha}}dv(z)<\infty \tag{12}$$

where sup is taken over all $S(\xi,\delta)$.

THEOREM 2. *Let φ be a holomorphic self-map of B, $\frac{n-1}{n}<p<\frac{n}{n-1}$, $g\in H(B)$ such that $g(0)=0$. Then $\mathscr{D}^{\alpha}_{\varphi,g}:\mathscr{B}\to\mathscr{Q}_{p,0}$ is bounded if and only if*

$$\lim_{|a|\to1}\int_B\frac{(1-|z|^2)^2|g(z)|^2}{(1-|\varphi(z)|^2)^{2\alpha}}(1-|\varphi_a(z)|^2)^{np}d\lambda(z)=0. \tag{13}$$

Proof. Assume that (13) holds. Similar to the proof of Theorem 4 of [7] or Lemma 2.6 in [16], we have

$$\sup_{a\in B}\int_B\frac{(1-|z|^2)^2|g(z)|^2}{(1-|\varphi(z)|^2)^{2\alpha}}(1-|\varphi_a(z)|^2)^{np}d\lambda(z)<\infty.$$

By Theorem 1, we know that $\mathscr{D}^{\alpha}_{\varphi,g}:\mathscr{B}\to\mathscr{Q}_p$ is bounded. We need only to prove that $\mathscr{D}^{\alpha}_{\varphi,g}f\in\mathscr{Q}_{p,0}$ for each $f\in\mathscr{B}$, and this follows from the inequality

$$\begin{aligned}&\int_B|\Re(\mathscr{D}^{\alpha}_{\varphi,g}f)(z)|^2(1-|z|^2)^2(1-|\varphi_a(z)|^2)^{np}d\lambda(z)\\ \leqslant\ & C\|f\|^2_{\mathscr{B}}\int_B\frac{(1-|z|^2)^2|g(z)|^2}{(1-|\varphi(z)|^2)^{2\alpha}}(1-|\varphi_a(z)|^2)^{np}d\lambda(z).\end{aligned}$$

Conversely, assume that $\mathscr{D}^{\alpha}_{\varphi,g}:\mathscr{B}\to\mathscr{Q}_{p,0}$ is bounded. Using a way similar to the proof of Theorem 1, we choose functions $f_i\in\mathscr{B}(1\leqslant i\leqslant M)$ such that

$$\sum_{i=1}^{M}|\mathscr{D}^{\alpha}f_i(z)|\geqslant\frac{1}{(1-|z|)^{\alpha}},\quad z\in B. \tag{14}$$

Then $\mathscr{D}^{\alpha}_{\varphi,g}f_i\in\mathscr{Q}_{p,0}(1\leqslant i\leqslant M)$. Therefore

$$\begin{aligned}&\lim_{|a|\to1}\int_B\frac{(1-|z|^2)^2|g(z)|^2}{(1-|\varphi(z)|)^{2\alpha}}(1-|\varphi_a(z)|^2)^{np}d\lambda(z)\\ \leqslant\ & C\sum_{i=1}^{M}\lim_{|a|\to1}\int_B|\Re(\mathscr{D}^{\alpha}_{\varphi,g}f_i)(z)|^2(1-|z|^2)^2(1-|\varphi_a(z)|^2)^{np}d\lambda(z)=0,\end{aligned}$$

which shows that (13) holds. □

REMARK 2. From [5] we know that $f\in\mathscr{Q}_{p,0}$ if and only if

$$|\Re f(z)|^2(1-|z|^2)^{np-n+1}dv(z) \tag{15}$$

is a vanishing $p-$Carleson measure. Using this characterization of $\mathscr{Q}_{p,0}$ and similarly to the proof of Theorem 2 we see that $\mathscr{D}^{\alpha}_{\varphi,g}:\mathscr{B}\to\mathscr{Q}_{p,0}$ is bounded if and only if

$$\lim_{\delta\to0}\frac{1}{\delta^{np}}\int_{S(\xi,\delta)}\frac{(1-|z|^2)^{np-n+1}|g(z)|^2}{(1-|\varphi(z)|^2)^{2\alpha}}dv(z)=0 \tag{16}$$

uniformly for $\xi\in\partial B$.

By [11], we see that $\mathscr{Q}_p=\mathscr{B}$ and $\mathscr{Q}_{p,0}=\mathscr{B}_0$ when $1<p<\frac{n}{n-1}$. Now we consider the special case of $1<p<\frac{n}{n-1}$.

THEOREM 3. *Let φ be a holomorphic self-map of B, $g\in H(B)$ such that $g(0)=0$. Then the following statements are equivalent.*

(i) The operator $\mathscr{D}^{\alpha}_{\varphi,g}:\mathscr{B}\to\mathscr{B}$ is bounded;

(ii)

$$M_1:=\sup_{z\in B}(1-|z|^2)|g(z)|(1-|\varphi(z)|^2)^{-\alpha}<\infty. \tag{17}$$

Proof. $(i)\Rightarrow(ii)$. From Lemma 2, there exists a positive integer M and there exist functions $f_i\in\mathscr{B}(1\leqslant i\leqslant M)$ such that

$$\sum_{i=1}^{M}|\mathscr{D}^{\alpha}f_i(z)|\geqslant\frac{1}{(1-|z|)^{\alpha}},\quad z\in B. \tag{18}$$

Setting $z = \varphi(w)$, we get

$$\sum_{i=1}^{M} |\mathscr{D}^{\alpha} f_i(\varphi(w))| \geqslant \frac{1}{(1-|\varphi(w)|)^{\alpha}}, \quad w \in B. \tag{19}$$

Hence

$$\begin{aligned}\frac{(1-|z|^2)|g(z)|}{(1-|\varphi(z)|)^{\alpha}} &\leqslant \sum_{i=1}^{M}(1-|z|^2)|g(z)||\mathscr{D}^{\alpha} f_i(\varphi(z))| \\ &= \sum_{i=1}^{M}(1-|z|^2)|\Re(\mathscr{D}^{\alpha}_{\varphi,g} f_i)(z)|. \end{aligned} \tag{20}$$

By the boundedness of $\mathscr{D}^{\alpha}_{\varphi,g} : \mathscr{B} \to \mathscr{B}$, we see that $\mathscr{D}^{\alpha}_{\varphi,g} f_i \in \mathscr{B}(i = 1,\cdots,M)$. Therefore the superemum over $z \in B$ of the right-hand side of (20) is finite, this implies that (17) holds.

$(ii) \Rightarrow (i)$. Suppose that (17) holds. Then for arbitrary $z \in B$ and $f \in \mathscr{B}$, by Lemma 1 and (1) we have

$$\begin{aligned}(1-|z|^2)|\Re(\mathscr{D}^{\alpha}_{\varphi,g} f)(z)| &= (1-|z|^2)|\mathscr{D}^{\alpha} f(\varphi(z))g(z)| \\ &= (1-|\varphi(z)|^2)^{\alpha}|\mathscr{D}^{\alpha} f(\varphi(z))|\frac{(1-|z|^2)|g(z)|}{(1-|\varphi(z)|^2)^{\alpha}} \\ &\leqslant C\|f\|_{\mathscr{B}}\frac{(1-|z|^2)|g(z)|}{(1-|\varphi(z)|^2)^{\alpha}}. \end{aligned} \tag{21}$$

By the assumption we see that $(\mathscr{D}^{\alpha}_{\varphi,g} f)(0) = 0$. On account of the condition (17), the boundedness of the operator $\mathscr{D}^{\alpha}_{\varphi,g} : \mathscr{B} \to \mathscr{B}$ follows from (21) by taking the supremum over B. This completes the proof of Theorem 3. □

THEOREM 4. *Let φ be a holomorphic self-map of B, $g \in H(B)$ such that $g(0) = 0$. Then the following statements are equivalent.*

(i) The operator $\mathscr{D}^{\alpha}_{\varphi,g} : \mathscr{B} \to \mathscr{B}_0$ is bounded;

(ii)

$$\lim_{|z|\to 1}(1-|z|^2)|g(z)|(1-|\varphi(z)|^2)^{-\alpha} = 0. \tag{22}$$

Proof. $(i) \Rightarrow (ii)$. From Lemma 2, there exists a positive integer M and there exist functions $f_i \in \mathscr{B}(1 \leqslant i \leqslant M)$ such that

$$\sum_{i=1}^{M} |\mathscr{D}^{\alpha} f_i(z)| \geqslant \frac{1}{(1-|z|)^{\alpha}}, \quad z \in B. \tag{23}$$

From the proof of Theorem 1,

$$\frac{(1-|z|^2)|g(z)|}{(1-|\varphi(z)|)^{\alpha}} \leqslant \sum_{i=1}^{M}(1-|z|^2)|\Re(\mathscr{D}^{\alpha}_{\varphi,g} f_i)(z)|. \tag{24}$$

By the boundedness of $\mathscr{D}^{\alpha}_{\varphi,g}:\mathscr{B}\to\mathscr{B}_0$, we see that $\mathscr{D}^{\alpha}_{\varphi,g}f_i\in\mathscr{B}_0(i=1,\cdots,M)$. Therefore the right-hand side of (24) tends to zero as $|z|\to 1$. Hence (22) is satisfied.

$(ii)\Rightarrow(i)$. Suppose that (22) holds. Then for arbitrary $z\in B$ and $f\in\mathscr{B}$, from the proof of Theorem 3 we have

$$(1-|z|^2)|\Re(\mathscr{D}^{\alpha}_{\varphi,g}f)(z)| = (1-|\varphi(z)|^2)^{\alpha}|\mathscr{D}^{\alpha}f(\varphi(z))|\frac{(1-|z|^2)|g(z)|}{(1-|\varphi(z)|^2)^{\alpha}}$$
$$\leqslant C\|f\|_{\mathscr{B}}\frac{(1-|z|^2)|g(z)|}{(1-|\varphi(z)|^2)^{\alpha}}\to 0 \tag{25}$$

as $|z|\to 1$, that is $\mathscr{D}^{\alpha}_{\varphi,g}f\in\mathscr{B}_0$. From the assumption and Theorem 3 we see that $\mathscr{D}^{\alpha}_{\varphi,g}:\mathscr{B}\to\mathscr{B}$ is bounded. Therefore $\mathscr{D}^{\alpha}_{\varphi,g}:\mathscr{B}\to\mathscr{B}_0$ is bounded. This completes the proof of Theorem 4. □

Finally, we consider the case $\alpha=1$. In this case, we completely characterize the boundedness and compactness of the operator $\mathscr{D}^{1}_{\varphi,g}$ on the Bloch space in the unit ball.

THEOREM 5. *Let φ be a holomorphic self-map of B, $g\in H(B)$ such that $g(0)=0$. Then the following statements are equivalent.*

(i) The operator $\mathscr{D}^{1}_{\varphi,g}:\mathscr{B}_0\to\mathscr{B}$ is bounded;

(ii)

$$M_1:=\sup_{z\in B}(1-|z|^2)|g(z)|(1-|\varphi(z)|^2)^{-1}<\infty. \tag{26}$$

Proof. $(ii)\Rightarrow(i)$. Assume that (26) holds. By Theorem 3, we see that $\mathscr{D}^{1}_{\varphi,g}:\mathscr{B}\to\mathscr{B}$ is bounded. Hence $\mathscr{D}^{1}_{\varphi,g}:\mathscr{B}_0\to\mathscr{B}$ is bounded.

$(i)\Rightarrow(ii)$. Suppose that $\mathscr{D}^{1}_{\varphi,g}:\mathscr{B}_0\to\mathscr{B}$ is bounded. Taking $f(z)=1$, then by the boundedness of the operator $\mathscr{D}^{1}_{\varphi,g}:\mathscr{B}_0\to\mathscr{B}$ we get

$$L_1:=\sup_{z\in B}(1-|z|^2)|g(z)|<\infty. \tag{27}$$

For $a\in B$, set

$$f_a(z)=\ln\frac{e}{1-\langle z,a\rangle}. \tag{28}$$

It is easy to check that $\sup_{a\in B}\|f_a\|_{\mathscr{B}}\leqslant 2$ and $f_a\in\mathscr{B}_0$ for each $a\in B$. Therefore we have

$$2\|\mathscr{D}^{1}_{\varphi,g}\|_{\mathscr{B}\to\mathscr{B}}\geqslant\|\mathscr{D}^{1}_{\varphi,g}f_{\varphi(b)}\|_{\mathscr{B}}=|(\mathscr{D}^{1}_{\varphi,g}f_{\varphi(b)})(0)|+\sup_{z\in B}(1-|z|^2)|\Re(\mathscr{D}^{1}_{\varphi,g}f_{\varphi(b)})(z)|$$
$$\geqslant\frac{(1-|b|^2)|g(b)||\varphi(b)|^2}{1-|\varphi(b)|^2}-(1-|b|^2)|g(b)|\ln\frac{e}{1-|\varphi(b)|^2} \tag{29}$$

for any $b\in B$. Here we used the fact $|(\mathscr{D}^{1}_{\varphi,g}f_{\varphi(b)})(0)|=0$.

For $a \in B$, set

$$h_a(z) = \ln \frac{e}{1-\langle z,a\rangle} - \frac{1-|a|^2}{1-\langle z,a\rangle}.$$

It is easy to check that $\sup_{a\in B}\|h_a\|_{\mathscr{B}} \leqslant 4$ and $h_a \in \mathscr{B}_0$ for each $a \in B$. Moreover

$$\Re h_a(z) = \frac{\langle z,a\rangle}{1-\langle z,a\rangle} - \frac{(1-|a|^2)\langle z,a\rangle}{(1-\langle z,a\rangle)^2}.$$

Therefore we have

$$\begin{aligned} 4\|\mathscr{D}^1_{\varphi,g}\|_{\mathscr{B}\to\mathscr{B}} &\geqslant \|\mathscr{D}^1_{\varphi,g}h_{\varphi(d)}\|_{\mathscr{B}} = \sup_{z\in B}(1-|z|^2)|\Re(\mathscr{D}^1_{\varphi,g}h_{\varphi(d)})(z)| \\ &\geqslant (1-|d|^2)|g(d)||h_{\varphi(d)}(\varphi(d))| - (1-|d|^2)|g(d)||\Re h_{\varphi(d)}(\varphi(d))| \\ &= (1-|d|^2)|g(d)|\Big(\ln\frac{e}{1-|\varphi(d)|^2}-1\Big), \end{aligned} \tag{30}$$

for any $d \in B$. From (30), we get

$$\sup_{z\in B}(1-|z|^2)|g(z)|\Big(\ln\frac{e}{1-|\varphi(z)|^2}-1\Big) < \infty, \tag{31}$$

which together with (27) imply

$$\sup_{z\in B}(1-|z|^2)|g(z)|\ln\frac{e}{1-|\varphi(z)|^2} < \infty. \tag{32}$$

From (29) and (32), we obtain

$$\sup_{z\in B}(1-|z|^2)|g(z)|\frac{|\varphi(z)|^2}{1-|\varphi(z)|^2} < \infty, \tag{33}$$

which implies

$$\begin{aligned} \sup_{\{z\in B,|\varphi(z)|>1/2\}} \frac{(1-|z|^2)|g(z)|}{1-|\varphi(z)|^2} &\leqslant \frac{1}{4}\sup_{\{z\in B,|\varphi(z)|>1/2\}}(1-|z|^2)|g(z)|\frac{|\varphi(z)|^2}{1-|\varphi(z)|^2} \\ &< \infty. \end{aligned} \tag{34}$$

From (27) we get

$$\sup_{\{z\in B,|\varphi(z)|\leqslant 1/2\}} \frac{(1-|z|^2)|g(z)|}{1-|\varphi(z)|^2} \leqslant \frac{4}{3}\sup_{z\in B}(1-|z|^2)|g(z)| < \infty. \tag{35}$$

Combining (34) with (35) we get (26). This completes the proof of this theorem. □

THEOREM 6. *Let φ be a holomorphic self-map of B, $g \in H(B)$ such that $g(0) = 0$. Then the following statements are equivalent.*

(i) The operator $\mathscr{D}^1_{\varphi,g} : \mathscr{B}_0 \to \mathscr{B}_0$ is bounded;

(ii) The operator $\mathscr{D}^1_{\varphi,g} : \mathscr{B}_0 \to \mathscr{B}$ is bounded and

$$\lim_{|z|\to 1}(1-|z|^2)|g(z)| = 0. \tag{36}$$

Proof. $(i) \Rightarrow (ii)$. Suppose that $\mathscr{D}^1_{\varphi,g} : \mathscr{B}_0 \to \mathscr{B}_0$ is bounded, then it is clear that $\mathscr{D}^1_{\varphi,g} : \mathscr{B}_0 \to \mathscr{B}$ is bounded. Taking the function $f(z) = 1$, and employing the boundedness of $\mathscr{D}^1_{\varphi,g} : \mathscr{B}_0 \to \mathscr{B}_0$, we obtain that (36) holds.

$(ii) \Rightarrow (i)$. Assume that $\mathscr{D}^1_{\varphi,g} : \mathscr{B}_0 \to \mathscr{B}$ is bounded and (36) holds. Then for each polynomial $p(z)$, we have

$$\begin{aligned}(1-|z|^2)|\Re(\mathscr{D}^1_{\varphi,g}p)(z)| &= (1-|z|^2)|\mathscr{D}p(\varphi(z))||g(z)| \\ &= (1-|z|^2)|p(\varphi(z)) + \Re p(\varphi(z))||g(z)| \\ &\leqslant (\|p\|_\infty + \|\Re p\|_\infty)(1-|z|^2)|g(z)|.\end{aligned}$$

From (36), it follows that for each polynomial p, $\mathscr{D}^1_{\varphi,g}(p) \in \mathscr{B}_0$. The set of all polynomials is dense in $\mathscr{B}_0$, thus for every $f \in \mathscr{B}_0$ there is a sequence of polynomials $(p_k)_{k\in\mathbb{N}}$ such that $\|p_k - f\|_{\mathscr{B}} \to 0$ as $k \to \infty$. From the boundedness of $\mathscr{D}^1_{\varphi,g} : \mathscr{B}_0 \to \mathscr{B}$, we have that

$$\|\mathscr{D}^1_{\varphi,g}p_k - \mathscr{D}^1_{\varphi,g}f\|_{\mathscr{B}} \leqslant \|\mathscr{D}^1_{\varphi,g}\|_{\mathscr{B}_0\to\mathscr{B}}\,\|p_k - f\|_{\mathscr{B}} \to 0, \quad \text{as } k \to \infty.$$

From this and since $\mathscr{B}_0$ is a closed subset of $\mathscr{B}$, we obtain $\mathscr{D}^1_{\varphi,g}f = \lim_{k\to\infty}\mathscr{D}^1_{\varphi,g}p_k \in \mathscr{B}_0$. Therefore $\mathscr{D}^1_{\varphi,g} : \mathscr{B}_0 \to \mathscr{B}_0$ is bounded. The proof is completed. □

THEOREM 7. *Let φ be a holomorphic self-map of B, $g \in H(B)$ such that $g(0) = 0$. Suppose that $\mathscr{D}^1_{\varphi,g} : \mathscr{B} \to \mathscr{B}$ is bounded, then the following statements are equivalent.*

(i) The operator $\mathscr{D}^1_{\varphi,g} : \mathscr{B} \to \mathscr{B}$ is compact;

(ii) The operator $\mathscr{D}^1_{\varphi,g} : \mathscr{B}_0 \to \mathscr{B}$ is compact;

(iii)

$$\lim_{|\varphi(z)|\to 1}(1-|z|^2)|g(z)|(1-|\varphi(z)|^2)^{-1} = 0. \tag{37}$$

Proof. $(i) \Rightarrow (ii)$. It is obvious.

$(ii) \Rightarrow (iii)$. Suppose that $\mathscr{D}^1_{\varphi,g} : \mathscr{B}_0 \to \mathscr{B}$ is compact. Let $(z_k)_{k\in\mathbb{N}}$ be a sequence in B such that $|\varphi(z_k)| \to 1$ as $k \to \infty$. Set

$$f_k(z) = \Big(\ln\frac{e}{1-|\varphi(z_k)|^2}\Big)^{-1}\Big(\ln\frac{e}{1-\langle z, \varphi(z_k)\rangle}\Big)^2, \ \ k \in \mathbb{N}.$$

From the proof of Theorem 5 we see that $f_k \in \mathscr{B}_0$ and $\sup_{k\in\mathbb{N}} \|f_k\|_{\mathscr{B}} \leqslant C$. Moreover, f_k converges to zero uniformly on compact subsets of B. By Lemma 4,

$$\begin{aligned}&\Big|\frac{2(1-|z_k|^2)|g(z_k)||\varphi(z_k)|^2}{1-|\varphi(z_k)|^2} - (1-|z_k|^2)|g(z_k)|\ln\frac{e}{1-|\varphi(z_k)|^2}\Big|\\ &\leqslant \sup_{z\in B}(1-|z|^2)|\Re(\mathscr{D}^1_{\varphi,g}f_k)(z)|\\ &= \|\mathscr{D}^1_{\varphi,g}f_k\|_{\mathscr{B}} \to 0,\end{aligned} \tag{38}$$

as $k\to\infty$, this implies

$$\lim_{k\to\infty}\frac{2(1-|z_k|^2)|g(z_k)||\varphi(z_k)|^2}{1-|\varphi(z_k)|^2} = \lim_{k\to\infty}(1-|z_k|^2)|g(z_k)|\ln\frac{e}{1-|\varphi(z_k)|^2}, \tag{39}$$

if any one of the limits exists.

Let $(z_k)_{k\in\mathbb{N}}$ be a sequence in B such that $|\varphi(z_k)|\to 1$ as $k\to\infty$. Set

$$h_k(z) = \Big(\ln\frac{e}{1-|\varphi(z_k)|^2}\Big)^{-1}\Big(\ln\frac{e}{1-\langle z,\varphi(z_k)\rangle}\Big)^2 - 2\frac{1-|\varphi(z_k)|^2}{1-\langle z,\varphi(z_k)\rangle}. \tag{40}$$

Analogous to the proof of Theorem 5 we see that $h_k \in \mathscr{B}_0$ and $\sup_{k\in\mathbb{N}}\|h_k\|_{\mathscr{B}} \leqslant C$. Moreover, h_k converges to zero uniformly on compact subsets of B. By Lemma 4,

$$\begin{aligned}(1-|z_k|^2)|g(z_k)|\Big(\ln\frac{e}{1-|\varphi(z_k)|^2}-2\Big) &\leqslant \sup_{z\in B}(1-|z|^2)|\Re(\mathscr{D}^1_{\varphi,g}h_k)(z)|\\ &= \|\mathscr{D}^1_{\varphi,g}h_k\|_{\mathscr{B}}\to 0,\end{aligned}$$

as $k\to\infty$, from which we get

$$\lim_{|\varphi(z_k)|\to 1}(1-|z_k|^2)|g(z_k)|\Big(\ln\frac{e}{1-|\varphi(z_k)|^2}-2\Big) = 0,$$

which implies

$$\lim_{|\varphi(z_k)|\to 1}(1-|z_k|^2)|g(z_k)|\ln\frac{e}{1-|\varphi(z_k)|^2} = 0. \tag{41}$$

By (39) and (41), we get

$$\lim_{|\varphi(z_k)|\to 1}\frac{2(1-|z_k|^2)|g(z_k)|}{1-|\varphi(z_k)|^2} = \lim_{|\varphi(z_k)|\to 1}\frac{2(1-|z_k|^2)|g(z_k)||\varphi(z_k)|^2}{1-|\varphi(z_k)|^2} = 0. \tag{42}$$

(37) follows from the last equality.

$(iii)\Rightarrow(i)$. Assume that the condition (37) holds. Let $(f_k)_{k\in\mathbb{N}}$ be a sequence in $\mathscr{B}$ with $\sup_{k\in\mathbb{N}}\|f_k\|_{\mathscr{B}}\leqslant K$ and $f_k\to 0$ uniformly on compact subsets of B as $k\to\infty$. From the assumption that $\mathscr{D}^1_{\varphi,g}:\mathscr{B}\to\mathscr{B}$ is bounded and the proof of Theorem 5 we see that (27) holds. By (37) we have that for every $\varepsilon>0$, there is a $\delta\in(0,1)$, such that

$$(1-|z|^2)|g(z)|(1-|\varphi(z)|^2)^{-1} < \varepsilon \tag{43}$$

when $\delta < |\varphi(z)| < 1$. Let $\Omega = \{w \in B : |w| \leqslant \delta\}$. From (27) and (43), we have

$$\|\mathscr{D}^1_{\varphi,g} f_k\|_{\mathscr{B}} = \sup_{z\in B}(1-|z|^2)|\Re(\mathscr{D}^1_{\varphi,g} f_k)(z)| = \sup_{z\in B}(1-|z|^2)|\mathscr{D} f_k(\varphi(z))g(z)|$$
$$\leqslant \Big(\sup_{\{z\in B:\,|\varphi(z)|\leqslant\delta\}} + \sup_{\{z\in B:\,\delta<|\varphi(z)|<1\}}\Big)(1-|z|^2)|g(z)||\mathscr{D} f_k(\varphi(z))|$$
$$\leqslant L_1 \sup_{w\in\Omega}|\mathscr{D} f_k(w)| + C\|f_k\|_{\mathscr{B}} \sup_{\{z\in B:\,\delta<|\varphi(z)|<1\}} \frac{(1-|z|^2)|g(z)|}{1-|\varphi(z)|^2}$$
$$\leqslant L_1 \sup_{w\in\Omega}(|f_k(w)| + |\Re f_k(w)|) + CK\varepsilon.$$

By the Cauchy's estimate and the assumption we see that the sequences f_k and $\Re f_k$ converge to zero on compact subsets of B as $k\to\infty$. Since Ω is compact, we get that $\lim_{k\to\infty}\sup_{w\in\Omega}|f_k(w)| = 0$ and $\lim_{k\to\infty}\sup_{w\in\Omega}|\Re f_k(w)| = 0$. Using this fact and letting $k\to\infty$ in the last inequality, we obtain that

$$\limsup_{k\to\infty}\|\mathscr{D}^1_{\varphi,g} f_k\|_{\mathscr{B}} \leqslant CK\varepsilon.$$

Since ε is an arbitrary positive number it follows that $\limsup_{k\to\infty}\|\mathscr{D}^1_{\varphi,g} f_k\|_{\mathscr{B}} = 0$. It follows from Lemma 4 that the result follows. The proof of this theorem is completed. □

Let $L : X \to Y$ be a linear operator, where X and Y are Banach spaces. Recall that L is weakly compact if for every bounded sequence $(x_n)_{n\in\mathbb{N}}$ in X, $(L(x_n))_{n\in\mathbb{N}}$ has a weakly convergent subsequence (see [3]).

THEOREM 8. *Let φ be a holomorphic self-map of B, $g \in H(B)$ such that $g(0) = 0$. Then the following statements are equivalent.*

(i) The operator $\mathscr{D}^1_{\varphi,g} : \mathscr{B} \to \mathscr{B}_0$ is compact;

(ii) The operator $\mathscr{D}^1_{\varphi,g} : \mathscr{B}_0 \to \mathscr{B}_0$ is compact;

(iii) The operator $\mathscr{D}^1_{\varphi,g} : \mathscr{B}_0 \to \mathscr{B}_0$ is weakly compact;

(iv)

$$\lim_{|z|\to1}(1-|z|^2)|g(z)|(1-|\varphi(z)|^2)^{-1} = 0. \tag{44}$$

Proof. $(i) \Rightarrow (ii)$. It is obvious.

$(ii) \Rightarrow (iii)$. Since every compact operator is weakly compact, the result follows.

$(iii) \Rightarrow (iv)$. Assume that $\mathscr{D}^1_{\varphi,g} : \mathscr{B}_0 \to \mathscr{B}_0$ is weakly compact. Now we adopt the method of the proof of Theorem 2 in [8]. Since $\mathscr{D}^1_{\varphi,g} : \mathscr{B}_0 \to \mathscr{B}_0$ is bounded and $(\mathscr{B}_0)^* = A^1$ (see [23]), then we see that $(\mathscr{D}^1_{\varphi,g})^* : A^1 \to A^1$ is bounded. Hence every linear operator L on $\mathscr{B}_0$ can be identified by a function $g \in A^1$, so that for every $f \in \mathscr{B}_0$ and $g \in A^1$,

$$\langle \mathscr{D}^1_{\varphi,g}(f), g\rangle = \langle f, (\mathscr{D}^1_{\varphi,g})^*(g)\rangle. \tag{45}$$

On the other hand, since $(A^1)^* = \mathscr{B}$, we see that $(\mathscr{D}^1_{\varphi,g})^{**} : \mathscr{B} \to \mathscr{B}$ is bounded. Hence every $f \in \mathscr{B}_0$ can be viewed as an element of the space $(A^1)^*$ and

$$\langle f, (\mathscr{D}^1_{\varphi,g})^*(g)\rangle = \langle (\mathscr{D}^1_{\varphi,g})^{**}(f), g\rangle.$$

From these two formulas we get

$$\langle \mathscr{D}^1_{\varphi,g}(f), g\rangle = \langle (\mathscr{D}^1_{\varphi,g})^{**}(f), g\rangle,$$

for every $g \in A^1$. By a well known consequence of Hann-Banach theorem we obtain $(\mathscr{D}^1_{\varphi,g})^{**}(f) = \mathscr{D}^1_{\varphi,g}(f)$ for every $f \in \mathscr{B}_0$. Since $\mathscr{B}_0$ is w^* dense in $\mathscr{B}$, it follows that $(\mathscr{D}^1_{\varphi,g})^{**}(f) = \mathscr{D}^1_{\varphi,g}(f)$ for every $f \in \mathscr{B}$.

Let X and Y be Banach spaces. Using Gantmacher's theorem (see [3]), we have that $L : X \to Y$ is weakly compact if and only if $L^{**}(X^{**}) \subseteq Y$, where L^{**} is the second adjoint of L. Hence $\mathscr{D}^1_{\varphi,g} : \mathscr{B}_0 \to \mathscr{B}_0$ is weakly compact if and only if $(\mathscr{D}^1_{\varphi,g})^{**}((\mathscr{B}_0)^{**}) \subseteq \mathscr{B}_0$. From the fact that $(\mathscr{D}^1_{\varphi,g})^{**}(f) = \mathscr{D}^1_{\varphi,g}(f)$ for every $f \in \mathscr{B}$ and $(\mathscr{B}_0)^{**} \cong \mathscr{B}$ ([23]), we see that $\mathscr{D}^1_{\varphi,g} : \mathscr{B}_0 \to \mathscr{B}_0$ is weakly compact if and only if $\mathscr{D}^1_{\varphi,g}(\mathscr{B}) \subseteq \mathscr{B}_0$. Then the result follows from Theorem 4.

$(iv) \Rightarrow (i)$. Suppose that (44) holds. By (21) we have

$$(1-|z|^2)|\Re(\mathscr{D}^1_{\varphi,g}f)(z)| \leqslant C\|f\|_{\mathscr{B}}(1-|z|^2)|g(z)|(1-|\varphi(z)|^2)^{-1}. \tag{46}$$

Taking the supremum in (46) over the unit ball of the space $\mathscr{B}$, then letting $|z| \to 1$, we obtain

$$\lim_{|z|\to 1} \sup_{\|f\|_{\mathscr{B}} \leqslant 1} (1-|z|^2)|\Re(\mathscr{D}^1_{\varphi,g}f)(z)| = 0. \tag{47}$$

From Lemma 3 and (47), we see that $\mathscr{D}^1_{\varphi,g} : \mathscr{B} \to \mathscr{B}_0$ is compact. The proof is completed. □

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