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ON LYAPUNOV–TYPE INEQUALITY FOR A CLASS OF NONLINEAR SYSTEMS

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Abstract. In this paper, we will try to find a new Lyapunov-type inequality for a class of nonlinear systems, special cases of which contain some well-known Hamiltonian system, Emden-Fowler, half-linear and linear differential equations of second order. Our result extends the Lyapunov-type inequality given in [X. Wang, Stability criteria for linear periodic Hamiltonian systems, J. Math. Anal. Appl. 367 (2010), 329-336.].

1. Introduction

The Lyapunov inequality and many of its generalizations have proved to be useful tools in oscillation theory, disconjugacy, eigenvalue problems, boundary value problems and numerous other applications for the theories of differential and difference equations. Before we continue the description of the content of this paper, a few hints concerning the literature on the Lyapunov-type inequalities might be in order.

In 1949, Lyapunov [15] obtained the following result.

THEOREM A. *If $x(t)$ is a solution of*

$$x'' + q(t)x = 0 \tag{1}$$

with $x(a) = 0 = x(b)$ where $a, b \in \mathbb{R}$ with $a < b$ be consecutive zeros and $x(t) \neq 0$ for $t \in (a,b)$, then the so-called Lyapunov inequality

$$\int_a^b |q(s)|\, ds > \frac{4}{b-a} \tag{2}$$

holds, and the constant 4 cannot be replaced by a large number.

As it was first noticed by Wintner [26], and subsequently by several other authors, application of Sturm Comparison Theorem allows the replacement of $|q(t)|$ in the inequality (2) by $q^+(t)$, where $q^+(t) = \max\{q(t), 0\}$ is the nonnegative part of $q(t)$.

In 1964, Hartman [11] generalized the classical Lyapunov inequality (2) for the linear differential equation

$$\big(r(t)x'\big)' + q(t)x = 0, \qquad r(t) > 0 \tag{3}$$

as follows.

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THEOREM B. *If $a,b \in \mathbb{R}$ with $a < b$ are consecutive zeros of nontrivial solution of equation (3), then*

$$\int_a^b q^+(s)ds > \frac{4}{\int_a^b r^{-1}(s)ds} \tag{4}$$

holds.

This inequality has been extended in many directions and its half-linear differential equation

$$\left(r(t)\left|x'\right|^{\alpha-2}x'\right)' + q(t)\left|x\right|^{\alpha-2}x = 0, \qquad r(t) > 0 \text{ and } \alpha > 1 \tag{5}$$

extension found in Došlý and Řehák's recent book [7] as follows.

THEOREM C. *Let $a,b \in \mathbb{R}$ with $a < b$ be consecutive zeros of nontrivial solution of equation (5). Then*

$$\int_a^b q^+(s)ds > \frac{2^\alpha}{\left(\int_a^b r^{1/(1-\alpha)}(s)ds\right)^{\alpha-1}} \tag{6}$$

holds.

Since the appearance of Lyapunov's fundamental paper [15], various proofs and generalizations or improvements have appeared in the literature. A thorough literature review of continuous and discrete Lyapunov inequalities and their applications can be found in the survey papers of Cheng [3] and Tiryaki [22]. For authors who contributed the Lyapunov-type inequalities, we refer to Brown and Hinton [1, 2], Çakmak [4], Eliason [8], Kwong [13], Lee et al. [14], Pachpatte [17, 18], Panigrahi [19], Parhi and Panigrahi [20], Yang et al. [27], Yang and Lo [29], and the references quoted therein.

The linear Hamiltonian system, in the case of two scalar linear differential equations, has the form

$$y' = JH(t)y \quad , \quad t \in \mathbb{R}, \tag{7}$$

where

$$y(t) = (y_1(t), y_2(t))^T, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad H(t) = \begin{pmatrix} h_{11}(t) & h_{12}(t) \\ h_{21}(t) & h_{22}(t) \end{pmatrix}$$

with $h_{jk}(t),\ j,k = 1,2$ are real-valued piece-wise continuous functions defined on $\mathbb{R}$ and $h_{12}(t) = h_{21}(t)$.

Setting $y_1(t) = x(t),\ y_2(t) = u(t),\ h_{11}(t) = \beta_2(t),\ h_{12}(t) = h_{21}(t) = \alpha_1(t)$ and $h_{22}(t) = \beta_1(t)$ in the system (7), one can easily obtains the following linear system

$$\left.\begin{aligned} x' &= \alpha_1(t)x + \beta_1(t)u \\ u' &= -\beta_2(t)x - \alpha_1(t)u \end{aligned}\right\} \tag{8}$$

which is a special case of the linear counterpart of the nonlinear system

$$\left.\begin{aligned} x' &= \alpha_1(t)x + \beta_1(t)\left|u\right|^{\gamma-2}u \\ u' &= -\beta_2(t)\left|x\right|^{\beta-2}x - \alpha_1(t)u \end{aligned}\right\} \tag{9}$$

with $\gamma = 2$ and $\beta = 2$. The special cases of nonlinear system (9) contain the well-known linear equation (3), half-linear equation (5) and Emden-Fowler equation

$$\left(r(t)\left|x'\right|^{\alpha-2}x'\right)' + q(t)\left|x\right|^{\beta-2}x = 0, \qquad r(t) > 0,\ \alpha > 1 \text{ and } \beta > 1, \tag{10}$$

where $r(t)$ and $q(t)$ are real-valued piece-wise continuous functions for all $t \in \mathbb{R}$.

Although there is an extensive literature on the Lyapunov-type inequalities for above mentioned linear and half-linear differential equations, there is not much done for linear system (8) and nonlinear system (9). Recently, the Lyapunov-type inequalities have been obtained by Guseinov and Kaymakçalan [9] and Guseinov and Zafer [10] for linear system (8), and Tiryaki et al. [21] for nonlinear system (9). The discrete and time scale analogues of Lyapunov-type inequalities for systems have been also found in the papers by Ünal et al. [23], Jiang and Zhou [12] and Ünal and Çakmak [24]. The estimates for eigenvalues of second order (p,q)-Laplacian systems under Dirichlet boundary conditions were studied in the paper of Napoli and Pinasco [16], and their results were generalized by Çakmak and Tiryaki [5, 6] and Yang et al. [28] to the more general nonlinear systems.

More recently, Wang [25] has interested in the following Lyapunov-type inequality, which is closely related to the stability criteria obtained in [25], for linear system (8) as follows.

THEOREM D. *Let $\beta_1(t) \geqslant 0$ for $t \in \mathbb{R}$. Assume that (8) has a real solution $(x(t),u(t))$ such that $x(a) = 0 = x(b)$ and $x(t)$ is not identically zero on $[a,b]$, where $a,b \in \mathbb{R}$ with $a < b$. Then the Lyapunov inequality*

$$\left[\int_a^b \beta_1(t)\exp\left(-2\int_\tau^t \alpha_1(s)ds\right)dt\right]\int_a^b \beta_2^+(t)dt \geqslant 4 \tag{11}$$

holds for some $\tau \in (a,b)$.

REMARK 1. We note that the function $\beta_1(t)$ should be positive in the linear system (8) for all $t \in \mathbb{R}$. If we take $\alpha_1(t) \equiv 0$ in the system (8), then this system reduces to the linear equation (3) with

$$r(t) = \frac{1}{\beta_1(t)} > 0 \quad \textit{and} \quad q(t) = \beta_2(t) \tag{12}$$

for all $t \in \mathbb{R}$. Thus, Theorem D is valid when the function $\beta_1(t) > 0$ for all $t \in \mathbb{R}$. Moreover, it is easy to see in the proof of Theorem D that τ is the maximum point of the absolute value of the first component of the solution $(x(t),u(t))$ of system (8) for all $t \in (a,b)$.

REMARK 2. Similarly, if we take $\alpha_1(t) \equiv 0$ in the nonlinear system (9) for all $t \in \mathbb{R}$, then this system reduces also to the Emden-Fowler equation (10) with

$$r(t) = (\beta_1(t))^{1-\alpha} > 0, \quad \alpha > 1 \quad \textit{and} \quad q(t) = \beta_2(t) \tag{13}$$

for all $t \in \mathbb{R}$.

The principal aim of this paper is to prove a Lyapunov-type inequality for nonlinear system (9) by using the same approach as Wang [25]. Our motivation comes from the recent papers of Tiryaki et al. [21] and Wang [25]. Our result is an extension of result by Wang [25] mentioned above Theorem D.

In this paper, we derive a Lyapunov-type inequality for nonlinear system (9), where the first component of the solution $(x(t), u(t))$ has consecutive zeros at the points $a, b \in \mathbb{R}$ with $a < b$ in $I = [t_0, \infty) \subset \mathbb{R}$. For the special cases of nonlinear system (9), we also derive some Lyapunov-type inequalities which not only relates points a and b in I at which the first component of the solution $(x(t), u(t))$ has consecutive zeros but also any point in (a, b) where the first component of the solution $(x(t), u(t))$ is maximized.

Since our attention is restricted to the Lyapunov-type inequality for nonlinear system of differential equations, we shall assume the existence of nontrivial solution $(x(t), u(t))$ of nonlinear system (9) and state our basic hypothesis with respect to the same system:

(i) $\gamma > 1$ and $\beta > 1$ are real constants.

(ii) $\alpha_1(t)$, $\beta_1(t)$ and $\beta_2(t)$ are real-valued piece-wise continuous functions such that $\beta_1(t) > 0$ for all $t \in \mathbb{R}$.

2. Main result

The main result of this paper is the following theorem.

THEOREM 1. *Let the hypotheses (i) and (ii) hold. If nonlinear system (9) has a real solution $(x(t), u(t))$ such that $x(a) = 0 = x(b)$ where $a, b \in \mathbb{R}$ with $a < b$ be consecutive zeros, and x is not identically zero on $[a, b]$, then the following inequality*

$$M^{\frac{\beta}{\alpha}-1}\left[\int_a^b \beta_1(t)\exp\left(-\gamma\int_\tau^t \alpha_1(s)ds\right)dt\right]^{\frac{1}{\gamma}}\left[\int_a^b \beta_2^+(t)dt\right]^{\frac{1}{\alpha}} \geqslant 2 \tag{14}$$

holds, where $\frac{1}{\gamma} + \frac{1}{\alpha} = 1$, $M = |x(\tau)| = \max_{a<t<b} |x(t)|$ *and* $\beta_2^+(t) = \max\{\beta_2(t), 0\}$ *is the nonnegative part of* $\beta_2(t)$.

Proof. It follows from $x(a) = 0 = x(b)$ where $a, b \in \mathbb{R}$ with $a < b$ be consecutive zeros, and x is not identically zero on $[a, b]$, one can choose $\tau \in (a, b)$ such that $|x(\tau)| = \max_{a<t<b} |x(t)| > 0$. From Rolle's theorem, clearly $x'(\tau) = 0$. Let $M = |x(\tau)|$. From the first equation of system (9), we have

$$\left[x(t)\exp\left(-\int_a^t \alpha_1(s)ds\right)\right]' = \beta_1(t)\exp\left(-\int_a^t \alpha_1(s)ds\right)|u(t)|^{\gamma-2}u(t) \tag{15}$$

and

$$\left[x(t)\exp\left(\int_t^b \alpha_1(s)ds\right)\right]' = \beta_1(t)\exp\left(\int_t^b \alpha_1(s)ds\right)|u(t)|^{\gamma-2}u(t) \tag{16}$$

for all $t \in \mathbb{R}$. Integrating (15) from a to τ and taking into account that $x(a)=0$, and (16) from τ to b and taking into account that $x(b)=0$, we have

$$x(\tau) = \int_a^\tau \beta_1(t)|u(t)|^{\gamma-2}u(t)\exp\left(-\int_\tau^t \alpha_1(s)ds\right)dt \tag{17}$$

and

$$-x(\tau) = \int_\tau^b \beta_1(t)|u(t)|^{\gamma-2}u(t)\exp\left(-\int_\tau^t \alpha_1(s)ds\right)dt, \tag{18}$$

respectively. Hence (17) and (18) give the following inequalities

$$|x(\tau)| \leqslant \int_a^\tau \beta_1(t)|u(t)|^{\gamma-1}\exp\left(-\int_\tau^t \alpha_1(s)ds\right)dt \tag{19}$$

and

$$|x(\tau)| \leqslant \int_\tau^b \beta_1(t)|u(t)|^{\gamma-1}\exp\left(-\int_\tau^t \alpha_1(s)ds\right)dt, \tag{20}$$

respectively. Summing up these last two inequalities gives

$$2|x(\tau)| \leqslant \int_a^b \beta_1(t)|u(t)|^{\gamma-1}\exp\left(-\int_\tau^t \alpha_1(s)ds\right)dt. \tag{21}$$

By using Hölder inequality on the integral of the right side of (21) with indices α and γ, we obtain

$$\int_a^b \beta_1(t)|u(t)|^{\gamma-1}\exp\left(-\int_\tau^t \alpha_1(s)ds\right)dt \leqslant$$

$$\left[\int_a^b \beta_1(t)\exp\left(-\gamma\int_\tau^t \alpha_1(s)ds\right)dt\right]^{\frac{1}{\gamma}}\left[\int_a^b \beta_1(t)|u(t)|^{\gamma}dt\right]^{\frac{1}{\alpha}} \tag{22}$$

where $\frac{1}{\gamma}+\frac{1}{\alpha}=1$. Therefore, substituting (22) into (21), we get

$$2|x(\tau)| \leqslant \left[\int_a^b \beta_1(t)\exp\left(-\gamma\int_\tau^t \alpha_1(s)ds\right)dt\right]^{\frac{1}{\gamma}}\left[\int_a^b \beta_1(t)|u(t)|^{\gamma}dt\right]^{\frac{1}{\alpha}}. \tag{23}$$

On the other hand, multiplying the first equation of system (9) by $u(t)$ and the second one by $x(t)$, and adding the result, we obtain

$$[x(t)u(t)]' = \beta_1(t)\,|u(t)|^{\gamma} - \beta_2(t)\,|x(t)|^{\beta}. \tag{24}$$

Integrating (24) from a to b and taking into account that $x(a) = 0 = x(b)$ yields

$$\int_a^b \beta_1(t)\,|u(t)|^{\gamma}\,dt = \int_a^b \beta_2(t)\,|x(t)|^{\beta}\,dt. \tag{25}$$

Substituting (25) into (23), we obtain

$$2\,|x(\tau)| \leqslant \left[\int_a^b \beta_1(t)\exp\left(-\gamma\int_\tau^t \alpha_1(s)ds\right)dt\right]^{\frac{1}{\gamma}} \left[\int_a^b \beta_2(t)\,|x(t)|^{\beta}\,dt\right]^{\frac{1}{\alpha}}. \tag{26}$$

Since $M = |x(\tau)| = \max_{a<t<b} |x(t)| > 0$ and $\beta_2^+(t) = \max\{\beta_2(t), 0\}$, it follows from (26) that

$$2 \leqslant M^{\frac{\beta}{\alpha}-1}\left[\int_a^b \beta_1(t)\exp\left(-\gamma\int_\tau^t \alpha_1(s)ds\right)dt\right]^{\frac{1}{\gamma}} \left[\int_a^b \beta_2^+(t)dt\right]^{\frac{1}{\alpha}} \tag{27}$$

which completes the proof. □

REMARK 3. We note that the inequality (14) should only be called a Lyapunov-type inequality for system (9) in the case $\beta = \alpha$ for otherwise it gives an upper bound for the maximum value of the absolute value of the first component of the solution $(x(t), u(t))$ of system (9) in the case $\alpha > \beta$ and a lower bound in the case $\alpha < \beta$.

REMARK 4. If we take $\alpha_1(t) = 0$ in the nonlinear system (9), then Theorem 1 in Tiryaki et al. [21] is exactly the same as our Theorem 1.

When $\beta = \alpha$ in nonlinear system (9), we have the following nonlinear system

$$\left.\begin{aligned} x' &= \alpha_1(t)x + \beta_1(t)\,|u|^{\gamma-2}u \\ u' &= -\beta_2(t)\,|x|^{\alpha-2}x - \alpha_1(t)u \end{aligned}\right\} \tag{28}$$

where $\frac{1}{\gamma} + \frac{1}{\alpha} = 1$. Thus, we shall arrive to the following result:

COROLLARY 1. *Let the hypotheses (i) and (ii) hold. If nonlinear system (28) has a real solution $(x(t), u(t))$ such that $x(a) = 0 = x(b)$ where $a, b \in \mathbb{R}$ with $a < b$ be consecutive zeros, and x is not identically zero on $[a, b]$, then the following inequality*

$$\left[\int_a^b \beta_1(t)\exp\left(-\gamma\int_\tau^t \alpha_1(s)ds\right)dt\right]^{\frac{1}{\gamma}} \left[\int_a^b \beta_2^+(t)dt\right]^{\frac{1}{\alpha}} \geqslant 2 \tag{29}$$

holds, where γ, α, τ and $\beta_2^+(t)$ are defined as before.

REMARK 5. If we take $\gamma = 2$ (or $\alpha = 2$) in the nonlinear system (28), then Corollary 1 reduces to the result of Wang [25] mentioned above Theorem D.

REMARK 6. Since $\beta_2^+(t) \leqslant |\beta_2(t)|$, the integral of $\int_a^b \beta_2^+(t)dt$ in the inequality (14) or (29) can be replaced by $\int_a^b |\beta_2(t)|\,dt$.

Now, we give an application of the obtained Lyapunov-type inequality for the following eigenvalue problem:

$$\left(r(t)\left|x'\right|^{\alpha-2}x'\right)' \pm \lambda q(t)\,|x|^{\alpha-2}x = 0, \qquad x(a) = 0 = x(b), \tag{30}$$

where $\alpha > 1$ is a constant, $r(t)$ and $q(t)$ are real-valued piece-wise continuous functions such that $r(t) > 0$ for all $t \in \mathbb{R}$.

Consider the following special case of nonlinear system (9), which is an equivalent system for the half-linear equation (30)

$$\left.\begin{array}{l} x' = \beta_1(t)\,|u|^{\gamma-2}u \\ u' = -\beta_2(t)\,|x|^{\alpha-2}x \end{array}\right\} \tag{31}$$

where $\beta_1(t) = r^{1-\gamma}(t)$, $\beta_2(t) = \pm\lambda q(t)$ and $\frac{1}{\gamma} + \frac{1}{\alpha} = 1$.

Obviously, Theorem 1 with Remark 6 for system (9) with $\alpha_1(t) = 0$ is satisfied for system (31). Therefore, we have

$$|\lambda| \geqslant \frac{2^{\alpha}}{\left(\int_a^b r^{1/(1-\alpha)}(t)dt\right)^{\alpha-1} \int_a^b |q(t)|\,dt}. \tag{32}$$

We also refer to the papers of Brown and Hinton [2], Cheng [3], Çakmak and Tiryaki [5], Guseinov and Kaymakçalan [9], Guseinov and Zafer [10], Tiryaki et al. [21], Wang [25] and the references quoted therein for the other applications of Lyapunov-type inequalities.

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