

LANDAU AND GRÜSS TYPE INEQUALITIES FOR INNER PRODUCT TYPE INTEGRAL TRANSFORMERS IN NORM IDEALS

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Abstract. For a probability measure μ and for square integrable fields (\mathcal{A}_t) and (\mathcal{B}_t) ($t \in \Omega$) of commuting normal operators we prove Landau type inequality

$$\begin{aligned} & \left\| \int_{\Omega} \mathcal{A}_t X \mathcal{B}_t d\mu(t) - \int_{\Omega} \mathcal{A}_t d\mu(t) X \int_{\Omega} \mathcal{B}_t d\mu(t) \right\| \\ & \leq \left\| \sqrt{\int_{\Omega} |\mathcal{A}_t|^2 d\mu(t) - \left| \int_{\Omega} \mathcal{A}_t d\mu(t) \right|^2} X \sqrt{\int_{\Omega} |\mathcal{B}_t|^2 d\mu(t) - \left| \int_{\Omega} \mathcal{B}_t d\mu(t) \right|^2} \right\| \end{aligned}$$

for all $X \in \mathcal{B}(\mathcal{H})$ and for all unitarily invariant norms $\|\cdot\|$.

For Schatten p -norms similar inequalities are given for arbitrary double square integrable fields. Also, for all bounded self-adjoint fields satisfying $C \leq \mathcal{A}_t \leq D$ and $E \leq \mathcal{B}_t \leq F$ for all $t \in \Omega$ and some bounded self-adjoint operators C, D, E and F , and for all $X \in \mathcal{C}_{\|\cdot\|}(\mathcal{H})$ we prove Grüss type inequality

$$\left\| \int_{\Omega} \mathcal{A}_t X \mathcal{B}_t d\mu(t) - \int_{\Omega} \mathcal{A}_t d\mu(t) X \int_{\Omega} \mathcal{B}_t d\mu(t) \right\| \leq \frac{\|D - C\| \cdot \|F - E\|}{4} \cdot \|X\|.$$

More general results for arbitrary bounded fields are also given.

1. Introduction

The Grüss inequality [10], as a complement of Chebyshev's inequality, states that if f and g are integrable real functions on $[a, b]$ such that $C \leq f(x) \leq D$ and $E \leq g(x) \leq F$ hold for some real constants C, D, E, F and for all $x \in [a, b]$, then

$$\left| \frac{1}{b-a} \int_a^b f(x)g(x)dx - \frac{1}{(b-a)^2} \int_a^b f(x)dx \int_a^b g(x)dx \right| \leq \frac{1}{4}(D-C)(F-E); \quad (1.1)$$

see [19] for several proofs of this inequality in the discrete form. It has been the subject of intensive investigation, in which conditions on functions are varied to obtain different estimates; see [8, 20] and references therein. This inequality has been investigated, applied and generalized by many authors in different areas of mathematics, among others in inner product spaces [7], quadrature formulae [5, 25], finite Fourier transforms [4], linear functionals [1, 12], matrix traces [24], inner product modules over H^* -algebras and C^* -algebras [2, 11], positive maps [21] and completely bounded maps [23].

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1.1. Symmetric gauge functions, unitarily invariant norms and their norm ideals

Let $\mathcal{B}(\mathcal{H})$ and $\mathcal{C}_\infty(\mathcal{H})$ denote respectively spaces of all bounded and all compact linear operators acting on a separable, complex Hilbert space \mathcal{H} . Each “symmetric gauge” (s.g.) function Φ on sequences gives rise to a unitarily invariant (u.i) norm on operators defined by $\|X\|_\Phi = \Phi(\{s_n(X)\}_{n=1}^\infty)$, with $s_1(X) \geq s_2(X) \geq \dots$ being the singular values of X , i.e., the eigenvalues of $|X| = (X^*X)^{\frac{1}{2}}$. We will denote by the symbol $\|\cdot\|$ any such norm, which is therefore defined on a naturally associated norm ideal $\mathcal{C}_{\|\cdot\|}(\mathcal{H})$ of $\mathcal{C}_\infty(\mathcal{H})$ and satisfies the invariance property $\|UXV\| = \|X\|$ for all $X \in \mathcal{C}_{\|\cdot\|}(\mathcal{H})$ and for all unitary operators $U, V \in \mathcal{B}(\mathcal{H})$.

Specially well known among u.i. norms are the Schatten p -norms defined for $1 \leq p < \infty$ as $\|X\|_p = \sqrt[p]{\sum_{n=1}^\infty s_n^p(X)}$, while $\|X\|_\infty = \|X\| = s_1(X)$ coincides with the operator norm $\|X\|$. Minimal and maximal u.i. norm are among Schatten norms, i.e., $\|X\|_\infty \leq \|X\| \leq \|X\|_1$ for all $X \in \mathcal{C}_1(\mathcal{H})$ (see inequality (IV.38) in [3]). For $f, g \in \mathcal{H}$, we will denote by $g^* \otimes f$ one dimensional operator $(g^* \otimes f)h = \langle h, g \rangle f$ for all $h \in \mathcal{H}$, known that the linear span of $\{g^* \otimes f \mid f, g \in \mathcal{H}\}$ is dense in each of $\mathcal{C}_p(\mathcal{H})$ for $1 \leq p \leq \infty$. Schatten p -norms are also classical examples of p -reconvexized norms. Namely, any u.i. norm $\|\cdot\|_\Phi$ could be p -reconvexized for any $p \geq 1$ by setting $\|A\|_{\Phi(p)} = \| |A|^p \|_\Phi^{\frac{1}{p}}$ for all $A \in \mathcal{B}(\mathcal{H})$ such that $|A|^p \in \mathcal{C}_\Phi(\mathcal{H})$. For the proof of the triangle inequality and other properties of these norms see preliminary section in [15]; for the characterization of the dual norm for p -reconvexized one see Th. 2.1 in [15].

1.2. Gel’fand integral of operator valued functions

In this section we will recall the basic properties and terminology related to the notion of Gel’fand integral, when it applies to operator valued (o.v.) functions. Since this theory is well known, we give those properties without the proof. Following [6], p.53., if $(\Omega, \mathfrak{M}, \mu)$ is a measure space, the mapping $\mathcal{A} : \Omega \rightarrow \mathcal{B}(\mathcal{H})$ will be called $[\mu]$ weakly $*$ -measurable if a scalar valued function $t \rightarrow \text{tr}(\mathcal{A}_t Y)$ is measurable for any $Y \in \mathcal{C}_1(\mathcal{H})$. In addition, if all these functions are in $L^1(\Omega, \mu)$, then according to the fact that $\mathcal{B}(\mathcal{H})$ is the dual space of $\mathcal{C}_1(\mathcal{H})$, for any $E \in \mathfrak{M}$ there will be the unique operator $\mathcal{I}_E \in \mathcal{B}(\mathcal{H})$, called the Gel’fand (Гельфанд) or weak $*$ -integral of \mathcal{A} over E , such that

$$\text{tr}(\mathcal{I}_E Y) = \int_E \text{tr}(\mathcal{A}_t Y) d\mu(t) \quad \text{for all } Y \in \mathcal{C}_1(\mathcal{H}). \tag{1.2}$$

We will denote it by $\int_E \mathcal{A}_t d\mu(t)$, $\int_E \mathcal{A} d\mu$ or exceptionally by $\mathcal{I}_E \mathcal{A} d\mu$, if the context requires to distinguish this one from other types of integration.

A practical tool for this type of integrability to deal with is the following

LEMMA 1.1. $\mathcal{A} : \Omega \rightarrow \mathcal{B}(\mathcal{H})$ is $[\mu]$ weakly $*$ -measurable (resp. $[\mu]$ weakly $*$ -integrable) iff the scalar valued functions $t \rightarrow \langle \mathcal{A}_t f, f \rangle$ are $[\mu]$ measurable (resp. integrable) for every $f \in \mathcal{H}$.

In view of Lemma 1.1, the basic definition (1.2) of Gel’fand integral for o.v. functions can be reformulated as follows:

LEMMA 1.2. *If $\langle \mathcal{A}f, f \rangle \in L^1(E, \mu)$ for all $f \in \mathcal{H}$, for some $E \in \mathfrak{M}$ and a $\mathfrak{B}(\mathcal{H})$ -valued function \mathcal{A} on E , then the mapping $f \rightarrow \int_E \langle \mathcal{A}_t f, f \rangle d\mu(t)$ represents a quadratic form of (the unique) bounded operator (denoted by) $\int_E \mathcal{A} d\mu$ or $\int_E \mathcal{A}_t d\mu(t)$, (we refer to it as to “intuitive” integral of \mathcal{A} over E), satisfying*

$$\left\langle \left(\int_E \mathcal{A}_t d\mu(t) \right) f, g \right\rangle = \int_E \langle \mathcal{A}_t f, g \rangle d\mu(t) \quad \text{for all } f, g \in \mathcal{H},$$

as well as

$$\text{tr} \left(\int_E \mathcal{A}_t d\mu(t) Y \right) = \int_E \text{tr}(\mathcal{A}_t Y) d\mu(t) \quad \text{for all } Y \in \mathfrak{C}_1(\mathcal{H}).$$

In other words, integrability of quadratic forms of an o.v. function assures its Gel’fand’s integrability and so the notions of “intuitive” and Gel’fand’s integral for o.v. functions coincide.

Following Ex.2 in [13], for a $[\mu]$ weakly*-measurable function $\mathcal{A} : \Omega \rightarrow \mathfrak{B}(\mathcal{H})$ we have that $\mathcal{A}^* \mathcal{A}$ is Gel’fand integrable iff $\int_\Omega \|\mathcal{A}_t f\|^2 d\mu(t) < \infty$ for all $f \in \mathcal{H}$. Moreover, for a $[\mu]$ weakly*-measurable function $\mathcal{A} : \Omega \rightarrow \mathfrak{B}(\mathcal{H})$ let us consider the operator of “vector valued functionalization” (of vectors), i.e., a linear transformation $\vec{\mathcal{A}} : D_{\vec{\mathcal{A}}} \rightarrow L^2(\Omega, \mu, \mathcal{H})$, with the domain $D_{\vec{\mathcal{A}}} = \left\{ f \in \mathcal{H} \mid \int_\Omega \|\mathcal{A}_t f\|^2 d\mu(t) < \infty \right\}$, defined by

$$(\vec{\mathcal{A}}f)(t) = \mathcal{A}_t f \quad \text{for } [\mu] \text{ a.e. } t \in \Omega \text{ and all } f \in D_{\vec{\mathcal{A}}}.$$

Now, another way for understanding Gel’fand’s integrability of $\mathcal{A}^* \mathcal{A}$ is provided by the following.

LEMMA 1.3. *$\vec{\mathcal{A}}$ is a closed operator; it is bounded if and only if $\mathcal{A}^* \mathcal{A}$ is Gel’fand integrable, and whenever this is the case, then $\|\vec{\mathcal{A}}\| = \sqrt{\int_\Omega \mathcal{A}_t^* \mathcal{A}_t d\mu(t)}$ and*

$$\|\vec{\mathcal{A}}\|_{\mathfrak{B}(\mathcal{H}, L^2(\Omega, \mu, \mathcal{H}))} = \left\| \int_\Omega \mathcal{A}_t^* \mathcal{A}_t d\mu(t) \right\|_{\mathfrak{C}_\Phi(\mathcal{H})}^{\frac{1}{2}}.$$

If additionally $\sqrt{\int_\Omega \mathcal{A}_t^* \mathcal{A}_t d\mu(t)} \in \mathfrak{C}_\Phi(\mathcal{H})$, then

$$\|\vec{\mathcal{A}}\|_{\mathfrak{C}_\Phi(\mathcal{H}, L^2(\Omega, \mu, \mathcal{H}))} = \left\| \sqrt{\int_\Omega \mathcal{A}_t^* \mathcal{A}_t d\mu(t)} \right\|_{\mathfrak{C}_\Phi(\mathcal{H})}. \tag{1.3}$$

Denoting $\|\cdot\|_{\mathfrak{C}_\Phi(\mathcal{H})}$ by $\|\|\cdot\|\|$, in view of both Definition 1 and equality (3) in [13] we now get

$$\begin{aligned} \|\|\mathcal{A}\|\|_2 &:= \left\| \int_\Omega \mathcal{A}_t^* \mathcal{A}_t d\mu(t) \right\|_{\mathfrak{C}_\Phi(\mathcal{H})}^{\frac{1}{2}} = \left\| \sqrt{\int_\Omega \mathcal{A}_t^* \mathcal{A}_t d\mu(t)} \right\|_{\mathfrak{C}_{\Phi(2)}(\mathcal{H})} \\ &= \|\|\vec{\mathcal{A}}\|\|_{\mathfrak{C}_{\Phi(2)}(\mathcal{H}, L^2(\Omega, \mu, \mathcal{H}))}. \end{aligned} \tag{1.4}$$

Thus we have recognized the space $L_G^2(\Omega, d\mu, \mathfrak{C}_\Phi(\mathcal{H}))$ of square integrable o.v. functions \mathcal{A} such that $\int_\Omega \mathcal{A}_t^* \mathcal{A}_t d\mu(t) \in \mathfrak{C}_\Phi(\mathcal{H})$ as the isometrically isomorphic to the norm ideal of operators $\mathfrak{C}_{\Phi(2)}(\mathcal{H}, L^2(\Omega, \mu, \mathcal{H})) \subseteq \mathfrak{B}_{\Phi(2)}(\mathcal{H}, L^2(\Omega, \mu, \mathcal{H}))$, associated to a (2-reconvexed) s.g function $\Phi^{(2)}$. Therefore the normability and the completeness of the space $L_G^2(\Omega, d\mu, \mathfrak{C}_\Phi(\mathcal{H}))$, as stated in Theorem 2.1 in [13], follow immediately. This effectively gives us a representation of $L_G^2(\Omega, d\mu, \mathfrak{C}_\Phi(\mathcal{H}))$, as a Hilbert module over a Banach *-algebra $\mathfrak{C}_\Phi(\mathcal{H})$ with its $\mathfrak{C}_\Phi(\mathcal{H})$ -valued inner product

$$\langle\langle \mathcal{A}, \mathcal{B} \rangle\rangle = \int_\Omega \mathcal{A}_t^* \mathcal{B}_t d\mu(t) \quad \text{for all } \mathcal{A}, \mathcal{B} \in L_G^2(\Omega, d\mu, \mathfrak{C}_\Phi(\mathcal{H})),$$

as a norm ideal $\mathfrak{C}_{\Phi(2)}(\mathcal{H}, L^2(\Omega, \mu, \mathcal{H}))$ of operators from \mathcal{H} into $L^2(\Omega, \mu, \mathcal{H})$ equipped with its $\mathfrak{C}_\Phi(\mathcal{H})$ -valued inner product

$$\langle\langle \vec{\mathcal{A}}, \vec{\mathcal{B}} \rangle\rangle = \vec{\mathcal{A}}^* \vec{\mathcal{B}} = \int_\Omega \mathcal{A}_t^* \mathcal{B}_t d\mu(t) \quad \text{for all } \vec{\mathcal{A}}, \vec{\mathcal{B}} \in \mathfrak{C}_{\Phi(2)}(\mathcal{H}, L^2(\Omega, \mu, \mathcal{H})).$$

1.3. Elementary operators and inner product type integral transformers in norm ideals

For weakly*-measurable o.v. functions $\mathcal{A}, \mathcal{B} : \Omega \rightarrow \mathfrak{B}(\mathcal{H})$ and for all $X \in \mathfrak{B}(\mathcal{H})$ the function $t \rightarrow \mathcal{A}_t X \mathcal{B}_t$ is also weakly*-measurable. If these functions are Gel'fand integrable for all $X \in \mathfrak{B}(\mathcal{H})$, then the inner product type linear transformation $X \rightarrow \int_\Omega \mathcal{A}_t X \mathcal{B}_t d\mu(t)$ will be called an inner product type (i.p.t.) transformer on $\mathfrak{B}(\mathcal{H})$ and denoted by $\int_\Omega \mathcal{A}_t \otimes \mathcal{B}_t d\mu(t)$ or $\mathcal{I}_{\mathcal{A}, \mathcal{B}}$. A special case when μ is the counting measure on \mathbb{N} is mostly known and widely investigated, and such transformers are known as elementary mappings or elementary operators.

As shown in Lemma 3.1 (a) in [13], a sufficient condition is provided when \mathcal{A}^* and \mathcal{B} are both in $L_G^2(\Omega, d\mu, \mathfrak{B}(\mathcal{H}))$. If each of families $(\mathcal{A}_t)_{t \in \Omega}$ and $(\mathcal{B}_t)_{t \in \Omega}$ consists of commuting normal operators, then by Theorem 3.2 in [13] the i.p.t. integral transformer $\int_\Omega \mathcal{A}_t \otimes \mathcal{B}_t d\mu(t)$ leaves every u.i. norm ideal $\mathfrak{C}_{\|\cdot\|}(\mathcal{H})$ invariant and the following Cauchy-Schwarz inequality holds:

$$\left\| \int_\Omega \mathcal{A}_t X \mathcal{B}_t d\mu(t) \right\| \leq \left\| \sqrt{\int_\Omega \mathcal{A}_t^* \mathcal{A}_t d\mu(t)} X \sqrt{\int_\Omega \mathcal{B}_t^* \mathcal{B}_t d\mu(t)} \right\| \quad (1.5)$$

for all $X \in \mathfrak{C}_{\|\cdot\|}(\mathcal{H})$.

Normality and commutativity condition can be dropped for Schatten p -norms as shown in Theorem 3.3 in [13]. In Theorem 3.1 in [14] a formula for the exact norm of the i.p.t. integral transformer $\int_\Omega \mathcal{A}_t \otimes \mathcal{B}_t d\mu(t)$ acting on $\mathfrak{C}_2(\mathcal{H})$ is found. In Theorem 2.1 in [14] the exact norm of the i.p.t. integral transformer $\int_\Omega \mathcal{A}_t^* \otimes \mathcal{A}_t d\mu(t)$ is given for two specific cases:

$$\left\| \int_\Omega \mathcal{A}_t^* \otimes \mathcal{A}_t d\mu(t) \right\|_{\mathfrak{B}(\mathcal{H}) \rightarrow \mathfrak{C}_\Phi(\mathcal{H})} = \left\| \int_\Omega \mathcal{A}_t^* \mathcal{A}_t d\mu(t) \right\|_{\mathfrak{C}_\Phi(\mathcal{H})}, \quad (1.6)$$

$$\left\| \int_\Omega \mathcal{A}_t^* \otimes \mathcal{A}_t d\mu(t) \right\|_{\mathfrak{C}_\Phi(\mathcal{H}) \rightarrow \mathfrak{C}_1(\mathcal{H})} = \left\| \int_\Omega \mathcal{A}_t \mathcal{A}_t^* d\mu(t) \right\|_{\mathfrak{C}_{\Phi^*}(\mathcal{H})},$$

where Φ_* stands for a s.g. function related to the dual space $(\mathcal{C}_\Phi(\mathcal{H}))^*$.

Also, as already noted in [14] at the end of page 2964, the norm appearing in (1.4) equals to a square root of the norm of the i.p.t. integral transformer $X \rightarrow \int_\Omega \mathcal{A}_t^* X \mathcal{A}_t d\mu(t)$ when acting from $\mathcal{B}(\mathcal{H})$ to $\mathcal{C}_\Phi(\mathcal{H})$. As this quantity actually presents a norm on the Banach space $L_G^2(\Omega, d\mu, \mathcal{B}(\mathcal{H}), \mathcal{C}_\Phi(\mathcal{H}))$ as elaborated in Theorem 2.2 in [14], therefore we conclude that spaces $L_G^2(\Omega, d\mu, \mathcal{B}(\mathcal{H}), \mathcal{C}_\Phi(\mathcal{H}))$ are both isometrically isomorphic to the norm ideal $\mathcal{C}_{\Phi(2)}(\mathcal{H}, L^2(\Omega, \mu, \mathcal{H}))$. As the objects of consideration in all those spaces are families of operators, from now on we will refer to such objects as to field of operators (for example $(\mathcal{A}_t)_{t \in \Omega}$ in $L^2(\Omega, \mu, \mathcal{C}_\Phi(\mathcal{H}))$). When we additionally require that the adjoint fields of operators $(\mathcal{A}_t^*)_{t \in \Omega}$ also belongs to $L^2(\Omega, \mu, \mathcal{C}_\Phi(\mathcal{H}))$, then we will say that $(\mathcal{A}_t)_{t \in \Omega}$ in doubly μ square integrable in $\mathcal{C}_\Phi(\mathcal{H})$ on Ω .

The norm appearing in (1.6) and its associated space $L_G^2(\Omega, d\mu, \mathcal{B}(\mathcal{H}), \mathcal{C}_\Phi(\mathcal{H}))$ present only a special case of norming a field $\mathcal{A} = (\mathcal{A}_t)_{t \in \Omega}$. A much wider class of norms $\|\cdot\|_{\Phi, \Psi}$ and their associated spaces $L_G^2(\Omega, d\mu, \mathcal{C}_\Phi(\mathcal{H}), \mathcal{C}_\Psi(\mathcal{H}))$ are given in [14] by

$$\|\mathcal{A}\|_{\Phi, \Psi} = \left\| \int_\Omega \mathcal{A}_t^* \otimes \mathcal{A}_t d\mu(t) \right\|_{\mathcal{B}(\mathcal{C}_\Phi(\mathcal{H}), \mathcal{C}_\Psi(\mathcal{H}))}^{\frac{1}{2}} \tag{1.7}$$

for an arbitrary pair of s.g. functions Φ and Ψ . For the proof of completeness of the space $L_G^2(\Omega, d\mu, \mathcal{C}_\Phi(\mathcal{H}), \mathcal{C}_\Psi(\mathcal{H}))$ see Theorem 2.2 in [14].

The potential for finding Grüss type inequalities for i.p.t. integral transformers relies on the fact that $\int_\Omega \mathcal{A}_t \otimes \mathcal{B}_t d\mu(t) - \int_\Omega \mathcal{A}_t d\mu(t) \otimes \int_\Omega \mathcal{B}_t d\mu(t)$ is also an i.p.t. integral transformer. As the representation for an i.p.t. integral transformer is not unique (as a rule), the successfulness of the application of some known inequalities to $\int_\Omega \mathcal{A}_t \otimes \mathcal{B}_t d\mu(t) - \int_\Omega \mathcal{A}_t d\mu(t) \otimes \int_\Omega \mathcal{B}_t d\mu(t)$ mainly depends on the right choice for its representation.

Before exposing the main results, we will draw our attention to the following lemma, which we will use in the sequel.

LEMMA 1.4. *If μ is a probability measure on Ω , then for every field $(\mathcal{A}_t)_{t \in \Omega}$ in $L^2(\Omega, \mu, \mathcal{B}(\mathcal{H}))$, for all $B \in \mathcal{B}(\mathcal{H})$, for all unitarily invariant norms $\|\cdot\|$ and for all $\theta > 0$,*

$$\int_\Omega |\mathcal{A}_t - B|^2 d\mu(t) = \int_\Omega \left| \mathcal{A}_t - \int_\Omega \mathcal{A}_t d\mu(t) \right|^2 d\mu(t) + \left| \int_\Omega \mathcal{A}_t d\mu(t) - B \right|^2 \tag{1.8}$$

$$\begin{aligned} &\geq \int_\Omega \left| \mathcal{A}_t - \int_\Omega \mathcal{A}_t d\mu(t) \right|^2 d\mu(t) \\ &= \int_\Omega |\mathcal{A}_t|^2 d\mu(t) - \left| \int_\Omega \mathcal{A}_t d\mu(t) \right|^2; \end{aligned} \tag{1.9}$$

$$\begin{aligned} \min_{B \in \mathcal{B}(\mathcal{H})} \left\| \int_{\Omega} |\mathcal{A}_t - B|^2 d\mu(t) \right\|^\theta &= \left\| \int_{\Omega} \left| \mathcal{A}_t - \int_{\Omega} \mathcal{A}_t d\mu(t) \right|^2 d\mu(t) \right\|^\theta \\ &= \left\| \int_{\Omega} |\mathcal{A}_t|^2 d\mu(t) - \left| \int_{\Omega} \mathcal{A}_t d\mu(t) \right|^2 \right\|^\theta. \end{aligned} \tag{1.10}$$

Thus, the considered minimum is always obtained for $B = \int_{\Omega} \mathcal{A}_t d\mu(t)$.

Proof. The expression in (1.8) equals

$$\begin{aligned} \int_{\Omega} |\mathcal{A}_t - B|^2 d\mu(t) &= \int_{\Omega} \left| \mathcal{A}_t - \int_{\Omega} \mathcal{A}_t d\mu(t) + \int_{\Omega} \mathcal{A}_t d\mu(t) - B \right|^2 d\mu(t) \\ &= \int_{\Omega} \left| \mathcal{A}_t - \int_{\Omega} \mathcal{A}_t d\mu(t) \right|^2 d\mu(t) + \int_{\Omega} \left| \int_{\Omega} \mathcal{A}_t d\mu(t) - B \right|^2 d\mu(t) \\ &\quad + 2\Re \int_{\Omega} \left(\mathcal{A}_t - \int_{\Omega} \mathcal{A}_t d\mu(t) \right)^* \left(\int_{\Omega} \mathcal{A}_t d\mu(t) - B \right) d\mu(t) \\ &= \int_{\Omega} \left| \mathcal{A}_t - \int_{\Omega} \mathcal{A}_t d\mu(t) \right|^2 d\mu(t) + \left| \int_{\Omega} \mathcal{A}_t d\mu(t) - B \right|^2, \end{aligned}$$

as

$$\begin{aligned} &\int_{\Omega} \left(\mathcal{A}_t - \int_{\Omega} \mathcal{A}_t d\mu(t) \right)^* \left(\int_{\Omega} \mathcal{A}_t d\mu(t) - B \right) d\mu(t) \\ &= \left(\int_{\Omega} \mathcal{A}_t^* d\mu(t) - \int_{\Omega} \mathcal{A}_t^* d\mu(t) \right) \left(\int_{\Omega} \mathcal{A}_t d\mu(t) - B \right) = 0. \end{aligned}$$

Inequality (1.9) follows from (1.8), while identity (1.9) is just a special case of Lemma 2.1 in [13] applied for $k = 1$ and $\delta_1 = \Omega$.

As $0 \leq A \leq B$ for $A, B \in \mathcal{C}_{\infty}(\mathcal{H})$ implies $s_n^\theta(A) \leq s_n^\theta(B)$ for all $n \in \mathbb{N}$, as well as $\|A^\theta\| \leq \|B^\theta\|$, then (1.10) follows.

2. Main results

Let us recall that for a pair of random real variables (Y, Z) its coefficient of correlation

$$\rho_{Y,Z} = \frac{|E(YZ) - E(Y)E(Z)|}{\sigma(Y)\sigma(Z)} = \frac{|E(YZ) - E(Y)E(Z)|}{\sqrt{E(Y^2) - E^2(Y)}\sqrt{E(Z^2) - E^2(Z)}}$$

always satisfies $|\rho_{Y,Z}| \leq 1$.

For square integrable functions f and g on $[0, 1]$ and $D(f, g) = \int_0^1 f(t)g(t) dt - \int_0^1 f(t) dt \int_0^1 g(t) dt$ Landau proved (see [17, 18])

$$|D(f, g)| \leq \sqrt{D(f, f)D(g, g)},$$

and the following theorem is a generalization of these facts to i.p.t. integral transformers.

THEOREM 2.1. (Landau type inequality for i.p.t. integral transformers in u.i. norm ideals) *Let μ be a probability measure on Ω , let both fields $(\mathcal{A}_t)_{t \in \Omega}$ and $(\mathcal{B}_t)_{t \in \Omega}$ in $L^2(\Omega, \mu, \mathfrak{B}(\mathcal{H}))$ consist of commuting normal operators and let*

$$\sqrt{\int_{\Omega} |\mathcal{A}_t|^2 d\mu(t) - \left| \int_{\Omega} \mathcal{A}_t d\mu(t) \right|^2} X \sqrt{\int_{\Omega} |\mathcal{B}_t|^2 d\mu(t) - \left| \int_{\Omega} \mathcal{B}_t d\mu(t) \right|^2} \in \mathfrak{C}_{\|\cdot\|}(\mathcal{H})$$

for some $X \in \mathfrak{B}(\mathcal{H})$. Then

$$\int_{\Omega} \mathcal{A}_t X \mathcal{B}_t d\mu(t) - \int_{\Omega} \mathcal{A}_t d\mu(t) X \int_{\Omega} \mathcal{B}_t d\mu(t) \in \mathfrak{C}_{\|\cdot\|}(\mathcal{H})$$

and

$$\begin{aligned} & \left\| \int_{\Omega} \mathcal{A}_t X \mathcal{B}_t d\mu(t) - \int_{\Omega} \mathcal{A}_t d\mu(t) X \int_{\Omega} \mathcal{B}_t d\mu(t) \right\| \\ & \leq \left\| \sqrt{\int_{\Omega} |\mathcal{A}_t|^2 d\mu(t) - \left| \int_{\Omega} \mathcal{A}_t d\mu(t) \right|^2} X \sqrt{\int_{\Omega} |\mathcal{B}_t|^2 d\mu(t) - \left| \int_{\Omega} \mathcal{B}_t d\mu(t) \right|^2} \right\|. \end{aligned} \quad (2.1)$$

Proof. First we note that we have the following Korkine type identity for i.p.t. integral transformers:

$$\begin{aligned} & \int_{\Omega} \mathcal{A}_t X \mathcal{B}_t d\mu(t) - \int_{\Omega} \mathcal{A}_t d\mu(t) X \int_{\Omega} \mathcal{B}_t d\mu(t) \\ & = \int_{\Omega} d\mu(s) \int_{\Omega} \mathcal{A}_t X \mathcal{B}_t d\mu(t) - \int_{\Omega} \int_{\Omega} \mathcal{A}_t X \mathcal{B}_s d\mu(s) d\mu(t) \\ & = \frac{1}{2} \int_{\Omega^2} (\mathcal{A}_s - \mathcal{A}_t) X (\mathcal{B}_s - \mathcal{B}_t) d(\mu \times \mu)(s, t). \end{aligned} \quad (2.2)$$

In this representation we have $(\mathcal{A}_s - \mathcal{A}_t)_{(s,t) \in \Omega^2}$ and $(\mathcal{B}_s - \mathcal{B}_t)_{(s,t) \in \Omega^2}$ to be in $L^2(\Omega^2, \mu \times \mu, \mathfrak{B}(\mathcal{H}))$ because by an application of the identity (2.2),

$$\begin{aligned} \frac{1}{2} \int_{\Omega^2} |\mathcal{A}_s - \mathcal{A}_t|^2 d(\mu \times \mu)(s, t) & = \int_{\Omega} |\mathcal{A}_t|^2 d\mu(t) - \left| \int_{\Omega} \mathcal{A}_t d\mu(t) \right|^2 \\ & = \int_{\Omega} \left| \mathcal{A}_t - \int_{\Omega} \mathcal{A}_t d\mu(t) \right|^2 d\mu(t) \in \mathfrak{B}(\mathcal{H}). \end{aligned} \quad (2.3)$$

Both families $(\mathcal{A}_s - \mathcal{A}_t)_{(s,t) \in \Omega^2}$ and $(\mathcal{B}_s - \mathcal{B}_t)_{(s,t) \in \Omega^2}$ consist of commuting normal operators and by Theorem 3.2 in [13]

$$\frac{1}{2} \int_{\Omega^2} (\mathcal{A}_s - \mathcal{A}_t) X (\mathcal{B}_s - \mathcal{B}_t) d(\mu \times \mu)(s, t) \in \mathfrak{C}_{\|\cdot\|}(\mathcal{H})$$

and

$$\begin{aligned} & \left\| \int_{\Omega} \mathcal{A}_t X \mathcal{B}_t d\mu(t) - \int_{\Omega} \mathcal{A}_t d\mu(t) X \int_{\Omega} \mathcal{B}_t d\mu(t) \right\| \\ &= \left\| \frac{1}{2} \int_{\Omega^2} (\mathcal{A}_s - \mathcal{A}_t) X (\mathcal{B}_s - \mathcal{B}_t) d(\mu \times \mu)(s, t) \right\| \\ &\leq \left\| \sqrt{\frac{1}{2} \int_{\Omega^2} |\mathcal{A}_s - \mathcal{A}_t|^2 d(\mu \times \mu)(s, t)} X \sqrt{\frac{1}{2} \int_{\Omega^2} |\mathcal{B}_s - \mathcal{B}_t|^2 d(\mu \times \mu)(s, t)} \right\| \\ &= \left\| \sqrt{\int_{\Omega} |\mathcal{A}_t|^2 d\mu(t) - \left| \int_{\Omega} \mathcal{A}_t d\mu(t) \right|^2} X \sqrt{\int_{\Omega} |\mathcal{B}_t|^2 d\mu(t) - \left| \int_{\Omega} \mathcal{B}_t d\mu(t) \right|^2} \right\|, \end{aligned}$$

due to identities (2.2) and (2.3). And so the conclusion (2.1) follows.

LEMMA 2.2. *Let μ (resp. ν) be a probability measure on Ω (resp. \mathfrak{U}), let both families $\{\mathcal{A}_s, \mathcal{C}_t\}_{(s,t) \in \Omega \times \mathfrak{U}}$ and $\{\mathcal{B}_s, \mathcal{D}_t\}_{(s,t) \in \Omega \times \mathfrak{U}}$ consist of commuting normal operators and let*

$$\begin{aligned} & \sqrt{\int_{\Omega} |\mathcal{A}_s|^2 d\mu(s) \int_{\mathfrak{U}} |\mathcal{C}_t|^2 d\nu(t) - \left| \int_{\Omega} \mathcal{A}_s d\mu(s) \int_{\mathfrak{U}} \mathcal{C}_t d\nu(t) \right|^2} \cdot X \cdot \\ & \sqrt{\int_{\Omega} |\mathcal{B}_s|^2 d\mu(s) \int_{\mathfrak{U}} |\mathcal{D}_t|^2 d\nu(t) - \left| \int_{\Omega} \mathcal{B}_s d\mu(s) \int_{\mathfrak{U}} \mathcal{D}_t d\nu(t) \right|^2} \end{aligned}$$

be in $\mathfrak{C}_{\|\cdot\|, \|\cdot\|}(\mathcal{H})$ for some $X \in \mathfrak{B}(\mathcal{H})$. Then

$$\begin{aligned} & \int_{\Omega} \int_{\mathfrak{U}} \mathcal{A}_s \mathcal{C}_t X \mathcal{B}_s \mathcal{D}_t d\mu(s) d\nu(t) \\ & - \int_{\Omega} \mathcal{A}_s d\mu(s) \int_{\mathfrak{U}} \mathcal{C}_t d\nu(t) X \int_{\Omega} \mathcal{B}_s d\mu(s) \int_{\mathfrak{U}} \mathcal{D}_t d\nu(t) \in \mathfrak{C}_{\|\cdot\|, \|\cdot\|}(\mathcal{H}) \end{aligned}$$

and

$$\begin{aligned} & \left\| \int_{\Omega} \int_{\mathfrak{U}} \mathcal{A}_s \mathcal{C}_t X \mathcal{B}_s \mathcal{D}_t d\mu(s) d\nu(t) - \int_{\Omega} \mathcal{A}_s d\mu(s) \int_{\mathfrak{U}} \mathcal{C}_t d\nu(t) X \int_{\Omega} \mathcal{B}_s d\mu(s) \int_{\mathfrak{U}} \mathcal{D}_t d\nu(t) \right\| \\ & \leq \left\| \sqrt{\int_{\Omega} |\mathcal{A}_s|^2 d\mu(s) \int_{\mathfrak{U}} |\mathcal{C}_t|^2 d\nu(t) - \left| \int_{\Omega} \mathcal{A}_s d\mu(s) \int_{\mathfrak{U}} \mathcal{C}_t d\nu(t) \right|^2} X \right. \\ & \quad \cdot \left. \sqrt{\int_{\Omega} |\mathcal{B}_s|^2 d\mu(s) \int_{\mathfrak{U}} |\mathcal{D}_t|^2 d\nu(t) - \left| \int_{\Omega} \mathcal{B}_s d\mu(s) \int_{\mathfrak{U}} \mathcal{D}_t d\nu(t) \right|^2} \right\|. \end{aligned}$$

Proof. Apply Theorem 2.1 to the probability measure $\mu \times \nu$ on $\Omega \times \mathbb{U}$ and families $(\mathcal{A}_s \mathcal{C}_t)_{(s,t) \in \Omega \times \mathbb{U}}$ and $(\mathcal{B}_s \mathcal{D}_t)_{(s,t) \in \Omega \times \mathbb{U}}$ of normal commuting operators in $L_G^2(\Omega \times \mathbb{U}, d\mu \times \nu, \mathfrak{B}(\mathcal{H}))$, taking in account that

$$\begin{aligned} \int_{\Omega \times \mathbb{U}} \mathcal{A}_s \mathcal{C}_t d(\mu \times \nu)(s, t) &= \int_{\Omega} \mathcal{A}_s d\mu(s) \int_{\mathbb{U}} \mathcal{C}_t d\nu(t), \\ \int_{\Omega \times \mathbb{U}} \mathcal{B}_s \mathcal{D}_t d(\mu \times \nu)(s, t) &= \int_{\Omega} \mathcal{B}_s d\mu(s) \int_{\mathbb{U}} \mathcal{D}_t d\nu(t), \end{aligned}$$

and similarly

$$\begin{aligned} \int_{\Omega \times \mathbb{U}} |\mathcal{A}_s \mathcal{C}_t|^2 d(\mu \times \nu)(s, t) &= \int_{\Omega} |\mathcal{A}_s|^2 d\mu(s) \int_{\mathbb{U}} |\mathcal{C}_t|^2 d\nu(t), \\ \int_{\Omega \times \mathbb{U}} |\mathcal{B}_s \mathcal{D}_t|^2 d(\mu \times \nu)(s, t) &= \int_{\Omega} |\mathcal{B}_s|^2 d\mu(s) \int_{\mathbb{U}} |\mathcal{D}_t|^2 d\nu(t). \end{aligned}$$

By the use of the mathematical induction, the previous lemma enables us to get straightforwardly the following

COROLLARY 2.3. If μ , $(\mathcal{A}_t)_{t \in \Omega}$ and $(\mathcal{B}_t)_{t \in \Omega}$ are as in Theorem 2.1, then for all $n \in \mathbb{N}$ and for any $(*_1, \dots, *_2n) \in \{*, 1\}^{2n}$,

$$\begin{aligned} & \left\| \int_{\Omega^n} \prod_{k=1}^n \mathcal{A}_{t_k}^{*_k} X \prod_{k=1}^n \mathcal{B}_{t_{n+k}}^{*_k} \prod_{k=1}^n d\mu(t_k) \right. \\ & \quad \left. - \left(\int_{\Omega} \mathcal{A}_t^{*_i} d\mu(t) \right)^i \left(\int_{\Omega} \mathcal{A}_t d\mu(t) \right)^{n-i} X \left(\int_{\Omega} \mathcal{B}_t^{*_j} d\mu(t) \right)^j \left(\int_{\Omega} \mathcal{B}_t d\mu(t) \right)^{n-j} \right\| \\ & \leq \left\| \left(\int_{\Omega} |\mathcal{A}_t|^2 d\mu(t) - \left| \int_{\Omega} \mathcal{A}_t d\mu(t) \right|^2 \right)^{\frac{n-i}{2}} X \left(\int_{\Omega} |\mathcal{B}_t|^2 d\mu(t) - \left| \int_{\Omega} \mathcal{B}_t d\mu(t) \right|^2 \right)^{\frac{n-j}{2}} \right\|, \end{aligned}$$

where i (resp. j) stands for the cardinality of $\{k \in \mathbb{N} \mid 1 \leq k \leq n \ \& \ *_k = *\}$ and (resp. $\{k \in \mathbb{N} \mid 1 \leq k \leq n \ \& \ *_k = *\}$).

For the Schatten p -norms $\|\cdot\|_p$ normality and commutativity conditions can be dropped, at the inevitable expense of the simplicity for its formulation. So we have the following

THEOREM 2.4. (Landau type inequality for i.p.t. integral transformers in Schatten ideals) *Let μ be a probability measure on Ω , let $(\mathcal{A}_t)_{t \in \Omega}$ and $(\mathcal{B}_t)_{t \in \Omega}$ be μ -weak* measurable families of bounded Hilbert space operators such that*

$$\int_{\Omega} (\|\mathcal{A}_t f\|^2 + \|\mathcal{A}_t^* f\|^2 + \|\mathcal{B}_t f\|^2 + \|\mathcal{B}_t^* f\|^2) d\mu(t) < \infty \quad \text{for all } f \in \mathcal{H}$$

and let $p, q, r \geq 1$ such that $\frac{1}{p} = \frac{1}{2q} + \frac{1}{2r}$. Then for all $X \in \mathfrak{C}_p(\mathcal{H})$,

$$\begin{aligned} & \left\| \int_{\Omega} \mathcal{A}_t X \mathcal{B}_t d\mu(t) - \int_{\Omega} \mathcal{A}_t d\mu(t) X \int_{\Omega} \mathcal{B}_t d\mu(t) \right\|_p \tag{2.4} \\ & \leq \left\| \left(\int_{\Omega} \left| \int_{\Omega} \left| \mathcal{A}_t^* - \int_{\Omega} \mathcal{A}_t^* d\mu(t) \right|^2 d\mu(t) \right)^{\frac{q-1}{2}} \left(\mathcal{A}_t - \int_{\Omega} \mathcal{A}_t d\mu(t) \right) \right|^2 d\mu(t) \right)^{\frac{1}{2q}} \\ & \quad \times \left\| \left(\int_{\Omega} \left| \int_{\Omega} \left| \mathcal{B}_t - \int_{\Omega} \mathcal{B}_t d\mu(t) \right|^2 d\mu(t) \right)^{\frac{r-1}{2}} \left(\mathcal{B}_t^* - \int_{\Omega} \mathcal{B}_t^* d\mu(t) \right) \right|^2 d\mu(t) \right)^{\frac{1}{2r}} \right\|_p. \end{aligned}$$

Proof. According to identity (2.3), application of Theorem 3.3 in [13] to families $(\mathcal{A}_s - \mathcal{A}_t)_{(s,t) \in \Omega^2}$ and $(\mathcal{B}_s - \mathcal{B}_t)_{(s,t) \in \Omega^2}$ gives

$$\begin{aligned} & \left\| \int_{\Omega} \mathcal{A}_t X \mathcal{B}_t d\mu(t) - \int_{\Omega} \mathcal{A}_t d\mu(t) X \int_{\Omega} \mathcal{B}_t d\mu(t) \right\|_p \\ & = \left\| \frac{1}{2} \int_{\Omega^2} (\mathcal{A}_s - \mathcal{A}_t) X (\mathcal{B}_s - \mathcal{B}_t) d(\mu \times \mu)(s, t) \right\|_p \\ & \leq \left\| \left(\frac{1}{2} \int_{\Omega^2} (\mathcal{A}_s^* - \mathcal{A}_t^*) \left(\frac{1}{2} \int_{\Omega^2} |\mathcal{A}_s^* - \mathcal{A}_t^*|^2 (\mu \times \mu)(s, t) \right)^{q-1} (\mathcal{A}_s - \mathcal{A}_t) d(\mu \times \mu)(s, t) \right)^{\frac{1}{2q}} X \right. \\ & \quad \left. \left(\frac{1}{2} \int_{\Omega^2} (\mathcal{B}_s - \mathcal{B}_t) \left(\frac{1}{2} \int_{\Omega^2} |\mathcal{B}_s - \mathcal{B}_t|^2 (\mu \times \mu)(s, t) \right)^{r-1} (\mathcal{B}_s^* - \mathcal{B}_t^*) d(\mu \times \mu)(s, t) \right)^{\frac{1}{2r}} \right\|_p. \tag{2.5} \end{aligned}$$

By application of identity (2.3) once again, the last expression in (2.5) becomes

$$\begin{aligned} & \left\| \left(\frac{1}{2} \int_{\Omega^2} (\mathcal{A}_s - \mathcal{A}_t)^* \left(\int_{\Omega} \left| \mathcal{A}_t^* - \int_{\Omega} \mathcal{A}_t^* d\mu(t) \right|^2 d\mu(t) \right)^{q-1} (\mathcal{A}_s - \mathcal{A}_t) d(\mu \times \mu)(s, t) \right)^{\frac{1}{2q}} \right. \\ & \quad \left. \times \left(\frac{1}{2} \int_{\Omega^2} (\mathcal{B}_s - \mathcal{B}_t) \left(\int_{\Omega} \left| \mathcal{B}_t - \int_{\Omega} \mathcal{B}_t d\mu(t) \right|^2 d\mu(s) \right)^{r-1} (\mathcal{B}_s - \mathcal{B}_t)^* d(\mu \times \mu)(s, t) \right)^{\frac{1}{2r}} \right\|_p. \end{aligned}$$

Denoting $\left(\int_{\Omega} |\mathcal{A}_s^* - \int_{\Omega} \mathcal{A}_t^* d\mu(t)|^2 d\mu(s) \right)^{\frac{q-1}{2}}$ (resp. $\left(\int_{\Omega} |\mathcal{B}_s - \int_{\Omega} \mathcal{B}_t d\mu(t)|^2 d\mu(s) \right)^{\frac{r-1}{2}}$) by Y (resp. Z), then the expression in (2.5) becomes

$$\left\| \left(\frac{1}{2} \int_{\Omega^2} |Y \mathcal{A}_s - Y \mathcal{A}_t|^2 d(\mu \times \mu)(s, t) \right)^{\frac{1}{2q}} \left(\frac{1}{2} \int_{\Omega^2} |Z \mathcal{B}_s^* - Z \mathcal{B}_t^*|^2 d(\mu \times \mu)(s, t) \right)^{\frac{1}{2r}} \right\|_p. \tag{2.6}$$

By a new application of identity (2.3) to families $(Y_{\mathcal{A}_t})_{t \in \Omega}$ and $(Z_{\mathcal{B}_t^*})_{t \in \Omega}$ (2.6) becomes

$$\left\| \left(\int_{\Omega} \left| Y_{\mathcal{A}_t} - \int_{\Omega} Y_{\mathcal{A}_t} d\mu(t) \right|^2 d\mu(t) \right)^{\frac{1}{2q}} X \left(\int_{\Omega} \left| Z_{\mathcal{B}_t^*} - \int_{\Omega} Z_{\mathcal{B}_t^*} d\mu(t) \right|^2 d\mu(t) \right)^{\frac{1}{2r}} \right\|_p,$$

which obviously equals to the righthand side expression in (2.4).

A special case of an abstract Hölder inequality presented in Theorem 3.1.(e) in [13] gives us

THEOREM 2.5. (Cauchy-Schwarz inequality for o.v. functions in u.i. norm ideals) *Let μ be a measure on Ω , let $(\mathcal{A}_t)_{t \in \Omega}$ and $(\mathcal{B}_t)_{t \in \Omega}$ be μ -weak* measurable families of bounded Hilbert space operators such that $\left| \int_{\Omega} |\mathcal{A}_t|^2 d\mu(t) \right|^\theta$ and $\left| \int_{\Omega} |\mathcal{B}_t|^2 d\mu(t) \right|^\theta$ are in in $\mathcal{C}_{\|\cdot\|}(\mathcal{H})$ for some $\theta > 0$ and for some u.i. norm $\|\cdot\|$. Then,*

$$\left\| \left| \int_{\Omega} \mathcal{A}_t^* \mathcal{B}_t d\mu(t) \right|^\theta \right\| \leq \left\| \left| \int_{\Omega} \mathcal{A}_t^* \mathcal{A}_t d\mu(t) \right|^\theta \right\|^{\frac{1}{2}} \left\| \left| \int_{\Omega} \mathcal{B}_t^* \mathcal{B}_t d\mu(t) \right|^\theta \right\|^{\frac{1}{2}}.$$

Proof. Take Φ to be a s.g. function that generates u.i. norm $\|\cdot\|$, $\Phi_1 = \Phi$, $\Phi_2 = \Phi_3 = \Phi^{(2)}$ (2-reconvexization of Φ), $\alpha = 2\theta$ and $X = I$, and then apply 3.1.(e) from [13].

Now we can easily derive the following generalization of Landau inequality for Gel'fand integrals of o.v. functions:

THEOREM 2.6. (Landau type inequality for o.v. functions in u.i. norm ideals) *If μ is a probability measure on Ω , $\theta > 0$ and $(\mathcal{A}_t)_{t \in \Omega}$ and $(\mathcal{B}_t)_{t \in \Omega}$ are as in Theorem 2.5, then,*

$$\begin{aligned} & \left\| \left| \int_{\Omega} \mathcal{A}_t^* \mathcal{B}_t d\mu(t) - \int_{\Omega} \mathcal{A}_t^* d\mu(t) \int_{\Omega} \mathcal{B}_t d\mu(t) \right|^\theta \right\| \\ & \leq \left\| \left| \int_{\Omega} |\mathcal{A}_t|^2 d\mu(t) - \left| \int_{\Omega} \mathcal{A}_t d\mu(t) \right|^2 \right|^\theta \right\| \left\| \left| \int_{\Omega} |\mathcal{B}_t|^2 d\mu(t) - \left| \int_{\Omega} \mathcal{B}_t d\mu(t) \right|^2 \right|^\theta \right\|. \end{aligned} \tag{2.7}$$

Proof. Apply Theorem 2.5 to o.v. families $(\mathcal{A}_s - \mathcal{A}_t)_{(s,t) \in \Omega^2}$ and $(\mathcal{B}_s - \mathcal{B}_t)_{(s,t) \in \Omega^2}$ and use identity (2.2) once again.

For the more general inequality in an arbitrary Hilbert C^* -module see Theorem 3.4 in [11]. The case when $\theta = 1$ and $\|\cdot\| = \|\cdot\|$ in Theorem 2.6 offers the proof for Hilbert C^* -module $L_G^2(\Omega, d\mu, \mathcal{B}(\mathcal{H}))$ in case of the lifted projection $h(t) = I$ for all $t \in \Omega$.

The case when $\theta = 1$ and $\|\cdot\| = \|\cdot\|_1$ in Theorem 2.6 offers a proof for a stronger version of Theorem 3.3 for Hilbert H^* -module $L_G^2(\Omega, d\mu, \mathcal{C}_1(\mathcal{H}))$ for the same lifted projection $h(t) = I$ for all $t \in \Omega$.

COROLLARY 2.7. Under conditions of Theorem 2.6 we have

$$\begin{aligned}
 & \left| \operatorname{tr} \left(\int_{\Omega} \mathcal{A}_t^* \mathcal{B}_t d\mu(t) - \int_{\Omega} \mathcal{A}_t^* d\mu(t) \int_{\Omega} \mathcal{B}_t d\mu(t) \right) \right|^2 \\
 & \leq \left\| \int_{\Omega} \mathcal{A}_t^* \mathcal{B}_t d\mu(t) - \int_{\Omega} \mathcal{A}_t^* d\mu(t) \int_{\Omega} \mathcal{B}_t d\mu(t) \right\|_1^2 \tag{2.8} \\
 & \leq \left(\int_{\Omega} \|\mathcal{A}_t\|_2^2 d\mu(t) - \left\| \int_{\Omega} \mathcal{A}_t d\mu(t) \right\|_2^2 \right) \left(\int_{\Omega} \|\mathcal{B}_t\|_2^2 d\mu(t) - \left\| \int_{\Omega} \mathcal{B}_t d\mu(t) \right\|_2^2 \right) \\
 & = \left(\left\| \sqrt{\int_{\Omega} |\mathcal{A}_t|^2 d\mu(t)} \right\|_2^2 - \left\| \int_{\Omega} \mathcal{A}_t d\mu(t) \right\|_2^2 \right) \\
 & \quad \cdot \left(\left\| \sqrt{\int_{\Omega} |\mathcal{B}_t|^2 d\mu(t)} \right\|_2^2 - \left\| \int_{\Omega} \mathcal{B}_t d\mu(t) \right\|_2^2 \right).
 \end{aligned}$$

Proof. An application of (2.7) for $\theta = 1$ and $\|\cdot\| = \|\cdot\|_1$ justifies (2.9), while (2.8) and all the remaining identities in (2.10) are obtainable by a straightforward calculations, based on some elementary properties of the trace tr and Gel'fand integrals:

$$\begin{aligned}
 & \left\| \int_{\Omega} \mathcal{A}_t^* \mathcal{B}_t d\mu(t) - \int_{\Omega} \mathcal{A}_t^* d\mu(t) \int_{\Omega} \mathcal{B}_t d\mu(t) \right\|_1^2 \\
 & \leq \left\| \int_{\Omega} |\mathcal{A}_t|^2 d\mu(t) - \left| \int_{\Omega} \mathcal{A}_t d\mu(t) \right|^2 \right\|_1 \left\| \int_{\Omega} |\mathcal{B}_t|^2 d\mu(t) - \left| \int_{\Omega} \mathcal{B}_t d\mu(t) \right|^2 \right\|_1 \tag{2.9} \\
 & = \left(\operatorname{tr} \left(\int_{\Omega} |\mathcal{A}_t|^2 d\mu(t) \right) - \operatorname{tr} \left(\left| \int_{\Omega} \mathcal{A}_t d\mu(t) \right|^2 \right) \right) \\
 & \quad \cdot \left(\operatorname{tr} \left(\int_{\Omega} |\mathcal{B}_t|^2 d\mu(t) \right) - \operatorname{tr} \left(\left| \int_{\Omega} \mathcal{B}_t d\mu(t) \right|^2 \right) \right) \\
 & = \left(\int_{\Omega} \|\mathcal{A}_t\|_2^2 d\mu(t) - \left\| \int_{\Omega} \mathcal{A}_t d\mu(t) \right\|_2^2 \right) \left(\int_{\Omega} \|\mathcal{B}_t\|_2^2 d\mu(t) - \left\| \int_{\Omega} \mathcal{B}_t d\mu(t) \right\|_2^2 \right) \\
 & = \left(\left\| \sqrt{\int_{\Omega} |\mathcal{A}_t|^2 d\mu(t)} \right\|_2^2 - \left\| \int_{\Omega} \mathcal{A}_t d\mu(t) \right\|_2^2 \right) \\
 & \quad \cdot \left(\left\| \sqrt{\int_{\Omega} |\mathcal{B}_t|^2 d\mu(t)} \right\|_2^2 - \left\| \int_{\Omega} \mathcal{B}_t d\mu(t) \right\|_2^2 \right). \tag{2.10}
 \end{aligned}$$

For bounded field of operators $\mathcal{A} = (\mathcal{A}_t)_{t \in \Omega}$ one can easily check that the radius of the smallest disk that essentially contains its range is

$$r_{\infty}(\mathcal{A}) = \inf_{A \in \mathcal{B}(\mathcal{H})} \sup_{t \in \Omega} \|\mathcal{A}_t - A\| = \inf_{A \in \mathcal{B}(\mathcal{H})} \|\mathcal{A} - A\|_{\infty} = \min_{A \in \mathcal{B}(\mathcal{H})} \|\mathcal{A} - A\|_{\infty}$$

(from the triangle inequality we have $\|\|\mathcal{A}_t - A'\| - \|\mathcal{A}_t - A\|\| \leq \|A' - A\|$, so the mapping $A \rightarrow \sup_{t \in \Omega} \|\mathcal{A}_t - A\|$ is nonnegative and continuous on $\mathcal{B}(\mathcal{H})$; since $(\mathcal{A}_t)_{t \in \Omega}$ is a bounded field of operators, we also have $\|\mathcal{A}_t - A\| \rightarrow \infty$ when $\|A\| \rightarrow \infty$, so this mapping attains its minimum), and it actually attains at some $A_0 \in \mathcal{B}(\mathcal{H})$, which represents a center of the considered disk. Any such field of operators is of finite diameter

$$\text{diam}_\infty(\mathcal{A}) = \sup_{s,t \in \Omega} \|\mathcal{A}_s - \mathcal{A}_t\|,$$

with the simple inequalities $r_\infty(\mathcal{A}) \leq \text{diam}_\infty(\mathcal{A}) \leq 2r_\infty(\mathcal{A})$ relating those quantities. For such fields of operators we can now state the following stronger version of the Grüss inequality.

THEOREM 2.8. (Grüss type inequality for i.p.t. integral transformers in u.i. norm ideals) *Let μ be a σ -finite measure on Ω and let $\mathcal{A} = (\mathcal{A}_t)_{t \in \Omega}$ and $\mathcal{B} = (\mathcal{B}_t)_{t \in \Omega}$ be $[\mu]$ a.e. bounded fields of operators. Then for all $X \in \mathcal{C}_{\|\cdot\|}(\mathcal{H})$,*

$$\begin{aligned} \sup_{\mu(\delta) > 0} \left\| \frac{1}{\mu(\delta)} \int_\delta \mathcal{A}_t X \mathcal{B}_t d\mu(t) - \frac{1}{\mu(\delta)} \int_\delta \mathcal{A}_t d\mu(t) X \frac{1}{\mu(\delta)} \int_\delta \mathcal{B}_t d\mu(t) \right\| \\ \leq \min \left\{ r_\infty(\mathcal{A}) r_\infty(\mathcal{B}), \frac{\text{diam}_\infty(\mathcal{A}) \text{diam}_\infty(\mathcal{B})}{2} \right\} \cdot \|X\| \end{aligned} \quad (2.11)$$

(i.e. sup is taken over all measurable sets $\delta \subseteq \Omega$ such that $0 < \mu(\delta) < \infty$).

Proof. Let $r_\infty(\mathcal{A}) = \|\mathcal{A} - A_0\|_\infty = \min_{A \in \mathcal{B}(\mathcal{H})} \|\mathcal{A} - A\|_\infty$, let $r_\infty(\mathcal{B}) = \|\mathcal{B} - B_0\|_\infty = \min_{B \in \mathcal{B}(\mathcal{H})} \|\mathcal{B} - B\|_\infty$ and let us note that

$$\begin{aligned} \frac{1}{\mu(\delta)} \int_\delta |\mathcal{A}_t - A_0|^2 d\mu(t) &\leq \frac{1}{\mu(\delta)} \int_\delta \sup_{t \in \Omega} \|\mathcal{A}_t - A_0\|^2 \cdot I d\mu(t) \\ &= \|\mathcal{A} - A_0\|_\infty^2 \cdot I = r_\infty^2(\mathcal{A}) \cdot I. \end{aligned}$$

Therefore $\left\| \frac{1}{\mu(\delta)} \int_\delta |\mathcal{A}_t - A_0|^2 d\mu(t) \right\|^{1/2} \leq r_\infty(\mathcal{A})$ and $\left\| \frac{1}{\mu(\delta)} \int_\delta |\mathcal{B}_t - B_0|^2 d\mu(t) \right\|^{1/2} \leq r_\infty(\mathcal{B})$ goes similarly. By identity (2.2) applied to the probability measure $\frac{1}{\mu(\delta)} \mu$ on δ and Lemma 1.4 we then have

$$\begin{aligned} &\frac{1}{2\mu(\delta)^2} \int_{\delta^2} |\mathcal{A}_s - \mathcal{A}_t|^2 d(\mu \times \mu)(s,t) \\ &= \frac{1}{\mu(\delta)} \int_\delta \left| \mathcal{A}_t - \frac{1}{\mu(\delta)} \int_\delta \mathcal{A}_t d\mu(t) \right|^2 d\mu(t) \\ &= \frac{1}{\mu(\delta)} \int_\delta |\mathcal{A}_t - A_0|^2 - \left| \frac{1}{\mu(\delta)} \int_\delta \mathcal{A}_t d\mu(t) - A_0 \right|^2 d\mu(t) \\ &\leq \|\mathcal{A} - A_0\|_\infty^2 \cdot I - \left| \frac{1}{\mu(\delta)} \int_\delta \mathcal{A}_t d\mu(t) - A_0 \right|^2 \end{aligned}$$

and therefore

$$\begin{aligned} & \left\| \frac{1}{2\mu(\delta)^2} \int_{\delta^2} |\mathcal{A}_s - \mathcal{A}_t|^2 d(\mu \times \mu)(s, t) \right\| \\ & \leq \left\| \|\mathcal{A} - A_0\|_\infty^2 \cdot I - \left| \frac{1}{2\mu(\delta)} \int_\delta \mathcal{A}_t d\mu(t) - A_0 \right|^2 \right\|. \end{aligned} \quad (2.12)$$

Similarly,

$$\begin{aligned} & \left\| \frac{1}{\mu(\delta)^2} \int_{\delta^2} |\mathcal{B}_s - \mathcal{B}_t|^2 d(\mu \times \mu)(s, t) \right\| \\ & \leq \left\| \|\mathcal{B} - B_0\|_\infty^2 \cdot I - \left| \frac{1}{\mu(\delta)} \int_\delta \mathcal{B}_t d\mu(t) - B_0 \right|^2 \right\|. \end{aligned} \quad (2.13)$$

Those inequalities show that subfields $(\mathcal{A}_t - \mathcal{A}_s)_{(s,t) \in \delta \times \delta}$ and $(\mathcal{B}_t - \mathcal{B}_s)_{(s,t) \in \delta \times \delta}$ are in $L^2(\delta \times \delta, \frac{1}{\mu(\delta)}\mu \times \frac{1}{\mu(\delta)}\mu, \mathcal{B}(\mathcal{H}))$, and therefore according to identity (2.2) and Lemma 3.1(c) from [13],

$$\begin{aligned} & \left\| \frac{1}{\mu(\delta)} \int_\delta \mathcal{A}_t X \mathcal{B}_t d\mu(t) - \frac{1}{\mu(\delta)} \int_\delta \mathcal{A}_t d\mu(t) X \frac{1}{\mu(\delta)} \int_\delta \mathcal{B}_t d\mu(t) \right\| \\ & = \left\| \frac{1}{2\mu(\delta)^2} \int_{\delta^2} (\mathcal{A}_s - \mathcal{A}_t) X (\mathcal{B}_s - \mathcal{B}_t) d(\mu \times \mu)(s, t) \right\| \\ & \leq \left\| \frac{1}{2\mu(\delta)^2} \int_{\delta^2} |\mathcal{A}_s - \mathcal{A}_t|^2 d(\mu \times \mu)(s, t) \right\|^{\frac{1}{2}} \\ & \quad \cdot \left\| \frac{1}{2\mu(\delta)^2} \int_{\delta^2} |\mathcal{B}_s - \mathcal{B}_t|^2 d(\mu \times \mu)(s, t) \right\|^{\frac{1}{2}} \|X\| \\ & \leq \left\| \|\mathcal{A} - A_0\|_\infty^2 \cdot I - \left| \frac{1}{\mu(\delta)} \int_\delta \mathcal{A}_t d\mu(t) - A_0 \right|^2 \right\|^{\frac{1}{2}} \\ & \quad \cdot \left\| \|\mathcal{B} - B_0\|_\infty^2 \cdot I - \left| \frac{1}{\mu(\delta)} \int_\delta \mathcal{B}_t d\mu(t) - B_0 \right|^2 \right\|^{\frac{1}{2}} \|X\| \\ & \leq r_\infty(\mathcal{A}) r_\infty(\mathcal{B}) \|X\|, \end{aligned} \quad (2.14)$$

and the first half of inequality (2.11) is proved. The proof for the remaining part of (2.11) differs from the previous one only by use of obvious estimates

$$\begin{aligned} & \left\| \frac{1}{\mu(\delta)^2} \int_{\delta^2} |\mathcal{A}_s - \mathcal{A}_t|^2 d(\mu \times \mu)(s, t) \right\| \leq \frac{\text{diam}_\infty^2(\mathcal{A})}{2} \\ & \text{(resp. } \left\| \frac{1}{\mu(\delta)^2} \int_{\delta^2} |\mathcal{B}_s - \mathcal{B}_t|^2 d(\mu \times \mu)(s, t) \right\| \leq \frac{\text{diam}_\infty^2(\mathcal{B})}{2} \text{)} \end{aligned}$$

instead if (2.12) (resp. (2.13)) in (2.14).

Now we turn to the less general case when $(\mathcal{A}_t)_{t \in \Omega}$ and $(\mathcal{B}_t)_{t \in \Omega}$ are bounded fields of self-adjoint (generally non-commuting) operators, in which case the above inequality has the following form.

COROLLARY 2.9. Let μ be a probability measure on Ω , let C, D, E, F be bounded self-adjoint operators and let $(\mathcal{A}_t)_{t \in \Omega}$ and $(\mathcal{B}_t)_{t \in \Omega}$ be bounded self-adjoint fields satisfying $C \leq \mathcal{A}_t \leq D$ and $E \leq \mathcal{B}_t \leq F$ for all $t \in \Omega$. Then for all $X \in \mathfrak{C}_{\|\cdot\|}(\mathcal{H})$,

$$\left\| \int_{\Omega} \mathcal{A}_t X \mathcal{B}_t d\mu(t) - \int_{\Omega} \mathcal{A}_t d\mu(t) X \int_{\Omega} \mathcal{B}_t d\mu(t) \right\| \leq \frac{\|D - C\| \cdot \|F - E\|}{4} \cdot \|X\|. \tag{2.15}$$

Proof. As $\frac{C-D}{2} \leq \mathcal{A}_t - \frac{C+D}{2} \leq \frac{D-C}{2}$ for every $t \in \Omega$, then

$$\begin{aligned} \sup_{t \in \Omega} \left\| \mathcal{A}_t - \frac{C+D}{2} \right\| &= \sup_{t \in \Omega} \sup_{\|f\|=1} \left| \left\langle \left(\mathcal{A}_t - \frac{C+D}{2} \right) f, f \right\rangle \right| \\ &\leq \sup_{\|f\|=1} \left| \left\langle \frac{D-C}{2} f, f \right\rangle \right| = \frac{\|D-C\|}{2}, \end{aligned}$$

which implies $r_{\infty}(\mathcal{A}) \leq \frac{\|D-C\|}{2}$, and similarly $r_{\infty}(\mathcal{B}) \leq \frac{\|F-E\|}{2}$. Thus (2.15) follows directly from (2.11).

REMARK 2.10. Note that similar to the estimate (2.14) the righthand side of (2.15) can be improved to

$$\left\| \left\| \frac{D-C}{2} \right\|^2 - \left\| \int_{\Omega} \mathcal{A}_t d\mu(t) - \frac{C+D}{2} \right\|^2 \right\|^{\frac{1}{2}} \left\| \left\| \frac{F-E}{2} \right\|^2 - \left\| \int_{\Omega} \mathcal{B}_t d\mu(t) - \frac{E+F}{2} \right\|^2 \right\|^{\frac{1}{2}} \|X\|. \tag{2.16}$$

An estimate similar to (2.16) was given in a Grüss type inequality for square integrable Hilbert space valued functions in Theorem 3 in [4].

In case of $\mathcal{H} = \mathbb{C}$ and μ being the normalized Lebesgue measure on $[a, b]$ (i.e. $d\mu(t) = \frac{dt}{b-a}$), then (1.1) comes as an obvious corollary of Theorem 2.9. This special case also confirms the sharpness of the constant $\frac{1}{4}$ in the inequality (2.15).

Taking $\Omega = \{1, \dots, n\}$ and μ to be the normalized counting measure we get another corollary of Theorem 2.9, which gives us the following

COROLLARY 2.11. (Grüss type inequality for elementary operators)

Let $A_1, \dots, A_n, B_1, \dots, B_n, C, D, E$ and F be bounded linear self-adjoint operators acting on a Hilbert space \mathcal{H} such that $C \leq A_i \leq D$ and $E \leq B_i \leq F$ for all $i = 1, 2, \dots, n$ then for arbitrary $X \in \mathfrak{C}_{\|\cdot\|}(\mathcal{H})$,

$$\left\| \frac{1}{n} \sum_{i=1}^n A_i X B_i - \frac{1}{n^2} \sum_{i=1}^n A_i X \sum_{i=1}^n B_i \right\| \leq \frac{\|D-C\| \|F-E\|}{4} \|X\|.$$

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