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NECESSARY AND SUFFICIENT CONDITIONS FOR BOUNDEDNESS OF THE HARDY–TYPE OPERATOR FROM A WEIGHTED LEBESGUE SPACE TO A MORREY–TYPE SPACE

V. I. BURENKOV AND R. OINAROV

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Abstract. Necessary and sufficient conditions on functions u and w are established ensuring boundedness of the multi-dimensional Hardy-type operator $H_{n,\varphi}$ from a weighted Lebesgue space $L_{p,u}(\mathbb{R}^n)$ to a local Morrey-type space $LM_{q\theta,w}(\mathbb{R}^n)$ for a wide range of the numerical parameters p,q,θ.

1. Introduction

Let φ be a fixed non-negative measurable function on $(0,\infty)$ which is not equivalent to 0. In this paper we consider the multi-dimensional Hardy operator $H_{n,\varphi}$ defined for all functions $f \in L_1^{loc}(\mathbb{R}^n)$ by

$$(H_{n,\varphi}f)(x) = \varphi(|x|) \int\limits_{B_{|x|}} f(y)dy, \; x \in \mathbb{R}^n, \tag{1.1}$$

where $n \in \mathbb{N}$ and B_r is the open ball in $\mathbb{R}^n$ centered at the origin of radius $r > 0$.

Let $\Omega \subset \mathbb{R}^n$ be a measurable set, v be a non-negative measurable function defined on Ω, and $0 < p \leqslant \infty$. By $L_{p,v}(\Omega)$ we denote the weighted L_p-space, the space of all functions f measurable on Ω for which $\|f\|_{L_{p,v}(\Omega)} = \|vf\|_{L_p(\Omega)} < \infty$.

The problem of boundedness of the operator $H_{1,\varphi}$ from one weighted space $L_{p_1,v_1}(\mathbb{R})$ to another one $L_{p_2,v_2}(\mathbb{R})$ has been studied in detail and necessary and sufficient conditions on the weights v_1, v_2 ensuring boundedness of $H_{1,\varphi}$ from $L_{p_1,v_1}(\mathbb{R})$ to $L_{p_2,v_2}(\mathbb{R})$ have been obtained for all values of the parameters $0 < p_1, p_2 \leqslant \infty$. See books [9], [11], [8], [10] for formulations and proofs of these results, and for the history of the problem.

In the theory of partial differential equations, together with weighted L_p-spaces, Morrey spaces $M_{p,\lambda}$ are widely used. They were introduced by C. Morrey in 1938 [12] and defined as follows: For $0 \leqslant \lambda \leqslant n$, $1 \leqslant p \leqslant \infty$, $f \in M_{p,\lambda}$ if $f \in L_p^{loc}(\mathbb{R}^n)$ and

$$\|f\|_{M_{p,\lambda}} \equiv \|f\|_{M_{p,\lambda}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n,\, r>0} r^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x,r))} < \infty. \tag{1.2}$$

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In the last decade much attention was paid to studying properties of various operators in general Morrey-type spaces defined in following way. Let $0 < p, \theta \leqslant \infty$ and let w be a non-negative measurable function on $(0,\infty)$. We denote by $LM_{p\theta,w}$, $GM_{p\theta,w}$ respectively, the local Morrey-type spaces, the global Morrey-type spaces respectively, the spaces of all functions $f \in L_p^{loc}(\mathbb{R}^n)$ with finite quasinorms

$$\|f\|_{LM_{p\theta,w}} \equiv \|f\|_{LM_{p\theta,w}(\mathbb{R}^n)} = \|w(r)\|f\|_{L_p(B_r)}\|_{L_\theta(0,\infty)}, \tag{1.3}$$

$$\|f\|_{GM_{p\theta,w}} = \sup_{x\in\mathbb{R}^n} \|f(x+\cdot)\|_{LM_{p\theta,w}} \tag{1.4}$$

respectively. (Clearly $GM_{p\infty,r^{-\frac{\lambda}{p}}} \equiv M_{p,\lambda}$.)

In [3], [4], [5], [2], [6] for a wide range of the numerical parameters p_1, p_2, θ_1, θ_2 (but not for all admissible values of these parameters) necessary and sufficient conditions on functions w_1 and w_2 were established ensuring boundedness of the maximal operator, the fractional maximal operators, the Riesz potentials and the genuine singular integral operators from one local Morrey-type space $LM_{p_1\theta_1,w_1}$ to another one $LM_{p_2\theta_2,w_2}$.

In [7] the problem of boundedness from $LM_{p_1\theta_1,w_1}$ to $LM_{p_2\theta_2,w_2}$ was studied for the Hardy operator H_α, a particular case of the operator $H_{n,\varphi}$ in which $\varphi(r) = |B_r|^{1-\frac{\alpha}{n}}$, where $\alpha \in \mathbb{R}$ and $|B_r|$ is the Lebesque measure of the ball B_r. For all admissible values of the numerical parameters p_1, p_2, θ_1, θ_2 sufficient conditions on w_1 and w_2 were obtained ensuring the boundedness of H_α. Moreover, for a certain range of the numerical parameters and under certain regularity assumptions on w_2 these sufficient conditions coincide with the necessary ones. (See Section 4 for more details.) Under further a priori assumptions on w_1 and w_2 these conditions are also necessary and sufficient for the boundedness of H_α from $GM_{p_1\theta_1,w_1}$ to $GM_{p_2\theta_2,w_2}$.

The aim of this paper is investigation of boundedness of the operator $H_{n,\varphi}$ from one local Morrey-type space $LM_{p_1\theta_1,w_1}$ to another one $LM_{p_2\theta_2,w_2}$ under the assumption $\theta_1 = p_1$. One can easy verify that

$$\|f\|_{LM_{p_1p_1,w_1}} = \|f\|_{L_{p_1,u_1}} \tag{1.5}$$

where

$$u_1(x) = \|w_1\|_{L_{p_1}(|x|,\infty)}, \tag{1.6}$$

so the problem under consideration is a problem of boundedness of the operator $H_{n,\varphi}$ from a weighted space L_{p_1,u_1} with a non-negative radially symmetric non-increasing weight u_1 to a local Morrey-type space $LM_{p_2\theta_2,w_2}$. In fact we shall consider a more general case in which u_1 is a non-negative radially symmetric measurable weight, but not necessarily non-increasing.

2. Main results

Let $0 < \theta \leqslant \infty$. We denote by Ω_θ the set of all functions w which are non-negative, measurable on $(0,\infty)$, not equivalent to 0, and such that for some $t > 0$

$$\|w\|_{L_\theta(t,\infty)} < \infty. \tag{2.1}$$

Note that, given $0 < p,\theta \leqslant \infty$ and a function ω non-negative, measurable on $(0,\infty)$ and not equivalent to 0, the space $LM_{q\theta,w}$ is non-trivial, i.e. consists not only of functions equivalent to 0 on $\mathbb{R}^n$, if and only if $w \in \Omega_\theta$ [3].

In all statements below we assume that $u(x) = v(|x|)$, where v is a non-negative measurable function on $(0,\infty)$ and, for a given $0 < \theta \leqslant \infty$, w is a function of the class Ω_θ.

LEMMA 2.1. *Let $p \geqslant 1$, $0 < q,\theta \leqslant \infty$, and $c_1 > 0$. The inequality*

$$\|H_{n,\varphi} f\|_{LM_{q\theta,w}} \leqslant c_1 \|f\|_{L_{p,u}} \tag{2.2}$$

for all functions $f \in L_{p,u}$ is equivalent to the inequality

$$\left(\int_0^\infty w^\theta(r) \left(\int_0^r (H_{\tilde{\varphi}} g)^q dt \right)^{\frac{\theta}{q}} dr \right)^{\frac{1}{\theta}} \leqslant c_2 \|g\|_{L_{p,\tilde{u}}(0,\infty)} \tag{2.3}$$

for all non-negative functions $g \in L_{p,\tilde{u}}(0,\infty)$, where

$$H_{\tilde{\varphi}} g(t) = \tilde{\varphi}(t) \int_0^t g(s)ds, \tag{2.4}$$

$$\tilde{\varphi}(t) = \varphi(t) t^{\frac{n-1}{q}}, \qquad \tilde{u}(t) = v(t) t^{-\frac{n-1}{p'}}, \qquad c_2 = c_1 \sigma_n^{-\left(\frac{1}{p'}+\frac{1}{q}\right)},$$

σ_n is the surface area of the unit sphere S^{n-1} in $\mathbb{R}^n$, and $p' = \frac{p}{p-1}$.

Moreover

$$\|H_{n,\varphi}\|_{L_{p,u} \to LM_{q\theta,w}} = \sigma_n^{\frac{1}{p'}+\frac{1}{q}} \|H_{\tilde{\varphi}}\|_{L_{p,\tilde{u}} \to LM_{pq,w}}. \tag{2.5}$$

REMARK 2.2. If $\theta = q$ then inequality (2.3) takes the form

$$\left(\int_0^\infty \tilde{w}^q(t) (H_{\tilde{\varphi}} g)^q(t) dt \right)^{\frac{1}{q}} \leqslant c_2 \|g\|_{L_{p,\tilde{u}}(0,\infty)}$$

i.e.

$$\left\| \int_0^t g(s)ds \right\|_{L_{q,\tilde{\varphi}\tilde{w}}(0,\infty)} \leqslant c_2 \|g\|_{L_{p,\tilde{u}}(0,\infty)},$$

where

$$\tilde{w}(t) = \|w\|_{L_q(t,\infty)}, \quad 0 < t < \infty.$$

Hence, due to the known results [10], [11], [16] the operator $H_{n,\varphi}$ is bounded from $L_{p,u}$ to $LM_{qq,w}$ if and only if

1) for[1] $1 < p \leqslant q < \infty$

$$A_1 = \sup_{\beta>0} \left(\int_\beta^\infty (\tilde{\varphi}\tilde{w})^q dt \right)^{\frac{1}{q}} \left(\int_0^\beta \tilde{u}^{-p'} ds \right)^{\frac{1}{p'}} < \infty,$$

2) for $1 \leqslant q < p < \infty$

$$A_2 = \left(\int_0^\infty \left(\int_\beta^\infty (\tilde{\varphi}\tilde{w})^q dt \right)^{\frac{p}{p-q}} \left(\int_0^\beta \tilde{u}^{-p'} ds \right)^{\frac{p(q-1)}{p-q}} \tilde{u}^{-p'}(\beta) d\beta \right)^{\frac{p-q}{pq}} < \infty,$$

3) for $0 < q < 1 < p < \infty$

$$A_3 = \left(\int_0^\infty \left(\int_0^\beta \tilde{u}^{-p'} dt \right)^{\frac{q(p-1)}{p-q}} \left(\int_\beta^\infty (\tilde{\varphi}\tilde{w})^q ds \right)^{\frac{q}{p-q}} (\tilde{\varphi}(\beta)\tilde{w}(\beta))^q d\beta \right)^{\frac{p-q}{pq}} < \infty.$$

Moreover, the norm $\|H_{n,\varphi}\|_{L_{p,u}\to LM_{qq,w}}$ is equivalent to A_1, A_2, A_3 respectively, with equivalency constants depending only on n, p and q.

If $0 < q < 1 < p < \infty$ and $\int_0^\beta \tilde{u}^{-p'} ds < \infty$ for all $\beta > 0$, then integration by parts implies that

$$A_3 = \left(\frac{q(p-1)}{p} \right)^{\frac{p-q}{pq}} A_2.$$

If $\int_0^\beta \tilde{u}^{-p'} ds = \infty$ for some $\beta > 0$, then conditions $A_2 < \infty$ and $A_3 < \infty$ are not equivalent (see [16], page 93).

Taking into account this remark, in the sequel we focus on the case $\theta \neq q$.

[1] It may happen that $\int_0^\beta \tilde{u}^{-p'} ds = 0$ and $\int_\beta^\infty (\tilde{\varphi}\tilde{w})^q dt = \infty$. In this case, and in similar cases in the sequel, it is assumed that $\infty \cdot 0 = 0$.

THEOREM 2.3. *Let* $1 < p \leqslant q,\ \theta < \infty$. *Then the operator* $H_{n,\varphi}$ *is bounded from* $L_{p,u}$ *to* $LM_{q\theta,w}$ *if and only if*

$$B_1 = \sup_{\beta>0} \left(\int_\beta^\infty w^\theta(r) \left(\int_\beta^r \tilde{\varphi}^q ds \right)^{\frac{\theta}{q}} dr \right)^{\frac{1}{\theta}} \left(\int_0^\beta \tilde{u}^{-p'} dr \right)^{\frac{1}{p'}} < \infty. \tag{2.6}$$

Moreover,

$$\sigma_n^{\frac{1}{p'}+\frac{1}{q}} B_1 \leqslant \|H_{n,\varphi}\|_{L_{p,u}\to LM_{q\theta,w}} \leqslant 4\sigma_n^{\frac{1}{p'}+\frac{1}{q}} B_1.$$

REMARK 2.4. Since the functions w and φ are not equivalent to 0 on $(0,\infty)$ it follows from (2.6) that $\int_0^\beta \tilde{u}^{-p'} dr < \infty$ for all $\beta > 0$.

REMARK 2.5. Note that if $\theta = q$ then $B_1 = A_1$.

THEOREM 2.6. *Let* $0 < q < p \leqslant \theta < \infty$. *Assume that* $q \geqslant 1$ *or* $q < 1,\ p > 1$ *and* $\int_0^\beta \tilde{u}^{-p'} ds < \infty$ *for all* $\beta > 0$. *Then the operator* $H_{n,\varphi}$ *is bounded from* $L_{p,u}$ *to* $LM_{q\theta,w}$ *if and only if* $\max\{B_1, B_2\} < \infty$, *where*

$$B_2 = \sup_{\beta>0} \left(\int_\beta^\infty w^\theta dr \right)^{\frac{1}{\theta}} \left(\int_0^\beta \left(\int_t^\beta \widetilde{\varphi}^q dr \right)^{\frac{q}{p-q}} \widetilde{\varphi}^q(t) \left(\int_0^t \widetilde{u}^{-p'} dr \right)^{\frac{q(p-1)}{p-q}} dt \right)^{\frac{p-q}{pq}}.$$

Moreover, $\|H_{n,\varphi}\|_{L_{p,u}\to LM_{q\theta,w}}$ *is equivalent to* $\max\{B_1, B_2\}$, *briefly*

$$\|H_{n,\varphi}\|_{L_{p,u}\to LM_{q\theta,w}} \approx \max\{B_1, B_2\},$$

with the equivalency constants depending only on n, p, q *and* θ.

THEOREM 2.7. *Let* $0 < q < \theta < p < \infty$. *Assume that* $q \geqslant 1$ *or* $q < 1,\ \theta > 1$ *and* $\int_0^\beta \tilde{u}^{-p'} ds < \infty$ *for all* $\beta > 0$. *Then the operator* $H_{n,\varphi}$ *is bounded from* $L_{p,u}$ *to* $LM_{q\theta,w}$ *if and only if* $\max\{C_1, C_2\} < \infty$, *where*

$$C_1 = \left(\int_0^\infty \left(\int_\beta^\infty w^\theta(r) \left(\int_\beta^r \tilde{\varphi} ds \right)^{\frac{\theta}{q}} dr \right)^{\frac{p}{p-\theta}} \left(\int_0^\beta \tilde{u}^{-p'} dt \right)^{\frac{p(\theta-1)}{p-\theta}} \widetilde{u}^{-p'}(\beta) d\beta \right)^{\frac{p-\theta}{p\theta}},$$

$$C_2 = \left(\int_0^\infty \left(\int_\beta^\infty w^\theta dr \right)^{\frac{p}{p-\theta}} \left(\int_0^\beta \left(\int_t^\beta \tilde{\varphi} ds \right)^{\frac{q}{p-q}} \left(\int_0^t \tilde{u}^{-p'} d\tau \right)^{\frac{q(p-1)}{p-q}} dt \right)^{\frac{\theta(p-q)}{q(p-\theta)}} w^\theta(\beta) d\beta \right)^{\frac{p-\theta}{p\theta}} .$$

Moreover,

$$\|H_{n,\varphi}\|_{L_{p,u} \to LM_{q\theta,w}} \approx \max\{C_1, C_2\},$$

with the equivalency constants depending only on n, p, q *and* θ.

REMARK 2.8. Sometimes the variants of the Hardy operator of the following form are considered

$$(H_{n,\varphi,v} f)(x) = \varphi(|x|) v(|x|) \int_{B_{|x|}} \frac{f(y)}{v(|x|)} dx, \, x \in \mathbb{R}^n, \tag{2.7}$$

where v is a positive measurable function on $(0,\infty)$ when dealing with such operators one should keep in mind that

$$\|H_{n,\varphi,v}\|_{L_{p,u} \to LM_{q\theta,w}} = \|H_{n,\varphi v}\|_{L_{p,uv} \to LM_{q\theta,w}}.$$

Operators $H^\alpha_{n,v}$ of the form (2.7) with $\varphi(r) = r^{\alpha-n}$ where considered in [15], [14], where under a number of a priori assumptions on v and w necessary and sufficient conditions on v and w were found ensuring that $H^\alpha_{n,v} : LM_{p\infty,w} \to LM_{q\infty,w}$.

3. Proofs of the main results

Proof of Lemma 2.1

1. By taking the spherical coordinates and applying Hölder's inequality we get

$$\|H_{n,\varphi} f\|_{L_q(B_r)} = \left(\int_{B_r} \left| \varphi(|x|) \int_0^{|x|} \left(\int_{S^{n-1}} f(\sigma\rho) d\sigma \right) \rho^{n-1} d\rho \right|^q dx \right)^{\frac{1}{q}}$$

$$= \sigma_n^{\frac{1}{q}} \left(\int_0^r \left| \varphi(t) \int_0^t \left(\int_{S^{n-1}} f(\sigma\rho) d\sigma \right) \rho^{n-1} d\rho \right|^q t^{n-1} dt \right)^{\frac{1}{q}}$$

$$\leqslant \sigma_n^{\frac{1}{p'}+\frac{1}{q}} \left(\int_0^r \left(\varphi(t) t^{\frac{n-1}{q}} \int_0^t \left(\int_{S^{n-1}} |f(\sigma\rho)|^p d\sigma \right)^{\frac{1}{p}} \rho^{n-1} d\rho \right)^q dt \right)^{\frac{1}{q}}$$

$$= \sigma_n^{\frac{1}{p'}+\frac{1}{q}} \|\tilde{\varphi}(t) \int_0^t g(\rho) d\rho \|_{L_q(0,r)}, \tag{3.1}$$

where

$$g(\rho) = \left(\int_{S^{n-1}} |f(\sigma\rho)|^p d\sigma \right)^{\frac{1}{p}} \rho^{n-1}. \tag{3.2}$$

Furthermore,

$$\|f\|_{L_{p,u}} = \left(\int_{\mathbb{R}^n} |v(|x|)f(x)|^p dx \right)^{\frac{1}{p}} = \left(\int_0^\infty v(\rho)^p \left(\int_{S^{n-1}} |f(\sigma\rho)|^p d\sigma \right) \rho^{n-1} d\rho \right)^{\frac{1}{p}}$$
$$= \left(\int_0^\infty \left(v(\rho)\rho^{-\frac{n-1}{p'}} g(\rho) \right)^p d\rho \right)^{\frac{1}{p}} = \|g\|_{L_{p,\tilde{u}}(0,\infty)}. \tag{3.3}$$

Note that if $f(x) = \tilde{f}(|x|)$ where $\tilde{f}$ is a non-negative measurable function on $(0,\infty)$, then in (3.1) there is equality because in this case

$$\int_{S^{n-1}} f(\sigma\rho)d\sigma = \sigma_n \tilde{f}(\rho) = \sigma_n^{\frac{1}{p'}} \left(\int_{S^{n-1}} |f(\sigma\rho)|^p d\sigma \right)^{\frac{1}{p}}.$$

Also in this case by (3.2) we get

$$g(\rho) = \sigma_n^{\frac{1}{p}} \tilde{f}(\rho)\rho^{n-1}. \tag{3.4}$$

2. Assume that inequality (2.2) holds for all functions $f \in L_{p,u}$ and let g be an arbitrary non-negative function in $L_{p,\tilde{u}}(0,\infty)$. Taking into account equality (3.4) we put in (2.2) the function f defined by $f(x) = \sigma_n^{-\frac{1}{p}} g(|x|)|x|^{1-n}$, $x \in \mathbb{R}^n$. In this case there is equality in inequality (3.1) and this equality together with equality (3.3) imply inequality (2.3).

3. Assume that inequality (2.3) holds for all non-negative functions $g \in L_{p,\tilde{u}}(0,\infty)$, and let f be an arbitrary function in $L_{p,u}$. We put in (2.3) the function g defined by equality (3.2). By inequality (3.1) the left-hand side of inequality (2.3) is greater than or equal to $\sigma_n^{-(\frac{1}{p'}+\frac{1}{q})} \|H_{n,\varphi} f\|_{LM_{q\theta,w}}$, and by equality (3.3) the right-hand side of inequality (2.3) is equal to $c_2\|f\|_{L_{p,u}}$. Hence inequality (2.2) follows.

4. Equality (2.5) follows by inequality (3.1), equality (3.3) and the last part of Step 1. □

Proof of Theorem 2.3

Necessity. Assume that the operator $H_{n,\varphi}$ is bounded from $L_{p,u}$ to $LM_{q\theta,w}$. Then by Lemma 2.1 for some $c_2 > 0$ inequality (2.3) is satisfied. Let $0 < \alpha < \beta < \infty$ and $\varepsilon > 0$. Consider in (2.3) the test-functions g_ε defined by

$$g_\varepsilon(t) = v^{-p'}(t)t^{n-1}\chi_{(\alpha,\beta)}(t)\psi_\varepsilon(t),\ t \in (0,\infty),$$

where $\chi_{(\alpha,\beta)}$ is the characteristic function of the interval (α,β) and $\psi_\varepsilon(t)=1$ if $v(t)\geqslant\varepsilon$ and $\psi_\varepsilon(t)=0$ if $v(t)<\varepsilon$. Then

$$\|g_\varepsilon\|_{L_{p,\tilde{u}}}=\left(\int_\alpha^\beta\left(v^{-p'+1}(s)s^{(n-1)(1-\frac{1}{p'})}\psi_\varepsilon(s)\right)^p ds\right)^{\frac{1}{p}}$$
$$=\left(\int_\alpha^\beta v^{-p'}(s)s^{n-1}\psi_\varepsilon(s)ds\right)^{\frac{1}{p}}$$

and by (2.3)

$$\left(\int_\alpha^\beta v^{-p'}(s)s^{n-1}\psi_\varepsilon(s)ds\right)^{\frac{1}{p}}$$
$$\geqslant c_2^{-1}\left(\int_0^\infty w^\theta(r)\left(\int_0^r\left(\tilde{\varphi}(t)\int_0^t g_\varepsilon(s)ds\right)^q dt\right)^{\frac{\theta}{q}}dr\right)^{\frac{1}{\theta}}$$
$$\geqslant c_2^{-1}\left(\int_\beta^\infty w^\theta(r)\left(\int_\beta^r\tilde{\varphi}^q(t)\left(\int_\alpha^\beta v^{-p'}(s)s^{n-1}\psi_\varepsilon(s)ds\right)^q dt\right)^{\frac{\theta}{q}}dr\right)^{\frac{1}{\theta}}$$
$$=c_2^{-1}\left(\int_\beta^\infty w^\theta(r)\left(\int_\beta^r\varphi^q(t)t^{n-1}dt\right)^{\frac{\theta}{q}}dr\right)^{\frac{1}{\theta}}\left(\int_\alpha^\beta v^{-p'}(s)s^{n-1}\psi_\varepsilon(s)ds\right).$$

Since $\int_\alpha^\beta v^{-p'}(s)s^{n-1}\psi_\varepsilon(s)ds<\infty$ it follows that

$$\left(\int_\beta^\infty w^\theta(r)\left(\int_\beta^r\varphi^q(t)t^{n-1}dt\right)^{\frac{\theta}{q}}dr\right)^{\frac{1}{\theta}}\left(\int_\alpha^\beta v^{-p'}(s)s^{n-1}\psi_\varepsilon(s)ds\right)^{\frac{1}{p'}}\leqslant c_2.$$

By the Fatou Lemma this inequality implies that for all $\alpha\in(0,\beta)$

$$\left(\int_\beta^\infty w^\theta(r)\left(\int_\beta^r\varphi^q(t)t^{n-1}dt\right)^{\frac{\theta}{q}}dr\right)^{\frac{1}{q}}\left(\int_\alpha^\beta v^{-p'}(s)s^{n-1}ds\right)^{\frac{1}{p'}}\leqslant c_2,$$

which in its turn, by passing to the limit as $\alpha\to 0^+$ and by taking supremum with respect to $\beta>0$, implies condition (2.6).

Moreover, $B_1 \leqslant c_2$.

Sufficiency. **1.** By Lemma 2.1 it suffices to prove inequality (2.3). Let $B_1 < \infty$ and $0 \leqslant g \in L_{p,\tilde{u}}(0,\infty)$. First assume that g is not equivalent to 0 on $(0,\infty)$ and $g \in L_1(0,\infty)$. Consider the sequence $\{r_k\}_{k=-\infty}^{0}$, where $r_0 = +\infty$ and for integer $k < 0$ the numbers r_k are defined by

$$\int_0^{r_k} g(s)ds = 2^k \int_0^{\infty} g(s)ds.$$

Note that for all integer $k \leqslant 0$, $0 < r_{k-1} < r_k$ and

$$\int_0^{r_k} g(s)ds = 4 \int_{r_{k-2}}^{r_{k-1}} g(s)ds. \tag{3.5}$$

Let $\sigma = \sup\{x \in (0,\infty) : \int_0^x g(y)dy = 0\}$. Then

$$(0,\infty) = (0,\sigma] \bigcup \left(\bigcup_{k=-\infty}^{0} [r_{k-1}, r_k) \right).$$

Therefore by applying equality (3.5) we get

$$\begin{aligned}
I &= \int_0^\infty w^\theta(r) \left(\int_0^r \left(\tilde{\varphi}(t) \int_0^t g(s)ds \right)^q dt \right)^{\frac{\theta}{q}} dr \\
&= \sum_{k=-\infty}^{0} \int_{r_{k-1}}^{r_k} w^\theta(r) \left(\int_\sigma^r \left(\tilde{\varphi}(t) \int_0^t g(s)ds \right)^q dt \right)^{\frac{\theta}{q}} dr \\
&\leqslant \sum_{k=-\infty}^{0} \int_{r_{k-1}}^{r_k} w^\theta(r) \left(\sum_{i=-\infty}^{k} \int_{r_{i-1}}^{\min\{r,r_i\}} \left(\tilde{\varphi}(t) \int_0^t g(s)ds \right)^q dt \right)^{\frac{\theta}{q}} dr \\
&\leqslant \sum_{k=-\infty}^{0} \int_{r_{k-1}}^{r_k} w^\theta(r) \left(\sum_{i=-\infty}^{k} \left(\int_{r_{i-1}}^{\min\{r,r_i\}} \varphi^q(t) t^{n-1} dt \right) \left(\int_0^{r_i} g(s)ds \right)^q \right)^{\frac{\theta}{q}} dr \\
&= 4^\theta \sum_{k=-\infty}^{0} \int_{r_{k-1}}^{r_k} w^\theta(r) \left(\sum_{i=-\infty}^{k} \left(\int_{r_{i-1}}^{\min\{r,r_i\}} \varphi^q(t) t^{n-1} dt \right) \left(\int_{r_{i-2}}^{r_{i-1}} g(s)ds \right)^q \right)^{\frac{\theta}{q}} dr. \qquad (3.6)
\end{aligned}$$

2. If $\theta \leqslant q$ then by Jensen's inequality

$$I \leqslant 4^\theta \sum_{k=-\infty}^{0} \sum_{i=-\infty}^{k} \left(\int_{r_{i-2}}^{r_{i-1}} g(s)ds \right)^\theta \int_{r_{k-1}}^{r_k} w^\theta(r) \left(\int_{r_{i-1}}^{\min\{r,r_i\}} \varphi^q(t)t^{n-1}dt \right)^{\frac{\theta}{q}} dr$$

$$\leqslant 4^\theta \sum_{i=-\infty}^{0} \left(\int_{r_{i-2}}^{r_{i-1}} g(s)ds \right)^\theta \sum_{k=i}^{0} \int_{r_{k-1}}^{r_k} w^\theta(r) \left(\int_{r_{i-1}}^{r} \varphi^q(t)t^{n-1}dt \right)^{\frac{\theta}{q}} dr$$

$$= 4^\theta \sum_{i=-\infty}^{0} \left(\int_{r_{i-2}}^{r_{i-1}} g(s)ds \right)^\theta \int_{r_{i-1}}^{\infty} w^\theta(r) \left(\int_{r_{i-1}}^{r} \varphi^q(t)t^{n-1}dt \right)^{\frac{\theta}{q}} dr.$$

By Hölder's inequality, by the definition of B_1 and by Jensen's inequality we have

$$I \leqslant 4^\theta \sum_{i=-\infty}^{0} \left(\int_{r_{i-2}}^{r_{i-1}} |g(s)\tilde{u}(s)|^p ds \right)^{\frac{\theta}{p}}$$

$$\times \left(\int_{r_{i-2}}^{r_{i-1}} v^{-p'}(s)s^{n-1}ds \right)^{\frac{\theta}{p'}} \int_{r_{i-1}}^{\infty} w^\theta(r) \left(\int_{r_{i-1}}^{r} \varphi^q(t)t^{n-1}dt \right)^{\frac{\theta}{q}} dr$$

$$\leqslant (4B_1)^\theta \sum_{i=-\infty}^{0} \left(\int_{r_{i-2}}^{r_{i-1}} |g(s)\tilde{u}(s)|^p ds \right)^{\frac{\theta}{p}} \leqslant (4B_1)^\theta \left(\sum_{i=-\infty}^{0} \int_{r_{i-2}}^{r_{i-1}} |g(s)\tilde{u}(s)|^p ds \right)^{\frac{\theta}{p}}$$

$$= (4B_1)^\theta \left(\int_{0}^{r_{-1}} |g(s)\tilde{u}(s)|^p ds \right)^{\frac{\theta}{p}} \leqslant \left(4B_1 \|g\|_{L_{p,\tilde{u}}(0,\infty)} \right)^\theta.$$

Hence,

$$I^{\frac{1}{\theta}} \leqslant 4B_1 \|g\|_{L_{p,\tilde{u}}(0,\infty)}. \tag{3.7}$$

3. If $\theta > q$ then starting with inequality (3.6) and applying Minkowski's inequalities of the forms

$$\int_{r_{k-1}}^{r_k} \left(\sum_{i=-\infty}^{k} a_i(r) \right)^\sigma dr \leqslant \left(\sum_{i=-\infty}^{k} \left(\int_{r_{k-1}}^{r_k} a_i(r)^\sigma dr \right)^{\frac{1}{\sigma}} \right)^\sigma$$

and

$$\left(\sum_{k=-\infty}^{0} \left(\sum_{i=-\infty}^{k} b_i \right)^\sigma \right)^{\frac{1}{\sigma}} \leqslant \sum_{i=-\infty}^{0} \left(\sum_{k=i}^{0} b_i^\sigma \right)^{\frac{1}{\sigma}},$$

where $\sigma = \dfrac{\theta}{q} > 1$, we get

$$I^{\frac{q}{\theta}} \leqslant 4^q \left[\sum_{k=-\infty}^{0} \int_{r_{k-1}}^{r_k} \left(\sum_{i=-\infty}^{k} w^{\frac{1}{q}}(r) \left(\int_{r_{i-1}}^{\min\{r,r_i\}} \varphi^q(t) t^{n-1} dt \right) \left(\int_{r_{i-2}}^{r_{i-1}} g(s) ds \right)^q \right)^{\frac{\theta}{q}} dr \right]^{\frac{q}{\theta}}$$

$$\leqslant 4^q \left[\sum_{k=-\infty}^{0} \left[\sum_{i=-\infty}^{k} \left(\int_{r_{k-1}}^{r_k} w^\theta(r) \left(\int_{r_{i-1}}^{\min\{r,r_i\}} \varphi^q(t) t^{n-1} dt \right)^{\frac{\theta}{q}} \left(\int_{r_{i-2}}^{r_{i-1}} g(s) ds \right)^\theta dr \right)^{\frac{q}{\theta}} \right]^{\frac{\theta}{q}} \right]^{\frac{q}{\theta}}$$

$$\leqslant 4^q \sum_{i=-\infty}^{0} \left(\sum_{k=i}^{0} \int_{r_{k-1}}^{r_k} w^\theta(r) \left(\int_{r_{i-1}}^{r} \varphi^q(t) t^{n-1} dt \right)^{\frac{\theta}{q}} \left(\int_{r_{i-2}}^{r_{i-1}} g(s) ds \right)^\theta dr \right)^{\frac{q}{\theta}}$$

$$= 4^q \sum_{i=-\infty}^{0} \left(\int_{r_{i-2}}^{r_{i-1}} g(s) ds \right)^q \left(\sum_{k=i}^{0} \int_{r_{k-1}}^{r_k} w^\theta(r) \left(\int_{r_{i-1}}^{r} \varphi^q(t) t^{n-1} dt \right)^{\frac{\theta}{q}} dr \right)^{\frac{q}{\theta}}$$

$$= 4^q \sum_{i=-\infty}^{0} \left(\int_{r_{i-2}}^{r_{i-1}} g(s) ds \right)^q \left(\int_{r_{i-1}}^{\infty} w^\theta(r) \left(\int_{r_{i-1}}^{r} \varphi^q(t) t^{n-1} dt \right)^{\frac{\theta}{q}} dr \right)^{\frac{q}{\theta}}.$$

By Hölder's inequality, by the definition of B_1 and by Jensen's inequality we have

$$I^{\frac{q}{\theta}} \leqslant 4^q \sum_{i=-\infty}^{0} \left(\int_{r_{i-2}}^{r_{i-1}} |g(s)\tilde{u}(s)|^p ds \right)^{\frac{q}{p}} \left(\int_{r_{i-2}}^{r_{i-1}} v^{-p'}(s) s^{n-1} ds \right)^{\frac{q}{p'}}$$

$$\times \left(\int_{r_{i-1}}^{\infty} w^\theta(r) \left(\int_{r_{i-1}}^{r} \varphi^q(t) t^{n-1} dt \right)^{\frac{\theta}{q}} dr \right)^{\frac{q}{\theta}}$$

$$\leqslant (4B_1)^q \sum_{i=-\infty}^{0} \left(\int_{r_{i-2}}^{r_{i-1}} |g(s)\tilde{u}(s)|^p ds \right)^{\frac{q}{p}} \leqslant (4B_1)^q \left(\sum_{i=-\infty}^{0} \int_{r_{i-2}}^{r_{i-1}} |g(s)\tilde{u}(s)|^p ds \right)^{\frac{q}{p}}$$

$$= (4B_1)^q \left(\int_0^{r_{-1}} |g(s)\tilde{u}(s)|^p ds \right)^{\frac{q}{p}} \leqslant \left(4B_1 \|g\|_{L_{p,\tilde{u}}(0,\infty)} \right)^q .$$

Hence for $\theta > q$ inequality (3.7) also holds.

4. If $\int_0^\infty g(s)ds = \infty$ we consider the sequence $\{r_k\}_{k=-\infty}^{\infty}$, where for all integer k

$$\int_0^{r_k} g(s)ds = 2^k.$$

Note that in this case for all integer k $0 < r_{k-1} < r_k$ and equality (3.5) holds. By a similar argument we get for $\theta \leqslant q$

$$I \leqslant (4B_1)^\theta \left(\sum_{i=-\infty}^{\infty} \int_{r_{i-2}}^{r_{i-1}} |g(s)\tilde{u}(s)|^p ds \right)^{\frac{\theta}{p}} = \left(4B_1 \|g\|_{L_{p,\tilde{u}}(0,\infty)} \right)^\theta$$

and for $\theta > q$

$$I^{\frac{q}{\theta}} \leqslant (4B_1)^q \left(\sum_{i=-\infty}^{\infty} \int_{r_{i-2}}^{r_{i-1}} |g(s)\tilde{u}(s)|^p ds \right)^{\frac{q}{p}} = \left(4B_1 \|g\|_{L_{p,\tilde{u}}(0,\infty)} \right)^q .$$

So for all $0 \leqslant g \in L_{p,\tilde{u}}(0,\infty)$ inequality (3.7) holds which means that inequality (2.3) is valid with $c_2 = 4B_1$.

Therefore the statement of Theorem 2.3 follows by Lemma 2.1 □

Proof of Theorem 2.6

1. By Lemma 2.1 and the duality formula if suffices to estimate the quantity

$$C^q = \sup_{g \geqslant 0} \frac{\left(\int_0^\infty \left(w^q(r) \int_0^r (H_{\widetilde{\varphi}} g)^q dt \right)^{\frac{\theta}{q}} dr \right)^{\frac{q}{\theta}}}{\|\widetilde{u} g\|^q_{L_p(0,\infty)}}$$

$$= \sup_{g \geqslant 0} \sup_{h \geqslant 0} \frac{\int_0^\infty h(r) \left(\int_0^r \left(\widetilde{\varphi}(t) \int_0^t g(s)ds \right)^q dt \right) dr}{\|w^{-q}h\|_{L_{\frac{\theta}{\theta-q}}(0,\infty)} \|\widetilde{u} g\|^q_{L_p(0,\infty)}}$$

$$= \sup_{h \geqslant 0} \sup_{g \geqslant 0} \frac{\int_0^\infty \left(\widetilde{\varphi}(t) \int_0^t g(s)ds \right)^q \left(\int_t^\infty h(r)dr \right) dt}{\|w^{-q}h\|_{L_{\frac{\theta}{\theta-q}}(0,\infty)} \|\widetilde{u} g\|^q_{L_p(0,\infty)}} .$$

By Remark 2.2 it follows that

$$\sup_{g\geqslant 0}\frac{\int_0^\infty\left(\widetilde{\varphi}(t)\int_0^t g(s)ds\right)^q\int_t^\infty h(r)drdt}{\|\widetilde{u}g\|^q_{L_p(0,\infty)}}$$

$$\approx\left(\int_0^\infty\left(\int_t^\infty\widetilde{\varphi}^q(\tau)\left(\int_\tau^\infty h(r)dr\right)d\tau\right)^{\frac{p}{p-q}}\left(\int_0^t\widetilde{u}^{-p'}(s)ds\right)^{\frac{p(q-1)}{p-q}}\widetilde{u}^{-p'}(t)dt\right)^{\frac{p-q}{p}}$$

$$=\left(\int_0^\infty\left(\int_t^\infty\left(\int_t^r\widetilde{\varphi}^q(\tau)d\tau\right)h(r)dr\right)^{\frac{p}{p-q}}\left(\int_0^t\widetilde{u}^{-p'}(s)ds\right)^{\frac{p(q-1)}{p-q}}\widetilde{u}^{-p'}(t)dt\right)^{\frac{p-q}{p}},$$

where the equivalency constants depend only on p,q and θ.

Let

$$(Kh)(t)=\int_t^\infty k(r,t)h(r)dr,$$

where

$$k(r,t)=\int_t^r\tilde{\varphi}^q(\tau)d\tau.$$

Note that the kernel k satisfies the condition

$$k(r,t)=k(r,s)+k(s,t),\ r\geqslant s\geqslant t\geqslant 0. \tag{3.8}$$

Let

$$\widetilde{U}(t)=\left(\left(\int_0^t\tilde{u}^{-p'}(s)ds\right)^{\frac{p(q-1)}{p-q}}\tilde{u}^{-p'}(t)\right)^{\frac{p-q}{p}},\qquad\widetilde{W}(t)=w^{-q}(t).$$

Then by the above

$$C^q\approx\|K\|_{L_{\frac{\theta}{\theta-q},\widetilde{W}}(0,\infty)\to L_{\frac{p}{p-q},\widetilde{U}}(0,\infty)}, \tag{3.9}$$

where the equivalency constants depend only on p,q and θ.

2. By the assumptions on the parameters $1<\frac{\theta}{\theta-q}\leqslant\frac{p}{p-q}$, therefore due to condition (3.8) by [13] $C^q\approx\max\{\widetilde{B}_1,\widetilde{B}_2\}$, where the equivalency constants depend only on p,q and θ,

$$\widetilde{B}_1 = \sup_{\beta>0}\left(\int_0^\beta \widetilde{U}^{\frac{p}{p-q}}(t)dt\right)^{\frac{p-q}{q}}\left(\int_\beta^\infty \widetilde{W}^{-\frac{\theta}{q}}(r)k^{\frac{\theta}{q}}(r,\beta)dr\right)^{\frac{q}{\theta}}$$

$$= \sup_{\beta>0}\left(\int_0^\beta\left(\int_0^r \tilde{u}^{-p'}(s)ds\right)^{\frac{p(q-1)}{p-q}}\tilde{u}^{-p'}(r)dr\right)^{\frac{p-q}{p}}\left(\int_\beta^\infty w^\theta(r)\left(\int_\beta^r \widetilde{\varphi}^q(s)ds\right)^{\frac{\theta}{q}}dr\right)^{\frac{q}{\theta}}$$

$$= \left(\frac{p-q}{q(p-1)}\right)^{\frac{p-q}{p}}\sup_{\beta>0}\left(\int_\beta^\infty w^\theta(r)\left(\int_\beta^r \widetilde{\varphi}^q(s)ds\right)^{\frac{\theta}{q}}dr\right)^{\frac{q}{\theta}}\left(\int_0^\beta \tilde{u}^{-p'}(r)dr\right)^{\frac{q}{p'}}$$

$$= \left(\frac{p-q}{q(p-1)}\right)^{\frac{p-q}{p}}B_1^q$$

and

$$\widetilde{B}_2 = \sup_{\beta>0}\left(\int_0^\beta \widetilde{U}^{\frac{p}{p-q}}(t)k^{\frac{p}{p-q}}(\beta,t)dt\right)^{\frac{p-q}{q}}\left(\int_\beta^\infty \widetilde{W}^{-\frac{\theta}{q}}(r)dr\right)^{\frac{q}{\theta}}$$

$$= \sup_{\beta>0}\left(\int_0^\beta\left(\int_t^\beta \widetilde{\varphi}^q(s)ds\right)^{\frac{p}{p-q}}\left(\int_0^t \tilde{u}^{-p'}(r)dr\right)^{\frac{p(q-1)}{p-q}}\tilde{u}^{-p'}(t)dt\right)^{\frac{p-q}{p}}\left(\int_\beta^\infty w^\theta(r)dr\right)^{\frac{q}{\theta}}$$

$$= \left(\frac{p'}{q}\right)^{\frac{p-q}{p}}\sup_{\beta>0}\left(\int_0^\beta\left(\int_t^\beta \widetilde{\varphi}^q(s)ds\right)^{\frac{q}{p-q}}\widetilde{\varphi}^q(t)\left(\int_0^t \tilde{u}^{-p'}(r)dr\right)^{\frac{q(p-1)}{p-q}}dt\right)^{\frac{p-q}{p}}$$

$$\times\left(\int_\beta^\infty w^\theta(r)dr\right)^{\frac{q}{\theta}}$$

$$= \left(\frac{p'}{q}\right)^{\frac{p-q}{p}}B_2^q.$$

Taking into account equality (2.5) we get $C \approx \max\{B_1, B_2\}$, where the equivalency constants depend only on n, p, q and θ, hence the result. □

Proof of Theorem 2.7

By the assumptions on the parameters $1<\frac{p}{p-q}<\frac{\theta}{\theta-q}$, therefore due to formula (3.9) and condition (3.8) by [13] $C^q\approx\max\{\widetilde{C}_1,\widetilde{C}_2\}$, where

$$\widetilde{C}_1=\left(\int_0^\infty\left[\left(\int_\beta^\infty\widetilde{W}^{-\frac{\theta}{q}}(r)\left(\int_\beta^r\tilde{\varphi}(s)ds\right)^{\frac{\theta}{q}}dr\right)^{\frac{q}{\theta}}\left(\int_0^\beta\widetilde{U}^{\frac{p}{p-q}}\right)^{\frac{\theta-q}{\theta}}\right]^{\frac{p\theta}{q(p-\theta)}}\widetilde{U}^{\frac{p}{p-q}}(\beta)d\beta\right)^{\frac{q(p-\theta)}{p\theta}}$$

$$\approx\left(\int_0^\infty\left(\int_\beta^\infty w^\theta(r)\left(\int_\beta^r\tilde{\varphi}(s)ds\right)^{\frac{\theta}{q}}dr\right)^{\frac{p}{p-\theta}}\left(\int_0^\beta\tilde{u}^{-p'}(s)ds\right)^{\frac{p(\theta-1)}{p-\theta}}\tilde{u}^{-p'}(\beta)d\beta\right)^{\frac{q(p-\theta)}{p\theta}}$$

$$=C_1^q,$$

and

$$\widetilde{C}_2=\left(\int_0^\infty\left[\left(\int_0^\beta\left(\int_t^\beta\tilde{\varphi}(s)ds\right)^{\frac{p}{p-q}}\widetilde{U}^{\frac{p}{p-q}}(t)dt\right)^{\frac{p-q}{p}}\left(\int_\beta^\infty\widetilde{W}^{-\frac{\theta}{q}}(r)dr\right)^{\frac{q}{\theta}}\right]^{\frac{p\theta}{q(p-\theta)}}\widetilde{W}^{-\frac{\theta}{q}}(\beta)d\beta\right)^{\frac{q(p-\theta)}{p\theta}}$$

$$\approx\left(\int_0^\infty\left(\int_0^\beta\left(\int_t^\beta\tilde{\varphi}(s)ds\right)^{\frac{q}{p-q}}\left(\int_0^t\tilde{u}^{-p'}(\tau)d\tau\right)^{\frac{q(p-1)}{p-q}}\tilde{u}^{-p'}(t)dt\right)^{\frac{\theta(p-q)}{q(p-\theta)}}\right.$$

$$\left.\times\left(\int_\beta^\infty w^\theta(r)dr\right)^{\frac{p}{p-\theta}}w^\theta(\beta)d\beta\right)^{\frac{q(p-\theta)}{p\theta}}=C_2^q,$$

and the equivalency constants depend only p,q and θ. Hence the statement of Theorem 2.7 follows by equality (2.5). □

4. Corollaries of the main results

As noted in Introduction in [7] the problem of boundedness from $LM_{p_1\theta_1,w_1}$ to $LM_{p_2\theta_2,w_2}$ was studied for the Hardy-type operator H_α, where $\alpha\in\mathbb{R}$, defined by

$$(H_\alpha f)(x)=\frac{1}{|B_{|x|}|^{1-\frac{\alpha}{n}}}\int_{B_{|x|}}f(y)dy,\ x\in\mathbb{R}^n.$$

Since $H_\alpha \equiv H_{n,\varphi}$, where $\varphi(r) = v_n^{\frac{\alpha}{n}-1} r^{\alpha-n}$ and v_n is the volume of the unit ball in $\mathbb{R}^n$, Theorems 2.3, 2.6 and 2.7 are applicable to the operator H_α. We formulate the appropriate corollary for the case of Theorem 2.3, the other cases being similar.

COROLLARY 4.1. *Let* $1 < p_1 \leqslant p_2, \theta_2 < \infty$ *and* $\alpha \in \mathbb{R}$. *Then the operator* H_α *is bounded from* $LM_{p_1 p_1, w_1}$ *to* $LM_{p_2\theta_2,w_2}$ *if and only if*

$$D_1 = \sup_{\beta>0} \left(\int_\beta^\infty w_2^{\theta_2}(r) \left(\int_\beta^r s^{(\alpha-n)p_2+n-1} ds \right)^{\frac{\theta_2}{p_2}} dr \right)^{\frac{1}{\theta_2}} \left(\int_0^\beta \|w_1\|_{L_{p_1}(r,\infty)}^{-p_1'} r^{n-1} dr \right)^{\frac{1}{p_1'}} < \infty.$$

If $\alpha > \frac{n}{p_2'}$, then

$$D_1 \leqslant ((\alpha-n)p_2+n)^{-\frac{1}{p_2}} n^{-\frac{1}{p_1'}} E_1,$$

where

$$E_1 = \sup_{\beta>0} \|w_2(r) r^{\alpha-n(\frac{1}{p_1}-\frac{1}{p_2})}\|_{L_{\theta_2}(\beta,\infty)} \|w_1\|_{L_{p_1}(\beta,\infty)}^{-1}.$$

Hence the condition $E_1 < \infty$ is sufficient for the boundedness of H_α from $LM_{p_1p_1,w_1}$ to $LM_{p_2\theta_2,w_2}$ if $1 < p_1 \leqslant p_2, \theta_2 < \infty$ and $\alpha > \frac{n}{p_2'}$.

This condition is similar for the case $\theta_1 = p_1$ to the sufficient condition obtained in [7]. In that paper it is proved that if $1 \leqslant p_1 \leqslant \infty$, $0 < p_2, \theta_1, \theta_2 \leqslant \infty$, $\theta_1 \leqslant \theta_2$ and

$$\alpha \geqslant n\left(\frac{1}{p_1} - \frac{1}{p_2}\right) \text{ if } 1 < p_1 \leqslant p_2 < \infty \text{ or } p_1 = 1 \text{ and } p_2 = \infty \tag{4.1}$$

or

$$\alpha > n\left(\frac{1}{p_1} - \frac{1}{p_2}\right) \text{ if } p_1 = 1 \leqslant p_2 < \infty \text{ or } 0 < p_2 < p_1 \leqslant \infty, \tag{4.2}$$

then the condition

$$F_1 = \sup_{\beta>0} \|w_2(r) r^{\alpha-n(\frac{1}{p_1}-\frac{1}{p_2})}\|_{L_{\theta_2}(\beta,\infty)} \|w_1\|_{L_{\theta_1}(\beta,\infty)}^{-1} < \infty$$

is sufficient for the boundedness of H_α from $LM_{p_1\theta_1,w_1}$ to $LM_{p_2\theta_2,w_2}$. Moreover, under certain regularity assumption on the function w_2 (formulas (60) and (61) in [7]), the condition $F_1 < \infty$ is necessary and sufficient for the boundedness of H_α from $LM_{p_1\theta_1,w_1}$ to $LM_{p_2\theta_2,w_2}$.

Let us consider in more detail the case of power-type local Morrey space $LM_{p\theta}^\lambda$, where $0 < p, \theta \leqslant \infty$, $\lambda > 0$ if $\theta < \infty$ and $\lambda \geqslant 0$ if $\theta = \infty$, the spaces of all functions $f \in L_p^{loc}(\mathbb{R}^n)$ for which

$$\|f\|_{LM_{p\theta}^\lambda} = \left(\int_0^\infty \left(\frac{\|f\|_{L_p(B_r)}}{r^\lambda} \right)^\theta \frac{dr}{r} \right)^{\frac{1}{\theta}} < \infty$$

if $\theta < \infty$ and

$$\|f\|_{LM^{\lambda}_{p\infty}} = \sup_{r>0} \frac{\|f\|_{L_p(B_r)}}{r^{\lambda}} < \infty$$

if $\theta = \infty$. (If $\lambda \leqslant 0$ for $\theta < \infty$ or $\lambda < 0$ for $\theta = \infty$, then these spaces are trivial.) Clearly, $LM^{\lambda}_{p\theta} = LM_{p\theta,w}$ with $w(r) = r^{-\lambda-\frac{1}{\theta}}$ (hence $\|w\|_{L_\theta(r,\infty)} = (\lambda\theta)^{-\frac{1}{\theta}} r^{-\lambda}$).

First of all we note that the condition

$$\alpha = \lambda_2 - \lambda_1 + n\left(\frac{1}{p_1} - \frac{1}{p_2}\right) \tag{4.3}$$

is necessary for the boundedness of H_α from $LM^{\lambda_1}_{p_1\theta_1}$ to $LM^{\lambda_2}_{p_2\theta_2}$. This follows by the 'dilation' argument (see [7, Remark 9] for details.) The non-triviality assumptions on λ_1 and λ_2 imply that

$$\alpha < \lambda_2 + n\left(\frac{1}{p_1} - \frac{1}{p_2}\right) \text{ if } \theta_1 < \infty,\ \alpha \leqslant \lambda_2 + n\left(\frac{1}{p_1} - \frac{1}{p_2}\right) \text{ if } \theta_1 = \infty \tag{4.4}$$

and

$$\alpha > -\lambda_1 + n\left(\frac{1}{p_1} - \frac{1}{p_2}\right) \text{ if } \theta_2 < \infty,\ \alpha \geqslant -\lambda_1 + n\left(\frac{1}{p_1} - \frac{1}{p_2}\right) \text{ if } \theta_2 = \infty. \tag{4.5}$$

COROLLARY 4.2. *Let* $1 < p_1 \leqslant p_2, \theta_2 < \infty,\ \lambda_1, \lambda_2 > 0$ *and* $\alpha \in \mathbb{R}$. *Then the operator* H_α *is bounded from* $LM^{\lambda_1}_{p_1 p_1}$ *to* $LM^{\lambda_2}_{p_2\theta_2}$ *if and only if equality* (4.3) *holds.*

Proof. Follows by Corollary 4.1 if one takes into account inequalities (4.4). Indeed, it suffices to prove that equality (4.3) implies that $D_1 < \infty$. This follows since if $\alpha > \frac{n}{p_2'}$, then by (4.4) and (4.3)

$$D_1 \leqslant c \sup_{\beta>0} \left(\int_\beta^\infty r^{(-\lambda_2+\alpha-\frac{n}{p_2'})\theta_2-1} dr\right)^{\frac{1}{\theta_2}} \beta^{\lambda_1+\frac{n}{p_1'}} < \infty,$$

where $c = (\lambda_1 p_1)^{-\frac{1}{p_1}} (\lambda_1+n)^{-\frac{1}{p_1'}} |(\alpha-n)p_2+n|^{-\frac{1}{p_2}}$, if $\alpha < \frac{n}{p_2'}$, then by (4.3)

$$D_1 \leqslant c \sup_{\beta>0} \left(\int_\beta^\infty r^{-\lambda_2\theta_2-1} dr\right)^{\frac{1}{\theta_2}} \beta^{\alpha-\frac{n}{p_2'}+\lambda_1+\frac{n}{p_1'}} < \infty,$$

and if $\alpha = \frac{n}{p_2'}$, then by (4.3)

$$\begin{aligned} D_1 &= (\lambda_1 p_1)^{-\frac{1}{p_1}} (\lambda_1+n)^{-\frac{1}{p_1'}} \sup_{\beta>0} \left(\int_\beta^\infty r^{-\lambda_2\theta_2-1} \left(\ln\frac{r}{\beta}\right)^{\frac{\theta_2}{p_2}} dr\right)^{\frac{1}{\theta_2}} \beta^{\lambda_1+\frac{n}{p_1'}} \\ &= (\lambda_1 p_1)^{-\frac{1}{p_1}} (\lambda_1+n)^{-\frac{1}{p_1'}} \left(\int_1^\infty t^{-\lambda_2\theta_2-1} (\ln t)^{\frac{\theta_2}{p_2}} dt\right)^{\frac{1}{\theta_2}} \sup_{\beta>0} \beta^{-\lambda_2+\lambda_1+\frac{n}{p_1'}} < \infty. \quad \square \end{aligned}$$

For comparison, in [7] for a wider range of the parameters $p_1, p_2, \theta_1, \theta_2, \lambda_1, \lambda_2$, namely

$$1 \leqslant p_1 \leqslant \infty; 0 < p_2, \theta_1, \theta_2 \leqslant \infty; \lambda_i > 0 \text{ if } \theta_i < \infty,\ \lambda_i \geqslant 0 \text{ if } \theta_i = \infty (i = 1, 2), \qquad (4.6)$$

and

$$\theta_1 \leqslant \theta_2, \qquad (4.7)$$

it is proved that H_α is bounded from $LM^{\lambda_1}_{p_1\theta_1}$ to $LM^{\lambda_2}_{p_2\theta_2}$ if and only if equality (4.3) holds, but under the additional assumption that conditions (4.1) — (4.2) are satisfied. (These additional conditions are necessary for application of the method used in [7] which is based on proving first the boundedness of H_α from $L_{p_1}(B_r)$ to $L_{p_2}(B_r)$.)

Conditions (4.6) are maximal admissible assumptions on the parameters $p_1, p_2, \theta_1, \theta_2$ and λ_1, λ_2. As for inequality (4.7) it is likely that it is also a necessary condition for the boundedness of H_α from $LM^{\lambda_1}_{p_1\theta_1}$ to $LM^{\lambda_2}_{p_2\theta_2}$. If $\theta_1 = p_1$ and $\theta_2 = p_2$ this is true. (See [7, Remark 11].) If $\theta_1 = p_1$, $0 < p_2 < \theta_2 < \infty$ with $p_2 \geqslant 1$, or $p_2 < 1$ and $\theta > 1$, this follows by Theorem 2.7 because in the case $\theta_2 < \theta_1 = p_1$ $C_1 = \infty$.

So the conjecture is that under natural assumptions (4.6) the operator H_α is bounded from $LM^{\lambda_1}_{p_1\theta_1}$ to $LM^{\lambda_2}_{p_2\theta_2}$ if only if both inequality (4.7) and equality (4.3) are satisfied.

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Victor Burenkov
Faculty of Mechanics and Mathematics
L. N. Gumilyov Eurasian National University
5 Munaitpasov St
010008 Astana, Kazakhstan
e-mail: burenkov@cf.ac.uk

Ryskul Oinarov
Faculty of Mechanics and Mathematics
L. N. Gumilyov Eurasian National University
5 Munaitpasov St
010008 Astana, Kazakhstan
e-mail: o_ryskul@mail.ru