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ON THE INVARIANCE EQUATION FOR HEINZ MEANS

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Abstract. We solve the so-called invariance equation in the class of Heinz means, that is, we give necessary and sufficient conditions for the constants $0 \leqslant p,q,r \leqslant 1$ such that the identity

$$H_p(H_q(x,y),H_r(x,y)) = H_p(x,y) \quad (x,y \in \mathbb{R}^+)$$

holds true where the Heinz mean H_p is defined for $0 \leqslant p \leqslant 1$ as

$$H_p(x,y) = \frac{x^p y^{1-p} + x^{1-p} y^p}{2}.$$

The Taylor expansion of the Heinz mean is used.

1. Introduction

A continuous function $m\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is called a *mean* if

$$\min(x,y) \leqslant m(x,y) \leqslant \max(x,y) \quad (x,y \in \mathbb{R}^+). \tag{1}$$

A mean is *strict* if both inequalities in (1) are strict for $x \neq y$.

Let $M,N\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ be two means and $x,y \in \mathbb{R}^+$. Then we can define the iteration

$$\begin{aligned} x_0 &:= x, & y_0 &:= y, \\ x_{n+1} &:= M(x_n,y_n), & y_{n+1} &:= N(x_n,y_n) \end{aligned} \tag{2}$$

which is called the *Gaussian mean iteration*. Such a recurrence was first considered by J. L. Lagrange in 1785 who defined it by using the arithmetic and geometric means. However, it is named after C. F. Gauss who rediscovered the recurrence in 1791 at the age of 14 and later in 1799 uncovered its connections to elliptic integrals, see [3].

It is well-known (see [3]) that if M and N are strict means then the iteration (2) converges for every $x,y \in \mathbb{R}^+$ and its limit is a strict mean which is called the *Gauss compound* of M and N, denoted by $M \otimes N$. The characterization of $M \otimes N$ is the following.

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THEOREM 1. (Invariance principle) *Suppose that* $M \otimes N$ *exists. Then* $M \otimes N$ *is the unique mean* Φ *satisfying the so-called* invariance equation*:*

$$\Phi(M(x,y),N(x,y)) = \Phi(x,y) \quad (x,y \in \mathbb{R}^+). \tag{3}$$

If the invariance equation (3) is satisfied one also says that the mean Φ is (M,N) *invariant* or the mean N is *complementary* to M with respect to Φ, see [13, 7].

The invariance equation in general mean classes has been studied by many authors. This problem was considered first for the class of quasi-arithmetic means by O. Sutô in [14] and later by J. Matkowski [13] and it was completely solved in [8]. The invariance problem was solved for the class of weighted quasi-arithmetic means in [12], for the class of Greek means in [15] and for weighted Lehmer means in [6]. Recently, a computer aided solution was given for the class of Gini means and Stolarsky means in [4, 5]. In what follows, we consider the invariance equation in the class of Heinz means.

2. Heinz means

The *Heinz mean* for $0 \leqslant p \leqslant 1$ is defined in [2] as

$$H_p(x,y) = \frac{x^p y^{1-p} + x^{1-p} y^p}{2}.$$

Notice that $H_p = H_{1-p}$ so later we may assume $0 \leqslant p \leqslant 1/2$. The Heinz means provide an interpolation between the arithmetic and the geometric means. Indeed, $H_0(x,y) = (x+y)/2$ and $H_{1/2}(x,y) = \sqrt{xy}$, further, it is easily seen, by using the inequality of the arithmetic and geometric means for the lower estimate and its weighted version for the upper estimate, that

$$\sqrt{xy} \leqslant H_p(x,y) \leqslant \frac{x+y}{2}. \tag{4}$$

The matrix version of the inequality (4) is

$$2\|A^{1/2}XB^{1/2}\| \leqslant \|A^p XB^{1-p} + A^{1-p}XB^p\| \leqslant \|AX + XB\|. \tag{5}$$

A special case of (5) was proved by E. Heinz in 1951 (see [11]), who used it to derive several inequalities in the perturbation theory of operators, the mean is named after him. We note that the mean H_p is called *symmetric mean* in [10], see also [16] for some sharp inequalities corresponding to (4).

The next proposition summarizes some properties of the Heinz means.

PROPOSITION 2. *For fixed* $x,y \in \mathbb{R}^+$ *the function* $p \mapsto H_p(x,y)$ *is convex on* $[0,1]$*, it is strictly decreasing for* $0 \leqslant p \leqslant 1/2$ *and strictly increasing for* $1/2 \leqslant p \leqslant 1$.

Furthermore, for $|x| < 1$,

$$H_p(1,1-x) = 1 - \frac{1}{2}x + \frac{p(p-1)}{2}x^2 + \frac{p(p-1)}{4}x^3 + \frac{p(p-1)(p(p-1)+4)}{24}x^4 + \frac{p(p-1)(p(p-1)+2)}{16}x^5 + \frac{p(p-1)(p(p-1)(p(p-1)+52)+72)}{720}x^6 + o(x^6). \tag{6}$$

where the coefficients of the Taylor expansion depend only on $p(p-1)$.

Proof. By simple calculation we obtain

$$\frac{\mathrm{d}}{\mathrm{d}p}H_p(1,x) = \frac{1}{2}\log x \cdot (x^p - x^{1-p}) = \frac{1}{2}x^{1-p}\log x \cdot (x^{2p-1} - 1),$$

which is negative for $0 < p < 1/2$ and positive for $1/2 < p < 1$. In addition,

$$\frac{\mathrm{d}^2}{\mathrm{d}p^2}H_p(1,x) = \frac{1}{2}\log^2 x \cdot (x^p + x^{1-p}) > 0,$$

hence $H_p(1,x)$ is a convex function of p. Since $H_p(x,y) = xH_p(1,y/x)$, it also possesses the above properties. Additionally, for $|x| < 1$ the binomial series implies that

$$H_p(1,1-x) = \frac{(1-x)^p + (1-x)^{1-p}}{2} = \frac{1}{2}\sum_{n=0}^{5}(-1)^n\left(\binom{p}{n} + \binom{1-p}{n}\right)x^n + o(x^6)$$

which, after some simple calculation, gives (6). Clearly, the coefficients a_n in the Taylor expansion are polynomials of p and they are invariant under the change of p to $(1-p)$. Therefore, $a_n(p) = a_n(1-p)$ thus $p = 1$ and $p = 0$ are roots of the polynomial $a_n(p) - a_n(0)$ hence $a_n(p) = p(p-1)b_n(p) + a_n(0)$ where b_n is a polynomial of p (with degree less than a_n) and it is also invariant under the change of p to $(1-p)$. Whence by induction we obtain that $a_n(p)$ depends only on $p(p-1)$. $\square$

REMARK 3. A further interesting property of the Heinz means is an integral representation for the logarithmic mean, namely,

$$L(x,y) = \frac{x-y}{\log x - \log y} = \int_0^1 x^p y^{1-p}\,\mathrm{d}p = \int_0^1 H_p(x,y)\,\mathrm{d}p.$$

Thus, by integrating the inequality (4) with respect to p we obtain the well-known inequality $\sqrt{xy} \leqslant L(x,y) \leqslant (x+y)/2$.

Now let us consider the invariance equation in the class of Heinz means:

$$H_p(H_q(x,y),H_r(x,y)) = H_p(x,y) \quad (x,y \in \mathbb{R}^+). \tag{7}$$

The main result of the paper is the following.

THEOREM 4. *Let $0 \leqslant p,q,r \leqslant 1/2$. Then the invariance equation (7) is valid if and only if $p = q = r$.*

In view of Proposition 1 it follows that the Gauss compound of two Heinz means is again a Heinz mean only in the trivial case, i.e., when the two means are equal.

COROLLARY 5. *The Gauss compound of the Heinz means H_q and H_r is again a Heinz mean if and only if $H_q = H_r$.*

By taking $p = 0$ (or $p = 1$) in the invariance equation (7) we obtain as a consequence of Theorem 4 the solution of the so-called Matkowski-Sutô equation in the class of Heinz means.

COROLLARY 6. *Let $0 \leqslant q,r \leqslant 1/2$. Then the Matkowski-Sutô equation*

$$H_q(x,y) + H_r(x,y) = x + y \quad (x,y \in \mathbb{R}^+)$$

is satisfied if and only if $q = r = 0$.

REMARK 7. The paper [1] introduces the term *unsymmetric Heinz mean* for the function $(x,y) \mapsto x^p y^{1-p}$ $(0 \leqslant p \leqslant 1)$ which is nothing but a weighted geometric mean. In the class of unsymmetric Heinz means the invariance equation (3) is readily solved. Indeed, it is equivalent to

$$\left(x^q y^{1-q}\right)^p \left(x^r y^{1-r}\right)^{1-p} = x^p y^{1-p}$$

which yields that $p = r/(r+1-q)$.

3. Proof of the main result

Proof. It is obvious that for $p = q = r$ the invariance equation holds.

To prove the other part we use the Taylor expansion of the Heinz means up to order 6. It is convenient to normalize both sides of (7), i.e., we take $x = 1$, $y = 1-x$ and consider the equation

$$H_p(H_q(1,1-x),H_r(1,1-x)) = H_p(1,1-x) \quad (x \in \mathbb{R}^+). \tag{8}$$

On the other hand, to make the formalism as simple as possible we introduce a notation. For a number $0 \leqslant v \leqslant 1$ we denote $\hat{v} = v(v-1)$. By Proposition 2 it follows that the coefficients of the Taylor expansion of the left-hand side of (8) will depend on $\hat{p},\hat{q},\hat{r}$ and the right-hand side depends on $\hat{p}$ which will make the calculation less complicate.

The Taylor expansion of the right-hand side of (8) is given by (6). In order to obtain the series expansion of the left-hand side of (8) we need the expansion of $H_q(1,1-x)^p$ that can be calculated by applying the following lemma (see [9]).

LEMMA 8. *Suppose that $f(x)=\sum_{n=0}^{\infty}a_nx^n$ and $f(x)^{\alpha}=\sum_{n=0}^{\infty}b_nx^n$ where $\alpha\in\mathbb{R}$ and f,f^{α} are analytic functions near $x=0$ (actually, also formal power series can be considered). Then for $n\geqslant 0$,*

$$\sum_{k=0}^{n}(k(\alpha+1)-n)a_kb_{n-k}=0.$$

Now, Lemma 8 with the expansion of $H_q(1,1-x)$ and with the fact that $H_q(1,1)=1$ implies that

$$\begin{aligned}
&H_q(1,1-x)^p\\
&=1-\frac{1}{2}px+\left(\frac{1}{8}\hat{p}+\frac{1}{2}p\hat{q}\right)x^2+\left(\frac{1}{4}p\hat{q}-\frac{1}{4}\hat{p}\hat{q}-\frac{1}{48}\hat{p}(p-2)\right)x^3\\
&\quad+\left(\frac{1}{24}p\hat{q}(\hat{q}+4)-\frac{1}{8}\hat{p}\hat{q}+\frac{1}{8}\hat{p}\hat{q}^2+\frac{1}{16}\hat{p}(p-2)\hat{q}+\frac{1}{384}\hat{p}(p-2)(p-3)\right)x^4\\
&\quad+\left(\frac{1}{32}\hat{p}(p-2)\hat{q}-\frac{1}{16}\hat{p}(p-2)\hat{q}^2+\frac{1}{8}\hat{p}\hat{q}^2-\frac{1}{48}\hat{p}\hat{q}(\hat{q}+4)+\frac{1}{16}p\hat{q}(\hat{q}+2)\right.\\
&\qquad\left.-\frac{1}{96}\hat{p}(p-2)(p-3)\hat{q}-\frac{1}{3840}\hat{p}(p-2)(p-3)(p-4)\right)x^5\\
&\quad+\left(\frac{1}{720}p\hat{q}(\hat{q}(\hat{q}+52)+72)+\frac{1}{192}\hat{p}(p-2)\hat{q}(\hat{q}+4)-\frac{1}{16}\hat{p}(p-2)\hat{q}^2\right.\\
&\qquad+\frac{1}{48}\hat{p}(p-2)\hat{q}^3-\frac{1}{32}\hat{p}\hat{q}(\hat{q}+2)+\frac{1}{48}\hat{p}\hat{q}^2(\hat{q}+4)+\frac{1}{32}\hat{p}\hat{q}^2\\
&\qquad-\frac{1}{192}\hat{p}(p-2)(p-3)\hat{q}+\frac{1}{64}\hat{p}(p-2)(p-3)\hat{q}^2\\
&\qquad+\frac{1}{768}\hat{p}(p-2)(p-3)(p-4)\hat{q}\\
&\qquad\left.+\frac{1}{46080}\hat{p}(p-2)(p-3)(p-4)(p-5)\right)x^6+o(x^6).
\end{aligned}\tag{9}$$

By replacing p by $(1-p)$ and $\hat{q}$ by $\hat{r}$ (and keeping $\hat{p}$ unchanged) from (9) we obtain the Taylor expansion of $H_r(1,1-x)^{1-p}$. We can then take the Cauchy product of $H_q(1,1-x)^p$ and $H_r(1,1-x)^{1-p}$ which yields

$$\begin{aligned}
&H_q(1,1-x)^pH_r(1,1-x)^{1-p}\\
&=1-\frac{1}{2}x+\frac{1}{2}(p\hat{q}+(1-p)\hat{r})x^2+\frac{1}{4}(p\hat{q}+(1-p)\hat{r})x^3\\
&\quad+\frac{1}{24}\left(p\hat{q}(\hat{q}+4)+(1-p)\hat{r}(\hat{r}+4)+3\hat{p}(\hat{q}-\hat{r})^2\right)x^4\\
&\quad+\frac{1}{16}\left(p\hat{q}(\hat{q}+2)+(1-p)\hat{r}(\hat{r}+2)+3\hat{p}(\hat{q}-\hat{r})^2\right)x^5\\
&\quad+\frac{1}{720}\big(p\hat{q}(\hat{q}(\hat{q}+52)+72)+(1-p)\hat{r}(\hat{r}(\hat{r}+52)+72)\\
&\qquad-15\hat{p}(\hat{q}-\hat{r})^2(p\hat{q}+(1-p)\hat{r}-10)\big)x^6+o(x^6).
\end{aligned}\tag{10}$$

Finally, replacing p by $(1-p)$ in (10) gives the Taylor expansion of the product $H_q(1,1-x)^{1-p}H_r(1,1-x)^p$ which together with (10) implies that the Taylor expansion of the left-hand side of (8) is

$$\begin{aligned}
&H_p(H_q(1,1-x),H_r(1,1-x))\\
&=1-\frac{1}{2}x+\frac{1}{4}(\hat{q}+\hat{r})x^2+\frac{1}{8}(\hat{q}+\hat{r})x^3\\
&\quad+\frac{1}{48}\left(\hat{q}(\hat{q}+4)+\hat{r}(\hat{r}+4)+6\hat{p}(\hat{q}-\hat{r})^2\right)x^4\\
&\quad+\frac{1}{32}\left(\hat{q}(\hat{q}+2)+\hat{r}(\hat{r}+2)+6\hat{p}(\hat{q}-\hat{r})^2\right)x^5\\
&\quad+\frac{1}{1440}\big(\hat{q}(\hat{q}(\hat{q}+52)+72)+\hat{r}(\hat{r}(\hat{r}+52)+72)\\
&\qquad\qquad-15\hat{p}(\hat{q}-\hat{r})^2(\hat{q}+\hat{r}-20)\big)x^6+o(x^6).
\end{aligned}$$

Now, comparing the coefficients of the Taylor expansion of both sides of (8) it follows that

$$\hat{p}=\frac{1}{2}(\hat{q}+\hat{r}), \tag{11}$$

$$\hat{p}(\hat{p}+4)=\frac{1}{2}\left(\hat{q}(\hat{q}+4)+\hat{r}(\hat{r}+4)+6\hat{p}(\hat{q}-\hat{r})^2\right), \tag{12}$$

$$\begin{aligned}\hat{p}(\hat{p}(\hat{p}+52)+72)=\frac{1}{2}\big(&\hat{q}(\hat{q}(\hat{q}+52)+72)+\hat{r}(\hat{r}(\hat{r}+52)+72)\\&-15\hat{p}(\hat{q}-\hat{r})^2(\hat{q}+\hat{r}-20)\big)\end{aligned} \tag{13}$$

By substituting equation (11) into (12) we obtain that

$$(1+6\hat{q}+6\hat{r})(\hat{q}-\hat{r})^2=0.$$

If $\hat{q}=\hat{r}$ then (11) yields that $\hat{p}=\hat{q}=\hat{r}$. Otherwise $2\hat{p}=\hat{q}+\hat{r}=-1/6$ thus after some simplification equation (13) reduces to $(\hat{q}+1/12)^2=0$ hence $\hat{q}=-1/12$ so $\hat{p}=\hat{q}=\hat{r}$ again. Therefore, $\hat{p}=\hat{q}=\hat{r}$ is necessary in order to the invariance equation (8) be valid. Since the function $x\mapsto x(x-1)$ is injective on $[0,1/2]$ thus it follows that $p=q=r$. $\square$

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