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POTENTIAL INEQUALITY REVISITED II: EQUALITY CASE AND HARDY TYPE INEQUALITIES

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Abstract. A detailed analysis of potential inequality from [6] and [1] is presented. Special attention is given to the equality case. This enables us to obtain some improvements and generalizations of classical Hardy's inequality.

1. Introduction

Potential inequality is a very general inequality which generates many important inequalities as special cases. Authors proved potential inequality in [1] for kernels and functions that satisfy the maximum principle (see below).

In this paper the equality case will be discussed in detail. Further, improvements and generalizations of Hardy type inequalities will be derived.

Let us start by introducing notation and the setup, see [1] for details. We say that $N(x,dy)$ is a *(positive) kernel* on X if $N : X\times\mathscr{B}(X)\to[0,+\infty]$ is a mapping such that, for every $x\in X$, $A\mapsto N(x,A)$ is a σ-finite measure, and, for every $A\in\mathscr{B}(X)$, $x\mapsto N(x,A)$ is a measurable function. For a measurable function f, the *potential of f with respect to N at a point $x\in X$* is

$$(Nf)(x)=\int_X f(y)N(x,dy),$$

whenever the integral exists. The class of functions that have the potential at every point is denoted by $\mathscr{POT}(N)$.

For a measure μ on $(X,\mathscr{B}(X))$ and a measurable set $C\in\mathscr{B}(X)$ we will denote by $\hat{N}_C\mu$ the measure defined by

$$(\hat{N}_C\mu)(dy)=\int_C N(x,dy)\mu(dx).$$

If $C=X$ we will omit the subscript, i. e. $\hat{N}\mu$ will denote the measure $\hat{N}_X\mu$.

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DEFINITION 1. Let N be a positive kernel on X and $\mathscr{R} \subset \mathscr{POT}(N)$. We say that N *satisfies the strong maximum principle on* $\mathscr{R}$ (with constant $M \geqslant 1$) if

$$(Nf)(x) \leqslant Mu + N[f^+ \mathbf{1}_{\{(Nf)\geqslant u\}}](x) \tag{1}$$

holds for every $x \in X$, $f \in \mathscr{R}$ and $u \geqslant 0$.

□

In [1, Theorem 5] the potential inequality is proved for the standard class of convex and concave functions on $(0,+\infty)$:

THEOREM 1. (The revised potential inequality) *Let $N(x,dy)$ be a positive kernel on X which satisfies the strong maximum principle on $\mathscr{R}$ with constant M. Let $f \in \mathscr{R}$, $x \in X$ and $z > 0$ be such that $z \leqslant (Nf)(x)/M$ and denote by B_z the set*

$$B_z = \{y \in X : (Nf)(y) \geqslant z\}.$$

Then, for a convex function $\Phi : (0,+\infty) \to \mathbb{R}$, the following inequality holds

$$\begin{aligned}\Phi\Big(\frac{1}{M}(Nf)(x)\Big) - \Phi(z) \leqslant{} & \frac{1}{M} N[f^+ \varphi(Nf)\mathbf{1}_{B_z}](x) \\ & + \frac{1}{M}\varphi(z) N[f - f^+ \mathbf{1}_{B_z}](x) - z\varphi(z).\end{aligned}$$

If Φ is a concave function, then the above inequality is reversed.

In the sequel, we shall also need the following result.

THEOREM 2. [1, Theorem 15] *Under the assumptions of Theorem 1, if f is non-negative and $\lim_{z\to 0} z\varphi(z) = 0$, then*

$$\Phi\Big(\frac{1}{M}(Nf)(x)\Big) - \Phi(0+) \leqslant \frac{1}{M} N[f\varphi(Nf)\mathbf{1}_B](x).$$

Furthermore, for a finite measure μ on $(X,\mathscr{B}(X))$, the following inequality holds

$$\int_B \Phi\Big(\frac{1}{M}(Nf)(x)\Big)\mu(dx) - \Phi(0+)\mu(B) \leqslant \frac{1}{M}\int_B f(x)\varphi((Nf)(x))(\hat{N}_B\mu)(dx).$$

2. Equality Case

In this section we will give a set of sufficient conditions on the kernel $N(x,dy)$ and the function $f \in \mathscr{R}$ under which the potential inequality from Theorem 1 holds with equality, not just for convex functions, but for a more general class of functions $\mathscr{K}$ given in Definition 2 below. Using these results, Hardy's inequality will be generalized.

DEFINITION 2. A function Φ belongs to $\mathscr{K}$ if there exists a function φ of locally bounded total variation such that $\Phi(t) = \int_0^t \varphi(u)\,du$ for every $t > 0$.

Due to the properties of φ, one can easily derive the following two properties: (i) a function $\Phi \in \mathscr{K}$ is absolutely continuous and, thus, satisfies the fundamental theorem of integral calculus (see [3], p. 286) and (ii) the measure $d\varphi$ generated by φ is a well-defined signed measure. These two properties guarantee that the key step in the proof of Theorem 1, the integration by parts given in [1, equation (1)], is still valid for a function $\Phi \in \mathscr{K}$. Hence, we can state and prove the following theorem

THEOREM 3. (The Potential equality) *Let $\Phi \in \mathscr{K}$. If a positive kernel $N(x,dy)$ on X, which satisfies the strong maximum principle on $\mathscr{R}$ with constant M, and $f \in \mathscr{R}$ satisfy the following two properties:*

(i) *for every $x \in X$ and u such that $0 \leqslant u \leqslant \frac{1}{M}(Nf)(x)$, the maximum principle holds with equality, i. e.*

$$(Nf)(x) = Mu + N[f^+\mathbf{1}_{\{Nf \geqslant u\}}](x), \quad \textit{for} \quad 0 \leqslant Mu \leqslant (Nf)(x)$$

(ii) *for every $x \in X$ we have $N(x, B_{\tau(x)}) = 0$, where $\tau(x) = \frac{1}{M}(Nf)(x)$,*

then the potential inequality from Theorem 1 holds with equality, i. e.

$$\Phi\Big(\frac{1}{M}(Nf)(x)\Big) - \Phi(z) = \frac{1}{M}N[f^+\varphi(Nf)\mathbf{1}_{B_z}](x) + \frac{1}{M}\varphi(z)N[f - f^+\mathbf{1}_{B_z}](x) - z\varphi(z),$$

for every $x \in X$ and $z > 0$ such that $z \leqslant (Nf)(x)/M$.

Proof. Inspection of the proof of Theorem 1 given in [1, Theorem 5] shows that sign of inequality is used in two steps. Under the property (i), the first inequality holds with equality, i. e.

$$\int_z^{\tau(x)} (\tau(x) - u)\,d\varphi(u) = \int_z^{\tau(x)} \frac{1}{M}N[f^+\mathbf{1}_{\{Nf \geqslant u\}}]\,d\varphi(u).$$

Under the property (ii), the measure $N(x,\cdot)$ is identically equal to zero on the set $B_{\tau(x)} = \{y \in X : (Nf)(y) \geqslant \tau(x)\}$, so

$$\int_X N(x,dy)\left[f^+(y)\int_z^{\tau(x)} \mathbf{1}_{\{(Nf)\geqslant u\}}(y)\,d\varphi(u)\right] = \int_X N(x,dy)\left[f^+(y)\int_z^{+\infty} \mathbf{1}_{\{(Nf)\geqslant u\}}(y)\,d\varphi(u)\right],$$

i. e., the second inequality in the proof of Theorem 1 also holds with equality. □

REMARK 4. If a kernel $N(x,dy)$ satisfies the assumptions (i) and (ii) from Theorem 3 and if the condition that Φ is convex is loosened to $\Phi \in \mathscr{K}$, then the inequalities in Corollaries 6, 9 and 12 and Theorems 11 and 15 of [1] hold with equality. The same is true for the first inequalities in Corollaries 8, 10, 13 and 16 of [1].

The following lemma gives sufficient conditions under which a kernel N satisfies the assumptions of Theorem 3 for all nonnegative functions f.

LEMMA 5. *Let a kernel N on X and a nonnegative $f : X \to [0,+\infty)$ satisfy the following three properties*

(i) *for every $x_1, x_2 \in X$, the measures $N(x_1,\cdot)$ and $N(x_2,\cdot)$ are identical on the set B_t^c, where $t = \min\big((Nf)(x_1),(Nf)(x_2)\big)$, i. e.*

$$N(x_1, C \cap B_t^c) = N(x_2, C \cap B_t^c), \qquad \textit{for every } C \in \mathscr{B}(X)$$

(ii) $N(x,B_t) = 0$ *for every $x \in X$, where $t = (Nf)(x)$*

(iii) *The range $\{(Nf)(x) : x \in X\}$ is a connected set in $\mathbb{R}$ containing 0.*

Then, the kernel N satisfies the maximum principle for the function f with constant $M = 1$ and the potential equality from Theorem 3 holds for every $\Phi \in \mathscr{K}$.

Proof. The inequality from the definition of the maximum principle holds trivially for $u > (Nf)(x)$. On the other hand, for $u \leqslant (Nf)(x)$, nonnegative f and $M = 1$, the inequality from the definition of the maximum principle is equivalent to

$$\int_X f(y)\mathbf{1}_{\{(Nf)<u\}}(y)N(x,dy) \leqslant u. \tag{2}$$

By property (iii), there exists $\tilde{x}$ such that $u = (Nf)(\tilde{x})$. By property (ii) we have

$$u = \int_X f(y)N(\tilde{x},dy) = \int_X f(y)\mathbf{1}_{\{(Nf)<u\}}(y)N(\tilde{x},dy),$$

while by property (i) we have

$$\int_X f(y)\mathbf{1}_{\{(Nf)<u\}}(y)N(\tilde{x},dy) = \int_X f(y)\mathbf{1}_{\{(Nf)<u\}}(y)N(x,dy).$$

Combining the last two equalities, we see that (2) holds with equality. Therefore, the property (i) from Theorem 3 (with $M = 1$) holds, while the property (ii) from that theorem with $M = 1$ is equivalent to the property (ii) of this lemma. Hence, the assumptions of Theorem 3 are satisfied, so the potential equality holds. □

3. Applications to Hardy-type inequalities

Hardy's inequality states that for nonnegative f and $p > 1$ we have

$$\left[\int_0^{+\infty}\left(\frac{1}{x}\int_0^x f(y)dy\right)^p dx\right]^{\frac{1}{p}} \leqslant \frac{p}{p-1}\left[\int_0^{+\infty} f^p(x)dx\right]^{\frac{1}{p}}. \tag{3}$$

To work in our setup, we will set $X=(0,+\infty)$ and define the kernel N by its density

$$G(x,y)=\begin{cases}1, & \text{if } y\leqslant x,\\ 0, & \text{otherwise.}\end{cases} \tag{4}$$

Notice that

$$F(x)=(Nf)(x)=\int_0^x f(y)dy$$

and that the kernel N satisfies the conditions in Lemma 5.

THEOREM 6. *Let f be nonnegative, $F(x)=\int_0^x f(y)dy$, $p\in\mathbb{R}\setminus\{1\}$, $q=p/(p-1)$, $\Phi\in\mathscr{K}$, $\Phi(0+)=0$, $\lim_{z\to 0} z\varphi(z)=0$ and let $\nu_i(dx)=\lambda_i(x)dx$, $i=1,2$, be two σ-finite measures with densities λ_i that satisfy*

$$\lambda_2(x)=\lambda_1^{1-p}(x)\Big[\int_x^{+\infty}\lambda_1(y)dy\Big]^p. \tag{5}$$

Then

$$\int_B \Phi(F(x))\lambda_1(x)dx=\int_B f(x)\varphi(F(x))\lambda_1^{1/q}(x)\lambda_2^{1/p}(x)dx,$$

where

$$B=(b,+\infty),\qquad \textit{for } b=\operatorname{ess\,inf}\{y:f(y)>0\}.$$

Proof. By the assumptions on the kernel N, the function f and the set B we see that $B=\{x\in(0,+\infty):(Nf)(x)>0\}$.

The kernel N satisfies the conditions in Lemma 5 and, taking into account Remark 4, we see that the second inequality from Theorem 2 holds with equality. Since $\Phi(0+)=0$ this is equivalent to

$$\int_B \Phi(F(x))\nu_1(dx)=\int_B f(x)\varphi(F(x))(\hat{N}_B\nu_1)(dx). \tag{6}$$

Notice that

$$(\hat{N}_B\nu_1)(dx)=\int_b^{+\infty}\Big[\int_0^y dx\Big]\lambda_1(y)dy=\int_0^{+\infty}\Big[\int_{\max(b,x)}^{+\infty}\lambda_1(y)dy\Big]dx,$$

i. e.

$$d(\hat{N}_B\nu_1)(x)=\int_{\max(b,x)}^{+\infty}\lambda_1(y)dy.$$

Due to (5), we see that for $x\in B$

$$d(\hat{N}_B\nu_1)(x)=\lambda_1^{1/q}(x)\lambda_2^{1/p}(x).$$

Plugging the last equality into (6) finishes the proof. □

COROLLARY 7. *Let f, F, B and ν_i, $i=1,2$, be as in Theorem 6. Then, for $p>1$ the following inequality holds*

$$\left[\int_B F^p(x)\lambda_1(x)dx\right]^{\frac{1}{p}} \leqslant p\left[\int_B f^p(x)\lambda_2(x)dx\right]^{\frac{1}{p}},$$

while for $0<p<1$ the inequality is reversed.

Proof. Applying Theorem 6 with $\Phi(\tau)=\tau^p$ for $p>0$ we get

$$\int_B F^p(x)\lambda_1(x)dx = p\int_B f(x)\lambda_2^{1/p}(x)F^{p-1}(x)\lambda_1^{1/q}(x)dx.$$

Finally, applying Hölder's inequality on the right hand side integral for functions $f\lambda_2^{1/p}$ and $F^{p-1}\lambda_1^{1/q}=(F^p\lambda_1)^{1/q}$ and multiplying the inequality by $\left[\int_B F^p(x)\lambda_1(x)dx\right]^{-1/q}$ we get the claim of the corollary. □

Similarly, let the kernel N be defined on the set $X=(0,+\infty)$ by its density

$$G(x,y)=\begin{cases}1, & \text{if } x<y,\\ 0, & \text{otherwise.}\end{cases} \tag{7}$$

Then

$$F(x)=(Nf)(x)=\int_x^{+\infty} f(y)dy$$

and that the kernel N also satisfies the conditions in Lemma 5.

THEOREM 8. *Let f be nonnegative, $F(x)=\int_x^{+\infty} f(y)dy$, $p\in\mathbb{R}\backslash\{1\}$, $q=p/(p-1)$, $\Phi\in\mathscr{K}$, $\Phi(0+)=0$, $\lim_{z\to 0} z\varphi(z)=0$ and let $\nu_i(dx)=\lambda_i(x)dx$, $i=1,2$, be two σ-finite measures with densities λ_i that satisfy*

$$\lambda_2(x)=\lambda_1^{1-p}(x)\left[\int_0^x \lambda_1(y)dy\right]^p. \tag{8}$$

Then

$$\int_B \Phi(F(x))\lambda_1(x)dx=\int_B f(x)\varphi(F(x))\lambda_1^{1/q}(x)\lambda_2^{1/p}(x)dx,$$

where

$$B=(0,b), \qquad \textit{for } b=\operatorname{ess\,sup}\{y: f(y)>0\}.$$

Proof. By the assumptions on the kernel N, the function f and the set B we see that $B=\{x\in(0,+\infty):(Nf)(x)>0\}$.

The kernel N satisfies Lemma 5 and, taking into account Remark 4, we see that the second inequality from Theorem 2 holds with equality. Since $\Phi(0+)=0$ this is equivalent to

$$\int_B \Phi(F(x))\nu_1(dx)=\int_B f(x)\varphi(F(x))(\hat{N}_B\nu_1)(dx). \tag{9}$$

Notice that

$$(\hat{N}_B \nu_1)(dx) = \int_0^b \Big[\int_y^{+\infty} dx\Big] \lambda_1(y)dy = \int_0^{+\infty} \Big[\int_0^{\min(b,x)} \lambda_1(y)dy\Big] dx,$$

i. e. $d(\hat{N}_B \nu_1)(x) = \int_0^{\min(b,x)} \lambda_1(y)dy$. Due to (8), we see that for $x \in B$

$$d(\hat{N}_B \nu_1)(x) = \lambda_1^{1/q}(x)\lambda_2^{1/p}(x).$$

Plugging the last equality into (9) finishes the proof. □

COROLLARY 9. *Let f, F, B and ν_i, $i = 1,2$, be as in Theorem 8. Then, for $p > 1$ the following inequality holds*

$$\left[\int_B F^p(x)\lambda_1(x)dx\right]^{\frac{1}{p}} \leqslant p\left[\int_B f^p(x)\lambda_2(x)dx\right]^{\frac{1}{p}},$$

while for $0 < p < 1$ the inequality is reversed.

Proof. Applying Theorem 8 with $\Phi(\tau) = \tau^p$ for $p > 0$ we get

$$\int_B F^p(x)\lambda_1(x)dx = p\int_B f(x)\lambda_2^{1/p}(x)F^{p-1}(x)\lambda_1^{1/q}(x)dx.$$

Finally, applying Hölder's inequality on the right hand side integral for functions $f\lambda_2^{1/p}$ and $F^{p-1}\lambda_1^{1/q} = (F^p\lambda_1)^{1/q}$ and multiplying the inequality by $\big[\int_B F^p(x)\lambda_1(x)dx\big]^{-1/q}$ we get the claim of the corollary. □

COROLLARY 10. *Let f be nonnegative, $p > 0$, $p \neq 1$ and $k \neq 1$. Define F by*

$$F(x) = \begin{cases} \int_0^x f(y)dy, & k > 1, \\ \int_x^{+\infty} f(y)dy, & k < 1. \end{cases} \tag{10}$$

Then, for $p > 1$ the following inequality holds

$$\left[\int_0^{+\infty} x^{-k}F^p(x)dx\right]^{\frac{1}{p}} \leqslant \frac{p}{|k-1|}\left[\int_0^{+\infty} x^{p-k}f^p(x)dx\right]^{\frac{1}{p}},$$

while for $0 < p < 1$ the inequality is reversed.

Proof. Since, for any choice of p and k, we have $f(x) = F(x) = 0$ for a.e. $x \notin B$, the integrals over $(0,+\infty)$ can be replaced with integrals over B. Then the inequality for $k > 1$ follows from Corollary 7, while for $k < 1$ from Corollary 9, by taking

$$\lambda_1(x) = x^{-k} \qquad \text{and} \qquad \lambda_2(x) = \frac{x^{p-k}}{|k-1|^p}. \qquad \square$$

The last corollary is a well-known result (see [2]) and it is a generalization of Hardy's inequality. Indeed, by taking $k = p > 1$, $\lambda_1(x) = x^{-p}$ and $\lambda_2(x) = 1$ we get inequality (3).

We will end this section with an elementary proof of the potential equality from Theorem 6, but with additional assumptions that $f \in L^1$ and that $\nu_1(dx) = \lambda_1(x)dx$ is a finite measure. Under the same additional assumptions, the dual potential equality from Theorem 8 can be proven in an analogous way.

Proof of Theorem 6. We have

$$\frac{d}{dx}\Phi(F(x)) = \varphi(F(x))f(x)$$

and condition (5) can be restated as

$$\lambda_1^{1/q}(x)\lambda_2^{1/p}(x) = \int_x^{+\infty} \lambda_1(y)dy, \tag{11}$$

so the integration by parts gives

$$\int_b^{+\infty} \Phi(F(x))\lambda_1(x)dx = \\ = -\Phi(F(x)) \int_x^{+\infty} \lambda_1(y)dy \Big|_{x=b}^{x=+\infty} + \int_b^{+\infty} f(x)\varphi(F(x))\Big(\int_x^{+\infty} \lambda_1(y)dy\Big)dx \tag{12}$$

Notice that, due to the additional assumptions, we have

$$\lim_{x\to+\infty} F(x) = \int_0^{+\infty} f(y)dy < +\infty,$$

so $\lim_{x\to+\infty}\Phi(F(x))$ is finite, while $\lim_{x\to b}\Phi(F(x)) = \Phi(0+) = 0$ due to the definition of b and properties of Φ.

On the other hand, since ν_1 is a finite measure, the integral $\int_b^{+\infty}\lambda_1(y)dy$ is finite and

$$\lim_{x\to+\infty}\int_x^{+\infty} \lambda_1(y)dy = 0,$$

so the first term on the right hand side of equality (12) vanishes.

Taking into account (11), we see that the second term on the right hand side of equality (12) is equal to

$$\int_B f(x)\varphi(F(x))\lambda_1^{1/q}(x)\lambda_2^{1/p}(x)dx,$$

so the equality from Theorem 6 holds. □

4. n-dimensional Hardy-type inequality

Hardy's inequality compares the L^p norm of the "averages" $\frac{1}{x}\int_0^x f(y)dy$ to the L^p norm of the function itself. We can generalize this to n dimensions by looking at the averages of a function over the n-dimensional balls $B_n(0,\|x\|)$ around the origin. To derive the results, we will look at $X=\mathbb{R}^n$ and the kernel N with the density

$$G(x,y)=\begin{cases}1, & \|y\|\leqslant\|x\|,\\ 0, & \text{otherwise.}\end{cases}$$

The kernel N satisfies the conditions of Lemma 5 and we have

$$F(x)=(Nf)(x)=\int_{B_n(0,\|x\|)} f(y)dy.$$

THEOREM 11. *Let* $f:\mathbb{R}^n\to[0,+\infty)$,

$$F(x)=\int_{B_n(0,\|x\|)} f(y)dy,$$

$p\in\mathbb{R}\backslash\{1\}$, $q=p/(p-1)$ *and* $\Phi\in\mathscr{K}$ *such that* $\Phi(0+)=0$ *and* $\lim_{z\to 0} z\varphi(z)=0$. *Let* $\nu_i(dx)=\lambda_i(x)dx$, $i=1,2$, *be two* σ*-finite measures on* $\mathbb{R}^n$ *with densities* λ_i *that satisfy*

$$\lambda_2(x)=\lambda_1^{1-p}(x)\Big[\int_{B_n(0,\|x\|)^c}\lambda_1(y)dy\Big]^p. \tag{13}$$

Then

$$\int_B \Phi(F(x))\lambda_1(x)dx=\int_B f(x)\varphi(F(x))\lambda_1^{1/q}(x)\lambda_2^{1/p}(x)dx,$$

where

$$B=B_n(0,b)^c, \qquad \textit{for } b=\sup\{b' : f(y)=0 \text{ for almost every } y\in B_n(0,b')\}.$$

Proof. By the assumptions on the kernel N, the function f and the set B we see that $B=\{x\in\mathbb{R}^n : (Nf)(x)>0\}$.

The kernel N satisfies Lemma 5 and, taking into account Remark 4, we see that the second inequality from Theorem 2 holds with equality. Since $\Phi(0+)=0$ this is equivalent to

$$\int_B \Phi(F(x))\nu_1(dx)=\int_B f(x)\varphi(F(x))(\hat{N}_B\nu_1)(dx). \tag{14}$$

Notice that

$$\begin{aligned}(\hat{N}_B\nu_1)(dx)&=\int_{B_n(0,b)^c}\Big[\int_{B_n(0,\|y\|)}dx\Big]\lambda_1(y)dy\\ &=\int_{\mathbb{R}^n}\Big[\int_{B_n(0,\max(b,\|x\|))^c}\lambda_1(y)dy\Big]dx,\end{aligned}$$

i. e. $d(\hat{N}_B \nu_1)(x) = \int_{B_n(0,\max(b,\|x\|))^c} \lambda_1(y)$. Due to (13), we see that for $x \in B$

$$d(\hat{N}_B \nu_1)(x) = \lambda_1^{1/q}(x)\lambda_2^{1/p}(x).$$

Plugging the last equality into (14) finishes the proof. □

COROLLARY 12. *Let f, F, B and ν_i, $i = 1,2$, be as in Theorem 11. Then, for $p > 1$ the following inequality holds*

$$\left[\int_B F^p(x)\lambda_1(x)dx\right]^{\frac{1}{p}} \leqslant p\left[\int_B f^p(x)\lambda_2(x)dx\right]^{\frac{1}{p}},$$

while for $0 < p < 1$ the inequality is reversed.

Proof. Applying Theorem 11 with $\Phi(\tau) = \tau^p$ for $p > 0$ we get

$$\int_B F^p(x)\lambda_1(x)dx = p\int_B f(x)\lambda_2^{1/p}(x)F^{p-1}(x)\lambda_1^{1/q}(x)dx.$$

Finally, applying Hölder's inequality on the right hand side integral to the functions $f\lambda_2^{1/p}$ and $F^{p-1}\lambda_1^{1/q} = (F^p\lambda_1)^{1/q}$, and multiplying the inequality by

$$\left[\int_B F^p(x)\lambda_1(x)dx\right]^{-1/q}$$

we get the claim of the corollary. □

Similarly, let the kernel N be defined on the set $X = \mathbb{R}^n$ by its density

$$G(x,y) = \begin{cases} 1, & \text{if } \|x\| < \|y\|, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$F(x) = (Nf)(x) = \int_{B_n(0,\|x\|)^c} f(y)dy.$$

and that the kernel N also satisfies the conditions of Lemma 5.

THEOREM 13. *Let $f : \mathbb{R}^n \to [0,+\infty)$,*

$$F(x) = \int_{B_n(0,\|x\|)^c} f(y)dy,$$

$p \in \mathbb{R}\backslash\{1\}$, $q = p/(p-1)$ and $\Phi \in \mathscr{K}$ such that $\Phi(0+) = 0$ and $\lim_{z\to 0} z\varphi(z) = 0$. Let $\nu_i(dx) = \lambda_i(x)dx$, $i = 1,2$, be two σ-finite measures on $\mathbb{R}^n$ with densities λ_i that satisfy

$$\lambda_2(x) = \lambda_1^{1-p}(x)\left[\int_{B_n(0,\|x\|)} \lambda_1(y)dy\right]^p. \tag{15}$$

Then

$$\int_B \Phi(F(x))\lambda_1(x)dx = \int_B f(x)\varphi(F(x))\lambda_1^{1/q}(x)\lambda_2^{1/p}(x)dx,$$

where

$$B = B_n(0,b), \qquad \textit{for } b = \inf\{b' : f(y) = 0 \text{ for almost every } y \in B_n(0,b')^c\}.$$

Proof. By the assumptions on the kernel N, the function f and the set B we see that $B = \{x \in \mathbb{R}^n : (Nf)(x) > 0\}$.

The kernel N satisfies Lemma 5 and, taking into account Remark 4, we see that the second inequality from Theorem 2 holds with equality. Since $\Phi(0+) = 0$ this is equivalent to

$$\int_B \Phi(F(x))\nu_1(dx) = \int_B f(x)\varphi(F(x))(\hat{N}_B\nu_1)(dx). \tag{16}$$

Notice that

$$(\hat{N}_B\nu_1)(dx) = \int_{B_n(0,b)} \left[\int_{B_n(0,\|y\|)^c} dx\right]\lambda_1(y)dy = \int_{\mathbb{R}^n} \left[\int_{B_n(0,\min(b,x))} \lambda_1(y)dy\right]dx,$$

i. e. $d(\hat{N}_B\nu_1)(x) = \int_{B_n(0,\min(b,x))} \lambda_1(y)dy$. Due to (15), we see that for $x \in B$

$$d(\hat{N}_B\nu_1)(x) = \lambda_1^{1/q}(x)\lambda_2^{1/p}(x).$$

Plugging the last equality into (16) finishes the proof. □

COROLLARY 14. *Let f, F, B and ν_i, $i = 1,2$, be as in Theorem 13. Then, for $p > 1$ the following inequality holds*

$$\left[\int_B F^p(x)\lambda_1(x)dx\right]^{\frac{1}{p}} \leqslant p\left[\int_B f^p(x)\lambda_2(x)dx\right]^{\frac{1}{p}},$$

while for $0 < p < 1$ the inequality is reversed.

Proof. Applying Theorem 11 with $\Phi(\tau) = \tau^p$ for $p > 0$ we get

$$\int_B F^p(x)\lambda_1(x)dx = p\int_B f(x)\lambda_2^{1/p}(x)F^{p-1}(x)\lambda_1^{1/q}(x)dx.$$

Finally, applying Hölder's inequality on the right hand side integral for functions $f\lambda_2^{1/p}$ and $F^{p-1}\lambda_1^{1/q} = (F^p\lambda_1)^{1/q}$ and multiplying the inequality by $\left[\int_B F^p(x)\lambda_1(x)dx\right]^{-1/q}$ we get the claim of the corollary. □

REMARK 15. If the densities of the measures ν_i, $i = 1,2$ are radial, i. e. $\nu_i(dx) = \tilde{\lambda}_i(\|x\|)dx$ with $\tilde{\lambda}_i : [0,+\infty) \to [0,+\infty)$, then the integrals in conditions (13) and (15) can be simplified by switching to the hyperspherical coordinates. Condition (13) is equivalent to

$$\tilde{\lambda}_2(r) = \omega_n^p \tilde{\lambda}_1^{1-p}(r)\Big[\int_r^{+\infty} \tilde{\lambda}_1(t)dt\Big]^p$$

holding for every $r \geqslant 0$, while condition (15) is equivalent to

$$\tilde{\lambda}_2(r) = \omega_n^p \tilde{\lambda}_1^{1-p}(r)\Big[\int_0^{r} \tilde{\lambda}_1(t)dt\Big]^p$$

holding for every $r \geqslant 0$, where ω_n is the area of the surface of the unit hypersphere S^{n-1} in $\mathbb{R}^n$.

COROLLARY 16. *Let* $f : \mathbb{R}^n \to [0,+\infty)$, $p > 0$, $p \neq 1$, $q = p/(p-1)$ *and* $k \neq 1$. *Define* F *by*

$$F(x) = \begin{cases} \int_{B_n(0,\|x\|)} f(y)dy, & k > 1, \\ \int_{B_n(0,\|x\|)^c} f(y)dy, & k < 1. \end{cases} \tag{17}$$

Then, for $p > 1$ *the following inequality holds*

$$\left[\int_{\mathbb{R}^n} \|x\|^{-nk} F^p(x)dx\right]^{\frac{1}{p}} \leqslant \frac{\omega_n p}{n|k-1|}\left[\int_{\mathbb{R}^n} \|x\|^{n(p-k)} f^p(x)dx\right]^{\frac{1}{p}},$$

while for $0 < p < 1$ *the inequality is reversed.*

Proof. Since, for any choice of p and k, we have $f(x) = F(x) = 0$ for a.e. $x \notin B$, the integrals over $\mathbb{R}^n$ can be replaced with integrals over B. Then, the inequality for $k > 1$ follows from Corollary 12, while for $k < 1$ from Corollary 14, by taking into account Remark 15 with

$$\tilde{\lambda}_1(x) = \|x\|^{-nk} \qquad \text{and} \qquad \tilde{\lambda}_2(x) = \frac{\omega_n^p \|x\|^{n(p-k)}}{n^p|k-1|^p}. \qquad \square$$

REMARK 17. Since the volume of the n-dimensional ball $B_n(0,\|x\|)$ satisfies

$$V(B_n(0,\|x\|)) = \|x\|^n \omega_n / n,$$

the inequality from Corollary 16 for $k = p > 1$ can be rewritten as

$$\left[\int_{\mathbb{R}^n} \left(\frac{1}{V(B_n(0,\|x\|))}\int_{B_n(0,\|x\|)} f(y)dy\right)^p dx\right]^{\frac{1}{p}} \leqslant \frac{p}{p-1}\left[\int_{\mathbb{R}^n} f^p(x)dx\right]^{\frac{1}{p}},$$

which can be interpreted as the n-dimensional analogon of Hardy's inequality.

As in the 1-dimensional case, we will end this section with a direct, elementary proof of the potential equality from Theorem 11, but with additional assumptions that $f \in L^1$ and that $\nu_1(dx) = \lambda_1(x)dx$ is a finite measure. Under the same additional assumptions, the dual potential equality from Theorem 13 can be proven in an analogous way.

In the proof we will make use of hyperspherical coordinates $(r,\theta) = (r,\theta_1,\dots,\theta_{n-1})$ in $\mathbb{R}^n$. Interchange between x and (r,θ) or use of the equation $x = (r,\theta)$ will mean that (r,θ) are the hyperspherical coordinates of a point $x \in \mathbb{R}^n$. Let us denote by

$$J(r,\theta) = r^{n-1}\sin^{n-2}(\theta_1)\sin^{n-3}(\theta_2)\cdots\sin(\theta_{n-2})$$

the Jacobian of the transformation from hyperspherical to Cartesian coordinates and let $S^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$ denote the unit sphere in $\mathbb{R}^n$.

Proof of Theorem 11. Notice that $F(x)$ depends only on the norm $\|x\|$, so we can interpret F as a function with the domain $[0,+\infty)$ and, in this proof, we will use both notations $F(x)$ and $F(\|x\|)$. With this understanding, we see that

$$F(r) = \int_{B_n(0,r)} f(y)dy = \int_0^r \int_{\theta\in S^{n-1}} f(\tilde{r},\theta)J(\tilde{r},\theta)d\theta d\tilde{r},$$

where the last integral is obtained by transferring to hyperspherical coordinates. Therefore,

$$F'(r) = \int_{\theta\in S^{n-1}} f(r,\theta)J(r,\theta)d\theta.$$

Let us introduce the functions g and G by

$$g(r) = \int_{\theta\in S^{n-1}} \lambda_1(r,\theta)J(r,\theta)d\theta$$

and

$$G(r) = \int_r^{+\infty} g(\tilde{r})d\tilde{r}.$$

By transferring to hyperspherical coordinates in the integral in condition (13), we see that that condition is equivalent to

$$\lambda_1^{1/q}(x)\lambda_2^{1/p}(x) = G(\|x\|). \tag{18}$$

By transferring to hyperspherical coordinates and using integration by parts we get

$$\begin{aligned}
&\int_{B_n(0,b)^c} \Phi(F(x))\lambda_1(x)dx = \\
&\quad= \int_b^{+\infty}\int_{\theta\in S^{n-1}} \Phi(F(r))\lambda_1(r,\theta)J(r,\theta)d\theta dr = \int_b^{+\infty} \Phi(F(r))g(r)dr \\
&\quad= -\Phi(F(r))G(r)\Big|_b^{+\infty} + \int_b^{+\infty} \varphi(F(r))\Big(\int_{\theta\in S^{n-1}} f(r,\theta)J(r,\theta)d\theta\Big)G(r)dr. \qquad (19)
\end{aligned}$$

Notice that, due to the additional assumptions, we have

$$\lim_{r\to+\infty} F(r) = \int_{B_n(0,r)} f(y)dy < +\infty,$$

so $\lim_{r\to+\infty}\Phi(F(r))$ is finite, while $\lim_{r\to b}\Phi(F(r)) = \Phi(0+) = 0$ due to the definition of b and properties of Φ.

On the other hand, notice that $G(r) = \nu_1(B_n(0,r)^c)$, so, since ν_1 is a finite measure, $G(b)$ is finite and $\lim_{r\to+\infty} G(r) = 0$, so the first term on the right hand side of equality (19) vanishes.

By transferring back to Cartesian coordinates and taking into account (18), we see that the second term on the right hand side of equality (19) is equal to

$$\int_b^{+\infty}\int_{\theta\in S^{n-1}} f(r,\theta)\varphi(F(r))G(r)J(r,\theta)d\theta dr$$
$$= \int_{B_n(0,b)^c} f(x)\varphi(F(x))\lambda_1^{1/p}(x)\lambda_2^{1/q}(x)J(x)J^{-1}(x)dx.$$

The Jacobians J and J^{-1} cancel out, so we see that the equality from Theorem 11 holds. $\square$

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