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CHARACTERIZATIONS OF OPERATOR ORDER FOR k STRICTLY POSITIVE OPERATORS

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Abstract. Let A_i $(i=1,2,\cdots,k)$ be bounded linear operators on a Hilbert space. This paper aims to show a characterization of operator order $A_k \geqslant A_{k-1} \geqslant \cdots \geqslant A_2 \geqslant A_1 > 0$ in terms of operator inequalities. Afterwards, an application of the characterization is given to operator equalities due to Douglas's majorization and factorization theorem.

1. Introduction

A capital letter (such as T) means a bounded linear operator on a Hilbert space $\mathscr{H}$. T is said to be positive (denoted by $T \geqslant 0$) if $(Tx,x) \geqslant 0$ for all $x \in \mathscr{H}$, and T is said to be strictly positive (denoted by $T > 0$) if T is positive and invertible. The usual order $S \geqslant T$ among selfadjoint operators on $\mathscr{H}$ is defined by $(Sx,x) \geqslant (Tx,x)$ for all $x \in \mathscr{H}$. Let I denote the identity operator.

As an historical and beautiful extension of the famous Löwner-Heinz inequality: $A \geqslant B \geqslant 0 \Rightarrow A^{\alpha} \geqslant B^{\alpha}$ if $\alpha \in [0,1]$, T. Furuta proved the following operator inequality in 1987.

THEOREM 1.1. (Furuta inequality, [9]) *If $A \geqslant B \geqslant 0$, then for each $r \geqslant 0$,*

$$(A^{\frac{r}{2}}A^pA^{\frac{r}{2}})^{\frac{1}{q}} \geqslant (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1}{q}}, \tag{1.1}$$

$$(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{1}{q}} \geqslant (B^{\frac{r}{2}}B^pB^{\frac{r}{2}})^{\frac{1}{q}} \tag{1.2}$$

hold for $p \geqslant 0$ and $q \geqslant 1$ with $(1+r)q \geqslant p+r$.

K. Tanahashi showed that the conditions for p and q in Figure 1 are best possible for each $r \geqslant 0$. See [19]. It is well-known that Furuta inequality has many applications. See [2, 3, 5, 10, 13, 14, 24, 25, 26].

In 1995, T. Furuta showed the following theorem which interpolates Furuta inequality and Ando-Hiai inequality for log majorization([1]).

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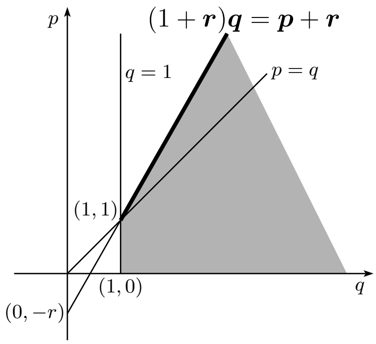

Figure 1: *Domain of Furuta inequality*

THEOREM 1.2. (The grand Furuta inequality, [11]) *If $A \geqslant B \geqslant 0$ with $A > 0$, then for each $t \in [0,1]$ and $p \geqslant 1$,*

$$A^{1-t+r} \geqslant \{A^{\frac{r}{2}}(A^{-\frac{t}{2}}B^{p}A^{-\frac{t}{2}})^{s}A^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}} \tag{1.3}$$

holds for $s \geqslant 1$ and $r \geqslant t$.

K. Tanahashi proved that the exponent value $\frac{1-t+r}{(p-t)s+r}$ of the grand Furuta inequality is the best possible in [20]. Afterwards, the proof was improved by T. Yamazaki and M. Fujii et al., respectively. See [22] and [7].

In 2003, the grand Furuta inequality was extended by M. Uchiyama in [21] as follows:

THEOREM 1.3. (Extended grand Furuta inequality, [21]) *If $A \geqslant B \geqslant C \geqslant 0$ with $B > 0$, then for each $t \in [0,1]$ and $p \geqslant 1$,*

$$A^{1-t+r} \geqslant \{A^{\frac{r}{2}}(B^{-\frac{t}{2}}C^{p}B^{-\frac{t}{2}})^{s}A^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}} \tag{1.4}$$

holds for $s \geqslant 1$ and $r \geqslant t$.

In 2008, the grand Furuta inequality was given another extension in [12] as follows:

THEOREM 1.4. (Extension of the grand Furuta inequality, [12]) *If $A \geqslant B \geqslant 0$ with $A > 0$, $t \in [0,1]$ and $p_1, p_2, \dots, p_{2n} \geqslant 1$ for any natural number n, then the following inequality*

$$\begin{aligned} A^{1-t+r} \geqslant \{A^{\frac{r}{2}}\big[A^{-\frac{t}{2}}\cdots\big[A^{-\frac{t}{2}}\{A^{\frac{t}{2}}(A^{-\frac{t}{2}}B^{p_1}A^{-\frac{t}{2}})^{p_2} \\ A^{\frac{t}{2}}\}^{p_3}A^{-\frac{t}{2}}\big]^{p_4}\cdots A^{-\frac{t}{2}}\big]^{p_{2n}}A^{\frac{r}{2}}\}^{\frac{1-t+r}{\phi(2n)-t+r}} \end{aligned} \tag{1.5}$$

holds for $r \geqslant t$, where $\phi(2n) = \{\cdots[\{[(p_1-t)p_2+t]p_3-t\}p_4+t]p_5-\cdots-t\}p_{2n}+t$.

We mention that some results related to the extension of the grand Furuta inequality are in [15], [16] and etc.

In 2010, C. Yang and Y. Wang [23] showed the following theorem which generalizes the extended grand Furuta inequality.

THEOREM 1.5. (Further extension of the grand Furuta inequality, [23]) *If* $A_{2n+1} \geqslant A_{2n} \geqslant A_{2n-1} \geqslant \cdots \geqslant A_3 \geqslant A_2 \geqslant A_1 \geqslant 0$ *with* $A_2 > 0$, $t_1, t_2, \ldots, t_{n-1}, t_n \in [0,1]$ *and* $p_1, p_2, \ldots, p_{2n-1}, p_{2n} \geqslant 1$ *for a natural number* n, *then the following inequality*

$$A_{2n+1}^{1-t_n+r} \geqslant \{A_{2n+1}^{\frac{r}{2}}[A_{2n}^{-\frac{t_n}{2}}\{A_{2n-1}^{\frac{t_{n-1}}{2}}\cdots A_5^{\frac{t_2}{2}}[A_4^{-\frac{t_2}{2}}\{A_3^{\frac{t_1}{2}}(A_2^{-\frac{t_1}{2}}A_1^{p_1}A_2^{-\frac{t_1}{2}})^{p_2} A_3^{\frac{t_1}{2}}\}^{p_3}A_4^{-\frac{t_2}{2}}]^{p_4}A_5^{\frac{t_2}{2}}\cdots A_{2n-1}^{\frac{t_{n-1}}{2}}\}^{p_{2n-1}}A_{2n}^{-\frac{t_n}{2}}]^{p_{2n}}A_{2n+1}^{\frac{r}{2}}\}^{\frac{1-t_n+r}{\psi[2n]-t_n+r}} \tag{1.6}$$

holds for $r \geqslant t_n$, *where* $\psi[2n] = \{\cdots[\{[(p_1-t_1)p_2+t_1]p_3-t_2\}p_4+t_2]p_5-\cdots-t_n\}p_{2n}+t_n$.

Recently, some beautiful results on characterizations of operator order have been shown, such as [8], [17] and [18]. C.-S. Lin, by using Furuta inequality, showed characterizations of operator order for two strictly positive operators in [17]. Afterwards, he and Y. J. Cho, by using the extended grand Furuta inequality, showed characterizations of operator order for three strictly positive operators in [18]. The aim of the present paper is to show a characterization of operator order $A_k \geqslant A_{k-1} \geqslant \cdots \geqslant A_2 \geqslant A_1 > 0$ for any positive integer k in terms of operator inequality via the further extension of the grand Furuta inequality. An application of the characterization is given to operator equalities due to Douglas's majorization and factorization theorem.

2. Main results and proofs

In this section, we show a characterization of operator order for k strictly positive operators. First, we assume that k is an odd integer ($k = 2n+1$).

THEOREM 2.1. *If* $A_1, A_2, A_3, \cdots, A_{2n-1}, A_{2n}, A_{2n+1}$ *are strictly positive operators, then the following two assertions are equivalent.*

(I) $A_{2n+1} \geqslant A_{2n} \geqslant A_{2n-1} \geqslant \cdots \geqslant A_3 \geqslant A_2 \geqslant A_1$.

(II) *If* $t_1, t_2, \cdots, t_n \in [0,1]$, $p_1, p_2, \cdots, p_{2n-1}, p_{2n} \geqslant 1$, $\psi[2n] = \{\cdots[\{[(p_1-t_1)p_2+t_1]p_3-t_2\}p_4+t_2]p_5-\cdots-t_n\}p_{2n}+t_n$, *then the following inequalities always hold for* $r \geqslant t_n$:

(II.1) $A_{2n+1}^{r-t_n} \geqslant \left\{A_{2n+1}^{\frac{r}{2}}\left[A_{2n}^{-\frac{t_n}{2}}\{A_{2n-1}^{\frac{t_{n-1}}{2}}\cdots A_5^{\frac{t_2}{2}}\left[A_4^{-\frac{t_2}{2}}\cdot\{A_3^{\frac{t_1}{2}}(A_2^{-\frac{t_1}{2}}A_1^{p_1}A_2^{-\frac{t_1}{2}})^{p_2}A_3^{\frac{t_1}{2}}\}^{p_3}\cdot A_4^{-\frac{t_2}{2}}\right]^{p_4}A_5^{\frac{t_2}{2}}\cdots A_{2n-1}^{\frac{t_{n-1}}{2}}\}^{p_{2n-1}}A_{2n}^{-\frac{t_n}{2}}\right]^{p_{2n}}A_{2n+1}^{\frac{r}{2}}\right\}^{\frac{r-t_n}{\psi[2n]-t_n+r}}$;

(II.2) $A_{2n+1}^{r-t_n} \geqslant \left\{A_{2n+1}^{\frac{r}{2}}\left[A_{2n+1}^{-\frac{t_n}{2}}\{A_{2n}^{\frac{t_{n-1}}{2}}\cdots A_6^{\frac{t_2}{2}}\left[A_5^{-\frac{t_2}{2}}\cdot\{A_4^{\frac{t_1}{2}}(A_3^{-\frac{t_1}{2}}A_2^{p_1}A_3^{-\frac{t_1}{2}})^{p_2}A_4^{\frac{t_1}{2}}\}^{p_3}\cdot A_5^{-\frac{t_2}{2}}\right]^{p_4}A_6^{\frac{t_2}{2}}\cdots A_{2n}^{\frac{t_{n-1}}{2}}\}^{p_{2n-1}}A_{2n+1}^{-\frac{t_n}{2}}\right]^{p_{2n}}A_{2n+1}^{\frac{r}{2}}\right\}^{\frac{r-t_n}{\psi[2n]-t_n+r}}$;

(II.3) $A_{2n+1}^{r-t_n} \geqslant \left\{A_{2n+1}^{\frac{r}{2}}\left[A_{2n+1}^{-\frac{t_n}{2}}\{A_{2n+1}^{\frac{t_{n-1}}{2}}\cdots A_7^{\frac{t_2}{2}}\left[A_6^{-\frac{t_2}{2}}\cdot\{A_5^{\frac{t_1}{2}}(A_4^{-\frac{t_1}{2}}A_3^{p_1}A_4^{-\frac{t_1}{2}})^{p_2}A_5^{\frac{t_1}{2}}\}^{p_3}\cdot A_6^{-\frac{t_2}{2}}\right]^{p_4}A_7^{\frac{t_2}{2}}\cdots A_{2n+1}^{\frac{t_{n-1}}{2}}\}^{p_{2n-1}}A_{2n+1}^{-\frac{t_n}{2}}\right]^{p_{2n}}A_{2n+1}^{\frac{r}{2}}\right\}^{\frac{r-t_n}{\psi[2n]-t_n+r}}$;

....

(II.n) $A_{2n+1}^{r-t_n} \geqslant \left\{A_{2n+1}^{\frac{r}{2}}\left[A_{2n+1}^{-\frac{t_n}{2}}\{A_{2n+1}^{\frac{t_{n-1}}{2}}\cdots A_{n+4}^{\frac{t_2}{2}}\left[A_{n+3}^{-\frac{t_2}{2}}\{A_{n+2}^{\frac{t_1}{2}}(A_{n+1}^{-\frac{t_1}{2}}A_n^{p_1}A_{n+1}^{-\frac{t_1}{2}})^{p_2}A_{n+2}^{\frac{t_1}{2}}\}^{p_3}\right.\right.\right.$

$$A_{n+3}^{-\frac{t_2}{2}}\Big]^{p_4} A_{n+4}^{\frac{t_2}{2}} \cdots A_{2n+1}^{\frac{t_{n-1}}{2}}\Big\}^{p_{2n-1}} A_{2n+1}^{-\frac{t_n}{2}}\Big]^{p_{2n}} A_{2n+1}^{\frac{r}{2}}\Big\}^{\frac{r-t_n}{\psi[2n]-t_n+r}};$$

(II.n+1) $$A_1^{r-t_n} \leqslant \Big\{A_1^{\frac{r}{2}}\Big[A_1^{-\frac{t_n}{2}}\Big\{A_1^{\frac{t_{n-1}}{2}} \cdots A_{n-2}^{\frac{t_2}{2}}\Big[A_{n-1}^{-\frac{t_2}{2}}\{A_n^{\frac{t_1}{2}}(A_{n+1}^{-\frac{t_1}{2}}A_{n+2}^{p_1}A_{n+1}^{-\frac{t_1}{2}})^{p_2}A_n^{\frac{t_1}{2}}\}^{p_3} A_{n-1}^{-\frac{t_2}{2}}\Big]^{p_4} A_{n-2}^{\frac{t_2}{2}} \cdots A_1^{\frac{t_{n-1}}{2}}\Big\}^{p_{2n-1}} A_1^{-\frac{t_n}{2}}\Big]^{p_{2n}} A_1^{\frac{r}{2}}\Big\}^{\frac{r-t_n}{\psi[2n]-t_n+r}};$$

....

(II.2n-2) $$A_1^{r-t_n} \leqslant \Big\{A_1^{\frac{r}{2}}\Big[A_1^{-\frac{t_n}{2}}\Big\{A_1^{\frac{t_{n-1}}{2}} \cdots A_{2n-5}^{\frac{t_2}{2}}\Big[A_{2n-4}^{-\frac{t_2}{2}}\{A_{2n-3}^{\frac{t_1}{2}} \cdot (A_{2n-2}^{-\frac{t_1}{2}}A_{2n-1}^{p_1}A_{2n-2}^{-\frac{t_1}{2}})^{p_2} \cdot A_{2n-3}^{\frac{t_1}{2}}\}^{p_3} A_{2n-4}^{-\frac{t_2}{2}}\Big]^{p_4} A_{2n-5}^{\frac{t_2}{2}} \cdots A_1^{\frac{t_{n-1}}{2}}\Big\}^{p_{2n-1}} A_1^{-\frac{t_n}{2}}\Big]^{p_{2n}} A_1^{\frac{r}{2}}\Big\}^{\frac{r-t_n}{\psi[2n]-t_n+r}};$$

(II.2n-1) $$A_1^{r-t_n} \leqslant \Big\{A_1^{\frac{r}{2}}\Big[A_1^{-\frac{t_n}{2}}\Big\{A_2^{\frac{t_{n-1}}{2}} \cdots A_{2n-4}^{\frac{t_2}{2}}\Big[A_{2n-3}^{-\frac{t_2}{2}}\{A_{2n-2}^{\frac{t_1}{2}} \cdot (A_{2n-1}^{-\frac{t_1}{2}}A_{2n}^{p_1}A_{2n-1}^{-\frac{t_1}{2}})^{p_2} \cdot A_{2n-2}^{\frac{t_1}{2}}\}^{p_3} A_{2n-3}^{-\frac{t_2}{2}}\Big]^{p_4} A_{2n-4}^{\frac{t_2}{2}} \cdots A_2^{\frac{t_{n-1}}{2}}\Big\}^{p_{2n-1}} A_1^{-\frac{t_n}{2}}\Big]^{p_{2n}} A_1^{\frac{r}{2}}\Big\}^{\frac{r-t_n}{\psi[2n]-t_n+r}};$$

(II.2n) $$A_1^{r-t_n} \leqslant \Big\{A_1^{\frac{r}{2}}\Big[A_2^{-\frac{t_n}{2}}\Big\{A_3^{\frac{t_{n-1}}{2}} \cdots A_{2n-3}^{\frac{t_2}{2}}\Big[A_{2n-2}^{-\frac{t_2}{2}}\{A_{2n-1}^{\frac{t_1}{2}} \cdot (A_{2n}^{-\frac{t_1}{2}}A_{2n+1}^{p_1}A_{2n}^{-\frac{t_1}{2}})^{p_2} \cdot A_{2n-1}^{\frac{t_1}{2}}\}^{p_3} A_{2n-2}^{-\frac{t_2}{2}}\Big]^{p_4} A_{2n-3}^{\frac{t_2}{2}} \cdots A_3^{\frac{t_{n-1}}{2}}\Big\}^{p_{2n-1}} A_2^{-\frac{t_n}{2}}\Big]^{p_{2n}} A_1^{\frac{r}{2}}\Big\}^{\frac{r-t_n}{\psi[2n]-t_n+r}}.$$

Proof. (I)$\Rightarrow$(II) Applying Löwner-Heinz inequality for $\frac{r-t_n}{1-t_n+r}$ to the further extension of the grand Furuta inequality, we obtain (II.1). Replacing $A_1, A_2, A_3, \cdots, A_{2n-1}$, A_{2n} by $A_2, A_3, A_4, \cdots, A_{2n}, A_{2n+1}$ in (II.1), respectively, we obtain (II.2). Replacing A_2, A_3, $A_4, \cdots, A_{2n-1}, A_{2n}$ by $A_3, A_4, A_5, \cdots, A_{2n}, A_{2n+1}$ in (II.2), respectively, we obtain (II.3). Similarly, we can obtain (II.4), (II.5), $\cdots$, (II.n).

If we replace $A_1, A_2, A_3 \cdots, A_{2n-1}, A_{2n}, A_{2n+1}$ by $A_{2n+1}^{-1}, A_{2n}^{-1}, A_{2n-1}^{-1}, \cdots, A_3^{-1}$, A_2^{-1}, A_1^{-1} in each of (II.1), (II.2), (II.3), $\cdots$, (II.n), respectively, and take inverses, then (II.2n), (II.2n-1), (II.2n-2), $\cdots$, (II.n+1) hold.

(II)$\Rightarrow$(I) Because each A_i is strictly positive and bounded, there exist u_i and v_i such that $+\infty > u_i I \geqslant A_i \geqslant v_i I > 0$ $(i = 1, 2, \cdots, 2n+1)$. If we take $p_1 = p_3 = p_4 = \cdots = p_{2n} = 1$, $t_1 = t_2 = \cdots = t_n = 1$, $r = 2$ in (II.1), then we have

$$\begin{aligned} &A_{2n+1} \\ &\geqslant \{A_{2n+1}A_{2n}^{-\frac{1}{2}}A_{2n-1}^{\frac{1}{2}} \cdots A_5^{\frac{1}{2}}A_4^{-\frac{1}{2}}A_3^{\frac{1}{2}}(A_2^{-\frac{1}{2}}A_1A_2^{-\frac{1}{2}})^{p_2}A_3^{\frac{1}{2}}A_4^{-\frac{1}{2}}A_5^{\frac{1}{2}} \cdots A_{2n-1}^{\frac{1}{2}}A_{2n}^{-\frac{1}{2}}A_{2n+1}\}^{\frac{1}{2}}. \end{aligned} \tag{2.1}$$

According to Theorem 6' in [6]: $X \geqslant Y > 0$ with $sI \geqslant X \geqslant tI > 0 \Rightarrow \frac{(s+t)^2}{4st}X^2 \geqslant Y^2$, we can obtain the following inequality by (2.1) and $u_{2n+1}I \geqslant A_{2n+1} \geqslant v_{2n+1}I > 0$.

$$\begin{aligned} &\frac{(u_{2n+1}+v_{2n+1})^2}{4u_{2n+1}v_{2n+1}}A_{2n+1}^2 \\ &\geqslant A_{2n+1}A_{2n}^{-\frac{1}{2}}A_{2n-1}^{\frac{1}{2}} \cdots A_5^{\frac{1}{2}}A_4^{-\frac{1}{2}}A_3^{\frac{1}{2}}(A_2^{-\frac{1}{2}}A_1A_2^{-\frac{1}{2}})^{p_2}A_3^{\frac{1}{2}}A_4^{-\frac{1}{2}}A_5^{\frac{1}{2}} \cdots A_{2n-1}^{\frac{1}{2}}A_{2n}^{-\frac{1}{2}}A_{2n+1}. \end{aligned} \tag{2.2}$$

Then we have

$$\begin{aligned}&\frac{(u_{2n+1}+v_{2n+1})^2}{4u_{2n+1}v_{2n+1}}\cdot\frac{u_{2n}u_{2n-2}\cdots u_6u_4}{v_{2n-1}v_{2n-3}\cdots v_5v_3}I\\ \geqslant\ &\frac{(u_{2n+1}+v_{2n+1})^2}{4u_{2n+1}v_{2n+1}}A_3^{-\frac{1}{2}}A_4^{\frac{1}{2}}A_5^{-\frac{1}{2}}\cdots A_{2n-1}^{-\frac{1}{2}}A_{2n}A_{2n-1}^{-\frac{1}{2}}\cdots A_5^{-\frac{1}{2}}A_4^{\frac{1}{2}}A_3^{-\frac{1}{2}}\\ \geqslant\ &(A_2^{-\frac{1}{2}}A_1A_2^{-\frac{1}{2}})^{p_2}.\end{aligned}\tag{2.3}$$

Thus,

$$\Big[\frac{(u_{2n+1}+v_{2n+1})^2}{4u_{2n+1}v_{2n+1}}\cdot\frac{u_{2n}u_{2n-2}\cdots u_6u_4}{v_{2n-1}v_{2n-3}\cdots v_5v_3}\Big]^{\frac{1}{p_2}}I\geqslant A_2^{-\frac{1}{2}}A_1A_2^{-\frac{1}{2}}\tag{2.4}$$

holds for any $p_2\geqslant 1$. $A_2\geqslant A_1$ is obtained by taking $p_2\to+\infty$.

Similarly, we can obtain $A_3\geqslant A_2, A_4\geqslant A_3,\cdots,A_{n+1}\geqslant A_n$ by (II.2), (II.3), $\cdots$, (II.n), respectively.

By the same setting for (2.2n), the following inequality holds according to Theorem 6 in [6]: $X\geqslant Y>0$ with $s'I\geqslant Y\geqslant t'I>0\Rightarrow\frac{(s'+t')^2}{4s't'}X^2\geqslant Y^2$.

$$\begin{aligned}&\frac{(u_1+v_1)^2}{4u_1v_1}A_1A_2^{-\frac{1}{2}}A_3^{\frac{1}{2}}\cdots A_{2n-2}^{-\frac{1}{2}}A_{2n-1}^{\frac{1}{2}}(A_{2n}^{-\frac{1}{2}}A_{2n+1}A_{2n}^{-\frac{1}{2}})^{p_2}A_{2n-1}^{\frac{1}{2}}A_{2n-2}^{-\frac{1}{2}}\cdots A_3^{\frac{1}{2}}A_2^{-\frac{1}{2}}A_1\\ \geqslant\ &A_1^2.\end{aligned}\tag{2.5}$$

Then we have

$$\begin{aligned}&A_{2n}^{-\frac{1}{2}}A_{2n+1}A_{2n}^{-\frac{1}{2}}\\ \geqslant\ &\Big[\frac{4u_1v_1}{(u_1+v_1)^2}A_{2n-1}^{-\frac{1}{2}}A_{2n-2}^{\frac{1}{2}}\cdots A_3^{-\frac{1}{2}}A_2A_3^{-\frac{1}{2}}\cdots A_{2n-2}^{\frac{1}{2}}A_{2n-1}^{-\frac{1}{2}}\Big]^{\frac{1}{p_2}}\\ \geqslant\ &\Big[\frac{4u_1v_1}{(u_1+v_1)^2}\cdot\frac{v_2v_4\cdots v_{2n-2}}{u_3u_5\cdots u_{2n-1}}\Big]^{\frac{1}{p_2}}I.\end{aligned}\tag{2.6}$$

$A_{2n+1}\geqslant A_{2n}$ is obtained by taking $p_2\to+\infty$ in (2.6).

Similarly, we can obtain $A_{2n}\geqslant A_{2n-1}, A_{2n-1}\geqslant A_{2n-2},\cdots,A_{n+2}\geqslant A_{n+1}$ by (II.2n-1), (II.2n-2), $\cdots$, (II.n+1), respectively. □

REMARK 2.1. If $n=1$, Theorem 2.1 is the main result of [18].

Next, we assume that k is an even integer ($k=2n$).

THEOREM 2.2. *If $A_1,A_2,A_3,\cdots,A_{2n-1},A_{2n}$ are strictly positive operators, then the following two assertions are equivalent:*

(I) $A_{2n}\geqslant A_{2n-1}\geqslant\cdots\geqslant A_3\geqslant A_2\geqslant A_1$.

(II) *If* $t_1,t_2,\cdots,t_n\in[0,1]$, $p_1,p_2,\cdots,p_{2n-1},p_{2n}\geqslant 1$, $\psi[2n]=\{\cdots[\{[(p_1-t_1)p_2+t_1]p_3-t_2\}p_4+t_2]p_5-\cdots-t_n\}p_{2n}+t_n$, *then the following inequalities always hold for* $r\geqslant t_n$:

(II.1) $A_{2n}^{r-t_n}\geqslant\Big\{A_{2n}^{\frac{r}{2}}\Big[A_{2n}^{-\frac{t_n}{2}}\{A_{2n-1}^{\frac{t_{n-1}}{2}}\cdots A_5^{\frac{t_2}{2}}\,[A_4^{-\frac{t_2}{2}}\cdot\{A_3^{\frac{t_1}{2}}(A_2^{-\frac{t_1}{2}}A_1^{p_1}A_2^{-\frac{t_1}{2}})^{p_2}A_3^{\frac{t_1}{2}}\}^{p_3}\cdot$

$A_4^{-\frac{t_2}{2}}]^{p_4}A_5^{\frac{t_2}{2}}\cdots A_{2n-1}^{\frac{t_{n-1}}{2}}\}^{p_{2n-1}}A_{2n}^{-\frac{t_n}{2}}\Big]^{p_{2n}}A_{2n}^{\frac{r}{2}}\Big\}^{\frac{r-t_n}{\psi[2n]-t_n+r}}$;

(II.2) $A_{2n}^{r-t_n} \geqslant \Big\{A_{2n}^{\frac{r}{2}}\Big[A_{2n}^{-\frac{t_n}{2}}\{A_{2n}^{\frac{t_{n-1}}{2}}\cdots A_6^{\frac{t_2}{2}}\big[A_5^{-\frac{t_2}{2}}\cdot\{A_4^{\frac{t_1}{2}}(A_3^{-\frac{t_1}{2}}A_2^{p_1}A_3^{-\frac{t_1}{2}})^{p_2}A_4^{\frac{t_1}{2}}\}^{p_3}\cdot$
$A_5^{-\frac{t_2}{2}}]^{p_4}A_6^{\frac{t_2}{2}}\cdots A_{2n}^{\frac{t_{n-1}}{2}}\}^{p_{2n-1}}A_{2n}^{-\frac{t_n}{2}}\Big]^{p_{2n}}A_{2n}^{\frac{r}{2}}\Big\}^{\frac{r-t_n}{\psi[2n]-t_n+r}}$;

(II.3) $A_{2n}^{r-t_n} \geqslant \Big\{A_{2n}^{\frac{r}{2}}\Big[A_{2n}^{-\frac{t_n}{2}}\{A_{2n}^{\frac{t_{n-1}}{2}}\cdots A_7^{\frac{t_2}{2}}\big[A_6^{-\frac{t_2}{2}}\cdot\{A_5^{\frac{t_1}{2}}(A_4^{-\frac{t_1}{2}}A_3^{p_1}A_4^{-\frac{t_1}{2}})^{p_2}A_5^{\frac{t_1}{2}}\}^{p_3}\cdot$
$A_6^{-\frac{t_2}{2}}]^{p_4}A_7^{\frac{t_2}{2}}\cdots A_{2n}^{\frac{t_{n-1}}{2}}\}^{p_{2n-1}}A_{2n}^{-\frac{t_n}{2}}\Big]^{p_{2n}}A_{2n}^{\frac{r}{2}}\Big\}^{\frac{r-t_n}{\psi[2n]-t_n+r}}$;

....

(II.n) $A_{2n}^{r-t_n} \geqslant \Big\{A_{2n}^{\frac{r}{2}}\Big[A_{2n}^{-\frac{t_n}{2}}\{A_{2n}^{\frac{t_{n-1}}{2}}\cdots A_{n+4}^{\frac{t_2}{2}}\big[A_{n+3}^{-\frac{t_2}{2}}\{A_{n+2}^{\frac{t_1}{2}}(A_{n+1}^{-\frac{t_1}{2}}A_n^{p_1}A_{n+1}^{-\frac{t_1}{2}})^{p_2}A_{n+2}^{\frac{t_1}{2}}\}^{p_3}$
$A_{n+3}^{-\frac{t_2}{2}}]^{p_4}A_{n+4}^{\frac{t_2}{2}}\cdots A_{2n}^{\frac{t_{n-1}}{2}}\}^{p_{2n-1}}A_{2n}^{-\frac{t_n}{2}}\Big]^{p_{2n}}A_{2n}^{\frac{r}{2}}\Big\}^{\frac{r-t_n}{\psi[2n]-t_n+r}}$;

(II.n+1) $A_1^{r-t_n} \leqslant \Big\{A_1^{\frac{r}{2}}\Big[A_1^{-\frac{t_n}{2}}\{A_1^{\frac{t_{n-1}}{2}}\cdots A_{n-2}^{\frac{t_2}{2}}\big[A_{n-1}^{-\frac{t_2}{2}}\{A_n^{\frac{t_1}{2}}(A_{n+1}^{-\frac{t_1}{2}}A_{n+2}^{p_1}A_{n+1}^{-\frac{t_1}{2}})^{p_2}A_n^{\frac{t_1}{2}}\}^{p_3}$
$A_{n-1}^{-\frac{t_2}{2}}]^{p_4}A_{n-2}^{\frac{t_2}{2}}\cdots A_1^{\frac{t_{n-1}}{2}}\}^{p_{2n-1}}A_1^{-\frac{t_n}{2}}\Big]^{p_{2n}}A_1^{\frac{r}{2}}\Big\}^{\frac{r-t_n}{\psi[2n]-t_n+r}}$;

....

(II.2n-2) $A_1^{r-t_n} \leqslant \Big\{A_1^{\frac{r}{2}}\Big[A_1^{-\frac{t_n}{2}}\{A_1^{\frac{t_{n-1}}{2}}\cdots A_{2n-5}^{\frac{t_2}{2}}\big[A_{2n-4}^{-\frac{t_2}{2}}\{A_{2n-3}^{\frac{t_1}{2}}\cdot(A_{2n-2}^{-\frac{t_1}{2}}A_{2n-1}^{p_1}A_{2n-2}^{-\frac{t_1}{2}})^{p_2}\cdot$
$A_{2n-3}^{\frac{t_1}{2}}\}^{p_3}A_{2n-4}^{-\frac{t_2}{2}}]^{p_4}A_{2n-5}^{\frac{t_2}{2}}\cdots A_1^{\frac{t_{n-1}}{2}}\}^{p_{2n-1}}A_1^{-\frac{t_n}{2}}\Big]^{p_{2n}}A_1^{\frac{r}{2}}\Big\}^{\frac{r-t_n}{\psi[2n]-t_n+r}}$;

(II.2n-1) $A_1^{r-t_n} \leqslant \Big\{A_1^{\frac{r}{2}}\Big[A_1^{-\frac{t_n}{2}}\{A_2^{\frac{t_{n-1}}{2}}\cdots A_{2n-4}^{\frac{t_2}{2}}\big[A_{2n-3}^{-\frac{t_2}{2}}\{A_{2n-2}^{\frac{t_1}{2}}\cdot(A_{2n-1}^{-\frac{t_1}{2}}A_{2n}^{p_1}A_{2n-1}^{-\frac{t_1}{2}})^{p_2}\cdot$
$A_{2n-2}^{\frac{t_1}{2}}\}^{p_3}A_{2n-3}^{-\frac{t_2}{2}}]^{p_4}A_{2n-4}^{\frac{t_2}{2}}\cdots A_2^{\frac{t_{n-1}}{2}}\}^{p_{2n-1}}A_1^{-\frac{t_n}{2}}\Big]^{p_{2n}}A_1^{\frac{r}{2}}\Big\}^{\frac{r-t_n}{\psi[2n]-t_n+r}}$.

Proof. Replace A_{2n+1} by A_{2n} in Theorem 2.1. □

Combining Theorem 2.1 with Theorem 2.2, we have shown a characterization of operator order $A_k \geqslant A_{k-1} \geqslant \cdots \geqslant A_2 \geqslant A_1 > 0$ for any positive integer k. For example, if $k=5$, we have the following result:

PROPOSITION 2.1. *If A_1, A_2, A_3, A_4 and A_5 are strictly positive operators, then $A_5 \geqslant A_4 \geqslant A_3 \geqslant A_2 \geqslant A_1$ if and only if the following four operator inequalities*

$$A_5^{r-t_2} \geqslant \Big\{A_5^{\frac{r}{2}}\Big[A_4^{-\frac{t_2}{2}}\big(A_3^{\frac{t_1}{2}}(A_2^{-\frac{t_1}{2}}A_1^{p_1}A_2^{-\frac{t_1}{2}})^{p_2}A_3^{\frac{t_1}{2}}\big)^{p_3}A_4^{-\frac{t_2}{2}}\Big]^{p_4}A_5^{\frac{r}{2}}\Big\}^{\frac{r-t_2}{\psi[4]-t_2+r}}, \tag{2.7}$$

$$A_5^{r-t_2} \geqslant \Big\{A_5^{\frac{r}{2}}\Big[A_5^{-\frac{t_2}{2}}\big(A_4^{\frac{t_1}{2}}(A_3^{-\frac{t_1}{2}}A_2^{p_1}A_3^{-\frac{t_1}{2}})^{p_2}A_4^{\frac{t_1}{2}}\big)^{p_3}A_5^{-\frac{t_2}{2}}\Big]^{p_4}A_5^{\frac{r}{2}}\Big\}^{\frac{r-t_2}{\psi[4]-t_2+r}}, \tag{2.8}$$

$$A_1^{r-t_2} \leqslant \Big\{A_1^{\frac{r}{2}}\Big[A_1^{-\frac{t_2}{2}}\big(A_2^{\frac{t_1}{2}}(A_3^{-\frac{t_1}{2}}A_4^{p_1}A_3^{-\frac{t_1}{2}})^{p_2}A_2^{\frac{t_1}{2}}\big)^{p_3}A_1^{-\frac{t_2}{2}}\Big]^{p_4}A_1^{\frac{r}{2}}\Big\}^{\frac{r-t_2}{\psi[4]-t_2+r}}, \tag{2.9}$$

$$A_1^{r-t_2} \leqslant \Big\{A_1^{\frac{r}{2}}\Big[A_2^{-\frac{t_2}{2}}\big(A_3^{\frac{t_1}{2}}(A_4^{-\frac{t_1}{2}}A_5^{p_1}A_4^{-\frac{t_1}{2}})^{p_2}A_3^{\frac{t_1}{2}}\big)^{p_3}A_2^{-\frac{t_2}{2}}\Big]^{p_4}A_1^{\frac{r}{2}}\Big\}^{\frac{r-t_2}{\psi[4]-t_2+r}} \tag{2.10}$$

hold for $p_1, p_2, p_3, p_4 \geqslant 1$, $t_1, t_2 \in [0,1]$ and $r \geqslant t_2$, where $\psi[4] = \{[(p_1-t_1)p_2 + t_1]p_3 - t_2\}p_4 + t_2$.

REMARK 2.2. It should be mentioned that we can not obtain $A_5 \geqslant A_4 \geqslant A_3 \geqslant A_2 \geqslant A_1$ only by (2.7) and (2.10). If $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{u} \end{pmatrix}$, $A_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $A_3 = \begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix}$, $A_4 = \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}$, $A_5 = \begin{pmatrix} u+\varepsilon & 0 \\ 0 & 1 \end{pmatrix}$, where $u > 1$ and $\varepsilon > 0$, then the five strictly positive operators satisfy (2.7) and (2.10), but do not satisfy $A_4 \geqslant A_3$.

3. An application

In what follows we give an application of the characterization in Theorem 2.1 and Theorem 2.2 to operator equalities.

THEOREM 3.1. *If $A_1, A_2, A_3, \cdots, A_{2n-1}, A_{2n}, A_{2n+1}$ are strictly positive operators, $t_1, t_2, \cdots, t_n \in [0,1]$, $p_1, p_2, \cdots, p_{2n} \geqslant 1$, $\psi[2n] = \{\cdots[\{[(p_1-t_1)p_2+t_1]p_3-t_2\}p_4+t_2]p_5-\cdots-t_n\}p_{2n}+t_n$, $r \geqslant t_n$, m is a positive integer such that $(r-t_n)m = \psi[2n]-t_n+r$ with $m \geqslant 2$, then the following assertions are mutually equivalent:*

(I) $A_{2n+1} \geqslant A_{2n} \geqslant A_{2n-1} \geqslant \cdots \geqslant A_3 \geqslant A_2 \geqslant A_1$.

(II) *The following operator inequalities hold:*

(II.1) $A_{2n+1}^{r-t_n} \geqslant \Big\{A_{2n+1}^{\frac{r}{2}}\Big[A_{2n}^{-\frac{t_n}{2}}\{A_{2n-1}^{\frac{t_{n-1}}{2}}\cdots A_5^{\frac{t_2}{2}}[A_4^{-\frac{t_2}{2}}\cdot\{A_3^{\frac{t_1}{2}}(A_2^{-\frac{t_1}{2}}A_1^{p_1}A_2^{-\frac{t_1}{2}})^{p_2}A_3^{\frac{t_1}{2}}\}^{p_3}\cdot A_4^{-\frac{t_2}{2}}]^{p_4}A_5^{\frac{t_2}{2}}\cdots A_{2n-1}^{\frac{t_{n-1}}{2}}\}^{p_{2n-1}}A_{2n}^{-\frac{t_n}{2}}\Big]^{p_{2n}}A_{2n+1}^{\frac{r}{2}}\Big\}^{\frac{1}{m}}$;

(II.2) $A_{2n+1}^{r-t_n} \geqslant \Big\{A_{2n+1}^{\frac{r}{2}}\Big[A_{2n+1}^{-\frac{t_n}{2}}\{A_{2n}^{\frac{t_{n-1}}{2}}\cdots A_6^{\frac{t_2}{2}}[A_5^{-\frac{t_2}{2}}\cdot\{A_4^{\frac{t_1}{2}}(A_3^{-\frac{t_1}{2}}A_2^{p_1}A_3^{-\frac{t_1}{2}})^{p_2}A_4^{\frac{t_1}{2}}\}^{p_3}\cdot A_5^{-\frac{t_2}{2}}]^{p_4}A_6^{\frac{t_2}{2}}\cdots A_{2n}^{\frac{t_{n-1}}{2}}\}^{p_{2n-1}}A_{2n+1}^{-\frac{t_n}{2}}\Big]^{p_{2n}}A_{2n+1}^{\frac{r}{2}}\Big\}^{\frac{1}{m}}$;

(II.3) $A_{2n+1}^{r-t_n} \geqslant \Big\{A_{2n+1}^{\frac{r}{2}}\Big[A_{2n+1}^{-\frac{t_n}{2}}\{A_{2n+1}^{\frac{t_{n-1}}{2}}\cdots A_7^{\frac{t_2}{2}}[A_6^{-\frac{t_2}{2}}\cdot\{A_5^{\frac{t_1}{2}}(A_4^{-\frac{t_1}{2}}A_3^{p_1}A_4^{-\frac{t_1}{2}})^{p_2}A_5^{\frac{t_1}{2}}\}^{p_3}\cdot A_6^{-\frac{t_2}{2}}]^{p_4}A_7^{\frac{t_2}{2}}\cdots A_{2n+1}^{\frac{t_{n-1}}{2}}\}^{p_{2n-1}}A_{2n+1}^{-\frac{t_n}{2}}\Big]^{p_{2n}}A_{2n+1}^{\frac{r}{2}}\Big\}^{\frac{1}{m}}$;

....

(II.n) $A_{2n+1}^{r-t_n} \geqslant \Big\{A_{2n+1}^{\frac{r}{2}}\Big[A_{2n+1}^{-\frac{t_n}{2}}\{A_{2n+1}^{\frac{t_{n-1}}{2}}\cdots A_{n+4}^{\frac{t_2}{2}}[A_{n+3}^{-\frac{t_2}{2}}\{A_{n+2}^{\frac{t_1}{2}}(A_{n+1}^{-\frac{t_1}{2}}A_n^{p_1}A_{n+1}^{-\frac{t_1}{2}})^{p_2}A_{n+2}^{\frac{t_1}{2}}\}^{p_3} A_{n+3}^{-\frac{t_2}{2}}]^{p_4}A_{n+4}^{\frac{t_2}{2}}\cdots A_{2n+1}^{\frac{t_{n-1}}{2}}\}^{p_{2n-1}}A_{2n+1}^{-\frac{t_n}{2}}\Big]^{p_{2n}}A_{2n+1}^{\frac{r}{2}}\Big\}^{\frac{1}{m}}$;

(II.n+1) $A_1^{r-t_n} \leqslant \Big\{A_1^{\frac{r}{2}}\Big[A_1^{-\frac{t_n}{2}}\{A_1^{\frac{t_{n-1}}{2}}\cdots A_{n-2}^{\frac{t_2}{2}}[A_{n-1}^{-\frac{t_2}{2}}\{A_n^{\frac{t_1}{2}}(A_{n+1}^{-\frac{t_1}{2}}A_{n+2}^{p_1}A_{n+1}^{-\frac{t_1}{2}})^{p_2}A_n^{\frac{t_1}{2}}\}^{p_3} A_{n-1}^{-\frac{t_2}{2}}]^{p_4}A_{n-2}^{\frac{t_2}{2}}\cdots A_1^{\frac{t_{n-1}}{2}}\}^{p_{2n-1}}A_1^{-\frac{t_n}{2}}\Big]^{p_{2n}}A_1^{\frac{r}{2}}\Big\}^{\frac{1}{m}}$;

....

(II.2n-2) $A_1^{r-t_n} \leqslant \Big\{A_1^{\frac{r}{2}}\Big[A_1^{-\frac{t_n}{2}}\{A_1^{\frac{t_{n-1}}{2}}\cdots A_{2n-5}^{\frac{t_2}{2}}[A_{2n-4}^{-\frac{t_2}{2}}\{A_{2n-3}^{\frac{t_1}{2}}\cdot(A_{2n-2}^{-\frac{t_1}{2}}A_{2n-1}^{p_1}A_{2n-2}^{-\frac{t_1}{2}})^{p_2}\cdot A_{2n-3}^{\frac{t_1}{2}}\}^{p_3}A_{2n-4}^{-\frac{t_2}{2}}]^{p_4}A_{2n-5}^{\frac{t_2}{2}}\cdots A_1^{\frac{t_{n-1}}{2}}\}^{p_{2n-1}}A_1^{-\frac{t_n}{2}}\Big]^{p_{2n}}A_1^{\frac{r}{2}}\Big\}^{\frac{1}{m}}$;

(II.2n-1) $A_1^{r-t_n} \leqslant \Big\{A_1^{\frac{r}{2}}\Big[A_1^{-\frac{t_n}{2}}\{A_2^{\frac{t_{n-1}}{2}}\cdots A_{2n-4}^{\frac{t_2}{2}}[A_{2n-3}^{-\frac{t_2}{2}}\{A_{2n-2}^{\frac{t_1}{2}}\cdot(A_{2n-1}^{-\frac{t_1}{2}}A_{2n}^{p_1}A_{2n-1}^{-\frac{t_1}{2}})^{p_2}\cdot A_{2n-2}^{\frac{t_1}{2}}\}^{p_3}A_{2n-3}^{-\frac{t_2}{2}}]^{p_4}A_{2n-4}^{\frac{t_2}{2}}\cdots A_2^{\frac{t_{n-1}}{2}}\}^{p_{2n-1}}A_1^{-\frac{t_n}{2}}\Big]^{p_{2n}}A_1^{\frac{r}{2}}\Big\}^{\frac{1}{m}}$;

(II.2n) $A_1^{r-t_n} \leqslant \Big\{A_1^{\frac{r}{2}}\Big[A_2^{-\frac{t_n}{2}}\{A_3^{\frac{t_{n-1}}{2}}\cdots A_{2n-3}^{\frac{t_2}{2}}[A_{2n-2}^{-\frac{t_2}{2}}\{A_{2n-1}^{\frac{t_1}{2}}\cdot(A_{2n}^{-\frac{t_1}{2}}A_{2n+1}^{p_1}A_{2n}^{-\frac{t_1}{2}})^{p_2}\cdot A_{2n-1}^{\frac{t_1}{2}}\}^{p_3}A_{2n-2}^{-\frac{t_2}{2}}]^{p_4}A_{2n-3}^{\frac{t_2}{2}}\cdots A_3^{\frac{t_{n-1}}{2}}\}^{p_{2n-1}}A_2^{-\frac{t_n}{2}}\Big]^{p_{2n}}A_1^{\frac{r}{2}}\Big\}^{\frac{1}{m}}$.

(III) *There exist strictly positive operators* $S_1,S_2,S_3,\cdots,S_{2n-2},S_{2n-1},S_{2n}$ *satisfying the following operator equalities, respectively, where each* S_i $(i=1,2,\cdots,2n)$ *is unique with* $\|S_i\|\leqslant 1$.

(III.1) $A_{2n+1}^{-\frac{t_n}{2}}S_1(A_{2n+1}^{r-t_n}S_1)^{m-1}A_{2n+1}^{-\frac{t_n}{2}} = A_{2n+1}^{-\frac{t_n}{2}}(S_1A_{2n+1}^{r-t_n})^{m-1}S_1A_{2n+1}^{-\frac{t_n}{2}} = \Big[A_{2n}^{-\frac{t_n}{2}}\{A_{2n-1}^{\frac{t_{n-1}}{2}}\cdots A_5^{\frac{t_2}{2}}[A_4^{-\frac{t_2}{2}}\{A_3^{\frac{t_1}{2}}(A_2^{-\frac{t_1}{2}}A_1^{p_1}A_2^{-\frac{t_1}{2}})^{p_2}A_3^{\frac{t_1}{2}}\}^{p_3}A_4^{-\frac{t_2}{2}}]^{p_4}A_5^{\frac{t_2}{2}}\cdots A_{2n-1}^{\frac{t_{n-1}}{2}}\}^{p_{2n-1}}A_{2n}^{-\frac{t_n}{2}}\Big]^{p_{2n}}$;

(III.2) $A_{2n+1}^{-\frac{t_n}{2}}S_2(A_{2n+1}^{r-t_n}S_2)^{m-1}A_{2n+1}^{-\frac{t_n}{2}} = A_{2n+1}^{-\frac{t_n}{2}}(S_2A_{2n+1}^{r-t_n})^{m-1}S_2A_{2n+1}^{-\frac{t_n}{2}} = \Big[A_{2n+1}^{-\frac{t_n}{2}}\{A_{2n}^{\frac{t_{n-1}}{2}}\cdots A_6^{\frac{t_2}{2}}[A_5^{-\frac{t_2}{2}}\{A_4^{\frac{t_1}{2}}(A_3^{-\frac{t_1}{2}}A_2^{p_1}A_3^{-\frac{t_1}{2}})^{p_2}A_4^{\frac{t_1}{2}}\}^{p_3}A_5^{-\frac{t_2}{2}}]^{p_4}A_6^{\frac{t_2}{2}}\cdots A_{2n}^{\frac{t_{n-1}}{2}}\}^{p_{2n-1}}A_{2n+1}^{-\frac{t_n}{2}}\Big]^{p_{2n}}$;

(III.3) $A_{2n+1}^{-\frac{t_n}{2}}S_3(A_{2n+1}^{r-t_n}S_3)^{m-1}A_{2n+1}^{-\frac{t_n}{2}} = A_{2n+1}^{-\frac{t_n}{2}}(S_3A_{2n+1}^{r-t_n})^{m-1}S_3A_{2n+1}^{-\frac{t_n}{2}} = \Big[A_{2n+1}^{-\frac{t_n}{2}}\{A_{2n+1}^{\frac{t_{n-1}}{2}}\cdots A_7^{\frac{t_2}{2}}[A_6^{-\frac{t_2}{2}}\{A_5^{\frac{t_1}{2}}(A_4^{-\frac{t_1}{2}}A_3^{p_1}A_4^{-\frac{t_1}{2}})^{p_2}A_5^{\frac{t_1}{2}}\}^{p_3}A_6^{-\frac{t_2}{2}}]^{p_4}A_7^{\frac{t_2}{2}}\cdots A_{2n+1}^{\frac{t_{n-1}}{2}}\}^{p_{2n-1}}A_{2n+1}^{-\frac{t_n}{2}}\Big]^{p_{2n}}$;

....

(III.n) $A_{2n+1}^{-\frac{t_n}{2}}S_n(A_{2n+1}^{r-t_n}S_n)^{m-1}A_{2n+1}^{-\frac{t_n}{2}} = A_{2n+1}^{-\frac{t_n}{2}}(S_nA_{2n+1}^{r-t_n})^{m-1}S_nA_{2n+1}^{-\frac{t_n}{2}} = \Big[A_{2n+1}^{-\frac{t_n}{2}}\{A_{2n+1}^{\frac{t_{n-1}}{2}}\cdots A_{n+4}^{\frac{t_2}{2}}[A_{n+3}^{-\frac{t_2}{2}}\{A_{n+2}^{\frac{t_1}{2}}(A_{n+1}^{-\frac{t_1}{2}}A_n^{p_1}A_{n+1}^{-\frac{t_1}{2}})^{p_2}A_{n+2}^{\frac{t_1}{2}}\}^{p_3}A_{n+3}^{-\frac{t_2}{2}}]^{p_4}A_{n+4}^{\frac{t_2}{2}}\cdots A_{2n+1}^{\frac{t_{n-1}}{2}}\}^{p_{2n-1}}A_{2n+1}^{-\frac{t_n}{2}}\Big]^{p_{2n}}$;

(III.n+1) $A_1^{-\frac{t_n}{2}}S_{n+1}^{-1}(A_1^{r-t_n}S_{n+1}^{-1})^{m-1}A_1^{-\frac{t_n}{2}} = A_1^{-\frac{t_n}{2}}(S_{n+1}^{-1}A_1^{r-t_n})^{m-1}S_{n+1}^{-1}A_1^{-\frac{t_n}{2}} = \Big[A_1^{-\frac{t_n}{2}}\{A_1^{\frac{t_{n-1}}{2}}\cdots A_{n-2}^{\frac{t_2}{2}}[A_{n-1}^{-\frac{t_2}{2}}\{A_n^{\frac{t_1}{2}}(A_{n+1}^{-\frac{t_1}{2}}A_{n+2}^{p_1}A_{n+1}^{-\frac{t_1}{2}})^{p_2}A_n^{\frac{t_1}{2}}\}^{p_3}A_{n-1}^{-\frac{t_2}{2}}]^{p_4}A_{n-2}^{\frac{t_2}{2}}\cdots A_1^{\frac{t_{n-1}}{2}}\}^{p_{2n-1}}A_1^{-\frac{t_n}{2}}\Big]^{p_{2n}}$;

....

(III.2n-2) $A_1^{-\frac{t_n}{2}}S_{2n-2}^{-1}(A_1^{r-t_n}S_{2n-2}^{-1})^{m-1}A_1^{-\frac{t_n}{2}} = A_1^{-\frac{t_n}{2}}(S_{2n-2}^{-1}A_1^{r-t_n})^{m-1}S_{2n-2}^{-1}A_1^{-\frac{t_n}{2}} = \Big[A_1^{-\frac{t_n}{2}}\{A_1^{\frac{t_{n-1}}{2}}\cdots[A_{2n-4}^{-\frac{t_2}{2}}\{A_{2n-3}^{\frac{t_1}{2}}(A_{2n-2}^{-\frac{t_1}{2}}A_{2n-1}^{p_1}A_{2n-2}^{-\frac{t_1}{2}})^{p_2}A_{2n-3}^{\frac{t_1}{2}}\}^{p_3}A_{2n-4}^{-\frac{t_2}{2}}]^{p_4}\cdots A_1^{\frac{t_{n-1}}{2}}\}^{p_{2n-1}}A_1^{-\frac{t_n}{2}}\Big]^{p_{2n}}$;

(III.2n-1) $A_1^{-\frac{t_n}{2}}S_{2n-1}^{-1}(A_1^{r-t_n}S_{2n-1}^{-1})^{m-1}A_1^{-\frac{t_n}{2}} = A_1^{-\frac{t_n}{2}}(S_{2n-1}^{-1}A_1^{r-t_n})^{m-1}S_{2n-1}^{-1}A_1^{-\frac{t_n}{2}} = \Big[A_1^{-\frac{t_n}{2}}\{A_2^{\frac{t_{n-1}}{2}}\cdots[A_{2n-3}^{-\frac{t_2}{2}}\{A_{2n-2}^{\frac{t_1}{2}}(A_{2n-1}^{-\frac{t_1}{2}}A_{2n}^{p_1}A_{2n-1}^{-\frac{t_1}{2}})^{p_2}A_{2n-2}^{\frac{t_1}{2}}\}^{p_3}A_{2n-3}^{-\frac{t_2}{2}}]^{p_4}\cdots A_2^{\frac{t_{n-1}}{2}}\}^{p_{2n-1}}A_1^{-\frac{t_n}{2}}\Big]^{p_{2n}}$;

(III.2n) $A_1^{-\frac{t_n}{2}}S_{2n}^{-1}(A_1^{r-t_n}S_{2n}^{-1})^{m-1}A_1^{-\frac{t_n}{2}} = A_1^{-\frac{t_n}{2}}(S_{2n}^{-1}A_1^{r-t_n})^{m-1}S_{2n}^{-1}A_1^{-\frac{t_n}{2}} =$

$$\Big[A_2^{-\frac{t_n}{2}}\{A_3^{\frac{t_{n-1}}{2}}\cdots[A_{2n-2}^{-\frac{t_2}{2}}\{A_{2n-1}^{\frac{t_1}{2}}(A_{2n}^{-\frac{t_1}{2}}A_{2n+1}^{p_1}A_{2n}^{-\frac{t_1}{2}})^{p_2}A_{2n-1}^{\frac{t_1}{2}}\}^{p_3}A_{2n-2}^{-\frac{t_2}{2}}]^{p_4}$$
$$\cdots A_3^{\frac{t_{n-1}}{2}}\}^{p_{2n-1}}A_2^{-\frac{t_n}{2}}\Big]^{p_{2n}}.$$

Proof. Because (I)$\Leftrightarrow$(II) holds obviously by Theorem 2.1, we only need to prove that (II)$\Leftrightarrow$(III).

Firstly, let us prove that (II.1)$\Rightarrow$(III.1). We recall Douglas's majorization and factorization theorem in [4]: $SS^* \leqslant \lambda^2 TT^* \Leftrightarrow$ there exists an operator Q such that $TQ = S$, where $\|Q\|^2 = \inf\{\mu : SS^* \leqslant \mu TT^*\}$.
By (II.1), there exists an operator E_1 with $\|E_1\| \leqslant 1$ such that

$$\begin{aligned}&A_{2n+1}^{\frac{r-t_n}{2}}E_1 = E_1^*A_{2n+1}^{\frac{r-t_n}{2}}\\ &= \Big\{A_{2n+1}^{\frac{r}{2}}\Big[A_{2n}^{-\frac{t_n}{2}}\{A_{2n-1}^{\frac{t_{n-1}}{2}}\cdots\{A_3^{\frac{t_1}{2}}(A_2^{-\frac{t_1}{2}}A_1^{p_1}A_2^{-\frac{t_1}{2}})^{p_2}A_3^{\frac{t_1}{2}}\}^{p_3}\cdots A_{2n-1}^{\frac{t_{n-1}}{2}}\}^{p_{2n-1}}A_{2n}^{-\frac{t_n}{2}}\Big]^{p_{2n}}A_{2n+1}^{\frac{r}{2}}\Big\}^{\frac{1}{2m}}.\end{aligned} \tag{3.1}$$

Taking $S_1 = E_1E_1^*$, we have

$$\begin{aligned}&A_{2n+1}^{\frac{r-t_n}{2}}S_1A_{2n+1}^{\frac{r-t_n}{2}}\\ &= \Big\{A_{2n+1}^{\frac{r}{2}}\Big[A_{2n}^{-\frac{t_n}{2}}\{A_{2n-1}^{\frac{t_{n-1}}{2}}\cdots\{A_3^{\frac{t_1}{2}}(A_2^{-\frac{t_1}{2}}A_1^{p_1}A_2^{-\frac{t_1}{2}})^{p_2}A_3^{\frac{t_1}{2}}\}^{p_3}\cdots A_{2n-1}^{\frac{t_{n-1}}{2}}\}^{p_{2n-1}}A_{2n}^{-\frac{t_n}{2}}\Big]^{p_{2n}}A_{2n+1}^{\frac{r}{2}}\Big\}^{\frac{1}{m}}.\end{aligned} \tag{3.2}$$

According to (3.2) and $S_1 = E_1E_1^*$, S_1 is unique and strictly positive with $\|S_1\| \leqslant 1$. (3.2) also implies that

$$\begin{aligned}&(A_{2n+1}^{\frac{r-t_n}{2}}S_1A_{2n+1}^{\frac{r-t_n}{2}})^m = A_{2n+1}^{\frac{r-t_n}{2}}S_1(A_{2n+1}^{r-t_n}S_1)^{m-1}A_{2n+1}^{\frac{r-t_n}{2}} = A_{2n+1}^{\frac{r-t_n}{2}}(S_1A_{2n+1}^{r-t_n})^{m-1}S_1A_{2n+1}^{\frac{r-t_n}{2}}\\ &= A_{2n+1}^{\frac{r}{2}}\Big[A_{2n}^{-\frac{t_n}{2}}\{A_{2n-1}^{\frac{t_{n-1}}{2}}\cdots\{A_3^{\frac{t_1}{2}}(A_2^{-\frac{t_1}{2}}A_1^{p_1}A_2^{-\frac{t_1}{2}})^{p_2}A_3^{\frac{t_1}{2}}\}^{p_3}\cdots A_{2n-1}^{\frac{t_{n-1}}{2}}\}^{p_{2n-1}}A_{2n}^{-\frac{t_n}{2}}\Big]^{p_{2n}}A_{2n+1}^{\frac{r}{2}}.\end{aligned} \tag{3.3}$$

Then (III.1) holds by (3.3).

Secondly, we prove that (III.1)$\Rightarrow$ (II.1). By (III.1),

$$\begin{aligned}&\Big\{A_{2n+1}^{\frac{r}{2}}\Big[A_{2n}^{-\frac{t_n}{2}}\{A_{2n-1}^{\frac{t_{n-1}}{2}}\cdots\{A_3^{\frac{t_1}{2}}(A_2^{-\frac{t_1}{2}}A_1^{p_1}A_2^{-\frac{t_1}{2}})^{p_2}A_3^{\frac{t_1}{2}}\}^{p_3}\cdots A_{2n-1}^{\frac{t_{n-1}}{2}}\}^{p_{2n-1}}A_{2n}^{-\frac{t_n}{2}}\Big]^{p_{2n}}A_{2n+1}^{\frac{r}{2}}\Big\}^{\frac{1}{m}}\\ &= \{A_{2n+1}^{\frac{r-t_n}{2}}S_1(A_{2n+1}^{r-t_n}S_1)^{m-1}A_{2n+1}^{\frac{r-t_n}{2}}\}^{\frac{1}{m}}\\ &= \{(A_{2n+1}^{\frac{r-t_n}{2}}S_1A_{2n+1}^{\frac{r-t_n}{2}})\cdot(A_{2n+1}^{\frac{r-t_n}{2}}S_1A_{2n+1}^{\frac{r-t_n}{2}})\cdots(A_{2n+1}^{\frac{r-t_n}{2}}S_1A_{2n+1}^{\frac{r-t_n}{2}})\}^{\frac{1}{m}}\\ &= A_{2n+1}^{\frac{r-t_n}{2}}S_1A_{2n+1}^{\frac{r-t_n}{2}}\\ &\leqslant A_{2n+1}^{r-t_n}.\end{aligned}$$

The inequality follows from the fact that $S_1 \leqslant \|S_1\|I \leqslant I$, and then (II.1) holds.

By using the same method mentioned above, we can prove that (II.2)$\Leftrightarrow$ (III.2), (II.3)$\Leftrightarrow$ (III.3), $\cdots$, (II.n)$\Leftrightarrow$ (III.n), respectively.

Next, we show that (II.2n)$\Leftrightarrow$(III.2n). Notice that for two strictly positive operators S and T, $S \geqslant T$ if and only if $T^{-1} \geqslant S^{-1}$. Then, (II.2n) is equivalent to

$$A_1^{-(r-t_n)} \geqslant \Big\{A_1^{-\frac{r}{2}}\Big[A_2^{-\frac{t_n}{2}}\{A_3^{\frac{t_{n-1}}{2}}\cdots A_{2n-3}^{\frac{t_2}{2}}[A_{2n-2}^{-\frac{t_2}{2}}\{A_{2n-1}^{\frac{t_1}{2}}(A_{2n}^{-\frac{t_1}{2}}A_{2n+1}^{p_1}A_{2n}^{-\frac{t_1}{2}})^{p_2} A_{2n-1}^{\frac{t_1}{2}}\}^{p_3}A_{2n-2}^{-\frac{t_2}{2}}]^{p_4}A_{2n-3}^{\frac{t_2}{2}}\cdots A_3^{\frac{t_{n-1}}{2}}\}^{p_{2n-1}}A_2^{-\frac{t_n}{2}}\Big]^{-p_{2n}}A_1^{-\frac{r}{2}}\Big\}^{\frac{1}{m}}. \tag{3.4}$$

The proof of (3.4)$\Leftrightarrow$(III.2n) is similar to the proof of (II.1)$\Leftrightarrow$(III.1), so we omit it here.

Repeat the method above, we can prove that (II.2n-1)$\Leftrightarrow$(III.2n-1), (II.2n-2)$\Leftrightarrow$ (III.2n-2), $\cdots$, (II.n+1)$\Leftrightarrow$(III.n+1). $\square$

THEOREM 3.2. *If $A_1,A_2,A_3,\cdots,A_{2n-1},A_{2n}$ are strictly positive operators, t_1,t_2, $\cdots$, $t_n \in [0,1]$, $p_1,p_2,\cdots,p_{2n} \geqslant 1$, $\psi[2n]=\{\cdots[\{[(p_1-t_1)p_2+t_1]p_3-t_2\}p_4+t_2]p_5-\cdots-t_n\}p_{2n}+t_n$, $r \geqslant t_n$, m is a positive integer such that $(r-t_n)m=\psi[2n]-t_n+r$ with $m \geqslant 2$, then the following assertions are mutually equivalent:*

(I) $A_{2n} \geqslant A_{2n-1} \geqslant \cdots \geqslant A_3 \geqslant A_2 \geqslant A_1$.

(II) *The following operator inequalities hold:*

(II.1) $A_{2n}^{r-t_n} \geqslant \Big\{A_{2n}^{\frac{r}{2}}\Big[A_{2n}^{-\frac{t_n}{2}}\{A_{2n-1}^{\frac{t_{n-1}}{2}}\cdots A_5^{\frac{t_2}{2}}[A_4^{-\frac{t_2}{2}}\cdot\{A_3^{\frac{t_1}{2}}(A_2^{-\frac{t_1}{2}}A_1^{p_1}A_2^{-\frac{t_1}{2}})^{p_2}A_3^{\frac{t_1}{2}}\}^{p_3}\cdot A_4^{-\frac{t_2}{2}}]^{p_4}A_5^{\frac{t_2}{2}}\cdots A_{2n-1}^{\frac{t_{n-1}}{2}}\}^{p_{2n-1}}A_{2n}^{-\frac{t_n}{2}}\Big]^{p_{2n}}A_{2n}^{\frac{r}{2}}\Big\}^{\frac{1}{m}}$;

(II.2) $A_{2n}^{r-t_n} \geqslant \Big\{A_{2n}^{\frac{r}{2}}\Big[A_{2n}^{-\frac{t_n}{2}}\{A_{2n}^{\frac{t_{n-1}}{2}}\cdots A_6^{\frac{t_2}{2}}[A_5^{-\frac{t_2}{2}}\cdot\{A_4^{\frac{t_1}{2}}(A_3^{-\frac{t_1}{2}}A_2^{p_1}A_3^{-\frac{t_1}{2}})^{p_2}A_4^{\frac{t_1}{2}}\}^{p_3}\cdot A_5^{-\frac{t_2}{2}}]^{p_4}A_6^{\frac{t_2}{2}}\cdots A_{2n}^{\frac{t_{n-1}}{2}}\}^{p_{2n-1}}A_{2n}^{-\frac{t_n}{2}}\Big]^{p_{2n}}A_{2n}^{\frac{r}{2}}\Big\}^{\frac{1}{m}}$;

(II.3) $A_{2n}^{r-t_n} \geqslant \Big\{A_{2n}^{\frac{r}{2}}\Big[A_{2n}^{-\frac{t_n}{2}}\{A_{2n}^{\frac{t_{n-1}}{2}}\cdots A_7^{\frac{t_2}{2}}[A_6^{-\frac{t_2}{2}}\cdot\{A_5^{\frac{t_1}{2}}(A_4^{-\frac{t_1}{2}}A_3^{p_1}A_4^{-\frac{t_1}{2}})^{p_2}A_5^{\frac{t_1}{2}}\}^{p_3}\cdot A_6^{-\frac{t_2}{2}}]^{p_4}A_7^{\frac{t_2}{2}}\cdots A_{2n}^{\frac{t_{n-1}}{2}}\}^{p_{2n-1}}A_{2n}^{-\frac{t_n}{2}}\Big]^{p_{2n}}A_{2n}^{\frac{r}{2}}\Big\}^{\frac{1}{m}}$;

....

(II.n) $A_{2n}^{r-t_n} \geqslant \Big\{A_{2n}^{\frac{r}{2}}\Big[A_{2n}^{-\frac{t_n}{2}}\{A_{2n}^{\frac{t_{n-1}}{2}}\cdots A_{n+4}^{\frac{t_2}{2}}[A_{n+3}^{-\frac{t_2}{2}}\{A_{n+2}^{\frac{t_1}{2}}(A_{n+1}^{-\frac{t_1}{2}}A_n^{p_1}A_{n+1}^{-\frac{t_1}{2}})^{p_2}A_{n+2}^{\frac{t_1}{2}}\}^{p_3} A_{n+3}^{-\frac{t_2}{2}}]^{p_4}A_{n+4}^{\frac{t_2}{2}}\cdots A_{2n}^{\frac{t_{n-1}}{2}}\}^{p_{2n-1}}A_{2n}^{-\frac{t_n}{2}}\Big]^{p_{2n}}A_{2n}^{\frac{r}{2}}\Big\}^{\frac{1}{m}}$;

(II.n+1) $A_1^{r-t_n} \leqslant \Big\{A_1^{\frac{r}{2}}\Big[A_1^{-\frac{t_n}{2}}\{A_1^{\frac{t_{n-1}}{2}}\cdots A_{n-2}^{\frac{t_2}{2}}[A_{n-1}^{-\frac{t_2}{2}}\{A_n^{\frac{t_1}{2}}(A_{n+1}^{-\frac{t_1}{2}}A_{n+2}^{p_1}A_{n+1}^{-\frac{t_1}{2}})^{p_2}A_n^{\frac{t_1}{2}}\}^{p_3} A_{n-1}^{-\frac{t_2}{2}}]^{p_4}A_{n-2}^{\frac{t_2}{2}}\cdots A_1^{\frac{t_{n-1}}{2}}\}^{p_{2n-1}}A_1^{-\frac{t_n}{2}}\Big]^{p_{2n}}A_1^{\frac{r}{2}}\Big\}^{\frac{1}{m}}$;

....

(II.2n-2) $A_1^{r-t_n} \leqslant \Big\{A_1^{\frac{r}{2}}\Big[A_1^{-\frac{t_n}{2}}\{A_1^{\frac{t_{n-1}}{2}}\cdots A_{2n-5}^{\frac{t_2}{2}}[A_{2n-4}^{-\frac{t_2}{2}}\{A_{2n-3}^{\frac{t_1}{2}}\cdot(A_{2n-2}^{-\frac{t_1}{2}}A_{2n-1}^{p_1}A_{2n-2}^{-\frac{t_1}{2}})^{p_2}\cdot A_{2n-3}^{\frac{t_1}{2}}\}^{p_3}A_{2n-4}^{-\frac{t_2}{2}}]^{p_4}A_{2n-5}^{\frac{t_2}{2}}\cdots A_1^{\frac{t_{n-1}}{2}}\}^{p_{2n-1}}A_1^{-\frac{t_n}{2}}\Big]^{p_{2n}}A_1^{\frac{r}{2}}\Big\}^{\frac{1}{m}}$;

(II.2n-1) $A_1^{r-t_n} \leqslant \Big\{A_1^{\frac{r}{2}}\Big[A_1^{-\frac{t_n}{2}}\{A_2^{\frac{t_{n-1}}{2}}\cdots A_{2n-4}^{\frac{t_2}{2}}[A_{2n-3}^{-\frac{t_2}{2}}\{A_{2n-2}^{\frac{t_1}{2}}\cdot(A_{2n-1}^{-\frac{t_1}{2}}A_{2n}^{p_1}A_{2n-1}^{-\frac{t_1}{2}})^{p_2}\cdot A_{2n-2}^{\frac{t_1}{2}}\}^{p_3}A_{2n-3}^{-\frac{t_2}{2}}]^{p_4}A_{2n-4}^{\frac{t_2}{2}}\cdots A_2^{\frac{t_{n-1}}{2}}\}^{p_{2n-1}}A_1^{-\frac{t_n}{2}}\Big]^{p_{2n}}A_1^{\frac{r}{2}}\Big\}^{\frac{1}{m}}$.

(III) *There exist strictly positive operators $S_1,S_2,S_3,\cdots,S_{2n-2},S_{2n-1}$ satisfying the following operator equalities, respectively, where each S_i $(i=1,2,\cdots,2n-1)$ is unique with $\|S_i\|\leqslant 1$.*

(III.1) $A_{2n}^{-\frac{t_n}{2}}S_1(A_{2n}^{r-t_n}S_1)^{m-1}A_{2n}^{-\frac{t_n}{2}} = A_{2n}^{-\frac{t_n}{2}}(S_1A_{2n}^{r-t_n})^{m-1}S_1A_{2n}^{-\frac{t_n}{2}} = \Big[A_{2n}^{-\frac{t_n}{2}}\{A_{2n-1}^{\frac{t_{n-1}}{2}}\cdots A_5^{\frac{t_2}{2}}[A_4^{-\frac{t_2}{2}}\{A_3^{\frac{t_1}{2}}(A_2^{-\frac{t_1}{2}}A_1^{p_1}A_2^{-\frac{t_1}{2}})^{p_2}A_3^{\frac{t_1}{2}}\}^{p_3}A_4^{-\frac{t_2}{2}}]^{p_4}A_5^{\frac{t_2}{2}}\cdots A_{2n-1}^{\frac{t_{n-1}}{2}}\}^{p_{2n-1}}A_{2n}^{-\frac{t_n}{2}}\Big]^{p_{2n}}$;

(III.2) $A_{2n}^{-\frac{t_n}{2}}S_2(A_{2n}^{r-t_n}S_2)^{m-1}A_{2n}^{-\frac{t_n}{2}} = A_{2n}^{-\frac{t_n}{2}}(S_2A_{2n}^{r-t_n})^{m-1}S_2A_{2n}^{-\frac{t_n}{2}} = \Big[A_{2n}^{-\frac{t_n}{2}}\{A_{2n}^{\frac{t_{n-1}}{2}}\cdots A_6^{\frac{t_2}{2}}[A_5^{-\frac{t_2}{2}}\{A_4^{\frac{t_1}{2}}(A_3^{-\frac{t_1}{2}}A_2^{p_1}A_3^{-\frac{t_1}{2}})^{p_2}A_4^{\frac{t_1}{2}}\}^{p_3}A_5^{-\frac{t_2}{2}}]^{p_4}A_6^{\frac{t_2}{2}}\cdots A_{2n}^{\frac{t_{n-1}}{2}}\}^{p_{2n-1}}A_{2n}^{-\frac{t_n}{2}}\Big]^{p_{2n}}$;

(III.3) $A_{2n}^{-\frac{t_n}{2}}S_3(A_{2n}^{r-t_n}S_3)^{m-1}A_{2n}^{-\frac{t_n}{2}} = A_{2n}^{-\frac{t_n}{2}}(S_3A_{2n}^{r-t_n})^{m-1}S_3A_{2n}^{-\frac{t_n}{2}} = \Big[A_{2n}^{-\frac{t_n}{2}}\{A_{2n}^{\frac{t_{n-1}}{2}}\cdots A_7^{\frac{t_2}{2}}[A_6^{-\frac{t_2}{2}}\{A_5^{\frac{t_1}{2}}(A_4^{-\frac{t_1}{2}}A_3^{p_1}A_4^{-\frac{t_1}{2}})^{p_2}A_5^{\frac{t_1}{2}}\}^{p_3}A_6^{-\frac{t_2}{2}}]^{p_4}A_7^{\frac{t_2}{2}}\cdots A_{2n}^{\frac{t_{n-1}}{2}}\}^{p_{2n-1}}A_{2n}^{-\frac{t_n}{2}}\Big]^{p_{2n}}$;

....

(III.n) $A_{2n}^{-\frac{t_n}{2}}S_n(A_{2n}^{r-t_n}S_n)^{m-1}A_{2n}^{-\frac{t_n}{2}} = A_{2n}^{-\frac{t_n}{2}}(S_nA_{2n}^{r-t_n})^{m-1}S_nA_{2n}^{-\frac{t_n}{2}} = \Big[A_{2n}^{-\frac{t_n}{2}}\{A_{2n}^{\frac{t_{n-1}}{2}}\cdots A_{n+4}^{\frac{t_2}{2}}[A_{n+3}^{-\frac{t_2}{2}}\{A_{n+2}^{\frac{t_1}{2}}(A_{n+1}^{-\frac{t_1}{2}}A_n^{p_1}A_{n+1}^{-\frac{t_1}{2}})^{p_2}A_{n+2}^{\frac{t_1}{2}}\}^{p_3}A_{n+3}^{-\frac{t_2}{2}}]^{p_4}A_{n+4}^{\frac{t_2}{2}}\cdots A_{2n}^{\frac{t_{n-1}}{2}}\}^{p_{2n-1}}A_{2n}^{-\frac{t_n}{2}}\Big]^{p_{2n}}$;

(III.n+1) $A_1^{-\frac{t_n}{2}}S_{n+1}^{-1}(A_1^{r-t_n}S_{n+1}^{-1})^{m-1}A_1^{-\frac{t_n}{2}} = A_1^{-\frac{t_n}{2}}(S_{n+1}^{-1}A_1^{r-t_n})^{m-1}S_{n+1}^{-1}A_1^{-\frac{t_n}{2}} = \Big[A_1^{-\frac{t_n}{2}}\{A_1^{\frac{t_{n-1}}{2}}\cdots A_{n-2}^{\frac{t_2}{2}}[A_{n-1}^{-\frac{t_2}{2}}\{A_n^{\frac{t_1}{2}}(A_{n+1}^{-\frac{t_1}{2}}A_{n+2}^{p_1}A_{n+1}^{-\frac{t_1}{2}})^{p_2}A_n^{\frac{t_1}{2}}\}^{p_3}A_{n-1}^{-\frac{t_2}{2}}]^{p_4}A_{n-2}^{\frac{t_2}{2}}\cdots A_1^{\frac{t_{n-1}}{2}}\}^{p_{2n-1}}A_1^{-\frac{t_n}{2}}\Big]^{p_{2n}}$;

....

(III.2n-2) $A_1^{-\frac{t_n}{2}}S_{2n-2}^{-1}(A_1^{r-t_n}S_{2n-2}^{-1})^{m-1}A_1^{-\frac{t_n}{2}} = A_1^{-\frac{t_n}{2}}(S_{2n-2}^{-1}A_1^{r-t_n})^{m-1}S_{2n-2}^{-1}A_1^{-\frac{t_n}{2}} = \Big[A_1^{-\frac{t_n}{2}}\{A_1^{\frac{t_{n-1}}{2}}\cdots[A_{2n-4}^{-\frac{t_2}{2}}\{A_{2n-3}^{\frac{t_1}{2}}(A_{2n-2}^{-\frac{t_1}{2}}A_{2n-1}^{p_1}A_{2n-2}^{-\frac{t_1}{2}})^{p_2}A_{2n-3}^{\frac{t_1}{2}}\}^{p_3}A_{2n-4}^{-\frac{t_2}{2}}]^{p_4}\cdots A_1^{\frac{t_{n-1}}{2}}\}^{p_{2n-1}}A_1^{-\frac{t_n}{2}}\Big]^{p_{2n}}$;

(III.2n-1) $A_1^{-\frac{t_n}{2}}S_{2n-1}^{-1}(A_1^{r-t_n}S_{2n-1}^{-1})^{m-1}A_1^{-\frac{t_n}{2}} = A_1^{-\frac{t_n}{2}}(S_{2n-1}^{-1}A_1^{r-t_n})^{m-1}S_{2n-1}^{-1}A_1^{-\frac{t_n}{2}} = \Big[A_1^{-\frac{t_n}{2}}\{A_2^{\frac{t_{n-1}}{2}}\cdots[A_{2n-3}^{-\frac{t_2}{2}}\{A_{2n-2}^{\frac{t_1}{2}}(A_{2n-1}^{-\frac{t_1}{2}}A_{2n}^{p_1}A_{2n-1}^{-\frac{t_1}{2}})^{p_2}A_{2n-2}^{\frac{t_1}{2}}\}^{p_3}A_{2n-3}^{-\frac{t_2}{2}}]^{p_4}\cdots A_2^{\frac{t_{n-1}}{2}}\}^{p_{2n-1}}A_1^{-\frac{t_n}{2}}\Big]^{p_{2n}}$.

Proof. Replace A_{2n+1} by A_{2n} in Theorem 3.1. □

Combining Theorem 3.1 with Theorem 3.2, we have given an application of the characterization of $A_k \geqslant A_{k-1} \geqslant \cdots \geqslant A_2 \geqslant A_1 > 0$ to operator equalities for any positive integer k.

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