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A NOTE ON SOME INEQUALITIES OF MARTINGALE SHARP FUNCTIONS

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Abstract. In this paper, some new inequalities for the sharp functions of martingales are established by use of rearrangement technique.

1. Introduction and preliminaries

Martingale inequalities of the sharp functions were studied by F. Weisz [1], in which the following martingale inequalities are formulated:

$$\| M(f) \|_p \leqslant C_p \| f^\sharp \|_p, \quad 1 < p < \infty;$$
$$\| S(f) \|_p \leqslant C_p \| f_r^S \|_p, \quad 0 < r < p < \infty;$$
$$\| s(f) \|_p \leqslant C_p \| f_r^s \|_p, \quad 0 < r < p < \infty.$$

These inequalities play an important role in the interpolation between martingale Hardy and BMO spaces. In this paper we adopt rearrangement technique to study these inequalities. Some new martingale inequalities of the sharp functions are obtained. In particular, our results show that the first inequality also holds for $p = 1$, and that there is no connection between p and r for the second and the third inequality to hold when $1 \leqslant p, r < \infty$.

The organization of this paper is divided into two sections. Some basic knowledge, which we will use, is collected in this section. Main results and proofs are given in the next section.

Let (Ω, μ) be a σ-finite measure space, $\mathscr{M}(\Omega)$ the space of all measurable functions on Ω. For $f \in \mathscr{M}(\Omega)$, denote its distribution function by

$$\lambda_f(t) = \mu(x : |f(x)| > t), \ t \geqslant 0,$$

and its decreasing rearrangement function f^* is defined as

$$f^*(t) = \inf\{s > 0 : \lambda_f(s) \leqslant t\}, \ t \geqslant 0.$$

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For $0 < p, q < \infty$, the Lorentz space $L_{p,q}$ is defined as

$$L_{p,q} = L_{p,q}(\Omega, \mathscr{F}, \mu) = \{f : \| f \|_{p,q} < \infty\},$$

where

$$\|f\|_{p,q} = \left(\int_0^\infty (f^*(t))^q t^{\frac{q}{p}} \frac{dt}{t} \right)^{\frac{1}{q}}.$$

We recall that $L_{p,p} = L_p$ for $0 < p < \infty$.

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a complete probability space, and $\{\mathscr{F}_n\}_{n\geqslant 0}$ a nondecreasing sequence of sub-σ-algebras of $\mathscr{F}$ such that $\mathscr{F} = \sigma(\bigcup_n \mathscr{F}_n)$. The conditional expectation operators relative to $\mathscr{F}_n$ are denoted by $\mathbb{E}_n$. For a martingale $f = (f_n)_{n\geqslant 0}$ relative to $(\Omega, \mathscr{F}, \mathbb{P}; (\mathscr{F}_n)_{n\geqslant 0})$, denote its martingale difference by $df_i = f_i - f_{i-1}$ ($i \geqslant 0$, with convention $df_0 = 0$), its maximal function, square function and conditional square function by

$$M_n(f) = \sup_{0\leqslant i\leqslant n} | f_i |, \qquad M(f) = \sup_{i\geqslant 0} | f_i |,$$

$$S_n(f) = (\sum_{i=0}^{n} |df_i|^2)^{\frac{1}{2}}, \qquad S(f) = (\sum_{i=0}^{\infty} |df_i|^2)^{\frac{1}{2}};$$

$$s_n(f) = (\sum_{i=1}^{n} \mathbb{E}_{i-1}|df_i|^2)^{\frac{1}{2}}, \qquad s(f) = (\sum_{i=1}^{\infty} \mathbb{E}_{i-1}|df_i|^2)^{\frac{1}{2}}.$$

For $0 < r < \infty$, the sharp functions of a martingale $f = (f_n)_{n\geqslant 0}$ are defined as

$$f^\sharp = \sup_{n\geqslant 0} \mathbb{E}_n | f - f_{n-1} |;$$

$$f_r^S = \sup_{n\geqslant 0} (\mathbb{E}_n [S^2(f) - S_{n-1}^2(f)]^{\frac{r}{2}})^{\frac{1}{r}};$$

$$f_r^s = \sup_{n\geqslant 0} (\mathbb{E}_n [s^2(f) - s_n^2(f)]^{\frac{r}{2}})^{\frac{1}{r}}.$$

Throughout this paper, we use C or C_p (depending only on p) to denote some constant and may be different at each occurrence.

2. Some martingale inequalities for the sharp functions

In this section, we first establish some martingale rearrangement inequalities. By use of these rearrangement inequalities and Hardy inequality, we formulate some new martingale inequalities of the sharp functions.

LEMMA 1. *Let $1 \leqslant r < \infty$. Then for any martingale $f = (f_n)_{n\geqslant 0}$ we have*

$$s(f)^*(t) \leqslant 4 f_r^{s*}\left(\frac{t}{2}\right) + s(f)^*(2t), \quad t > 0.$$

Proof. Set $\eta_n = (\mathbb{E}_n[s^2(f) - s_n^2(f)]^{\frac{r}{2}})^{\frac{1}{r}}$, define stopping times as follows:

$$\mu = \inf\{n : s_{n+1}(f) > s(f)^*(2t)\},$$

$$\nu = \inf\left\{n : s_{n+1}(f) > 4f_r^{s*}\left(\frac{t}{2}\right) + s(f)^*(2t)\right\},$$

$$\tau = \inf\left\{n : \eta_n > f_r^{s*}\left(\frac{t}{2}\right)\right\}.$$

Notice that $\mu \leqslant \nu$, and

$$\{\mu < \infty\} = \{s(f) > s(f)^*(2t)\},$$

$$\{\nu < \infty\} = \left\{s(f) > 4f_r^{s*}\left(\frac{t}{2}\right) + s(f)^*(2t)\right\},$$

$$\{\tau < \infty\} = \left\{f_r^s > f_r^{s*}\left(\frac{t}{2}\right)\right\},$$

$$\mathbb{P}(\mu < \infty) \leqslant 2t, \quad \mathbb{P}(\tau < \infty) \leqslant \frac{t}{2},$$

we have

$$\begin{aligned}\{\nu < \infty\} &= \{\nu < \infty, \mu < \tau\} \cup \{\nu < \infty, \mu \geqslant \tau\} \\ &\subseteq \{\tau < \infty\} \cup \{\nu < \infty, \mu < \tau\},\end{aligned}$$

$$\{\nu < \infty, \mu < \tau\} \subseteq \left\{\mu < \tau, s(f) - s_\mu(f) > 4f_r^{s*}\left(\frac{t}{2}\right)\right\}.$$

Therefore

$$\begin{aligned}\mathbb{P}(\nu < \infty, \mu < \tau) &\leqslant \frac{1}{4f_r^{s*}\left(\frac{t}{2}\right)} \int_{\{\mu<\tau\}} (s(f) - s_\mu(f))d\mathbb{P} \\ &\leqslant \frac{1}{4f_r^{s*}\left(\frac{t}{2}\right)} \int_{\{\mu<\tau\}} \mathbb{E}[s(f) - s_\mu(f) \mid \mathscr{F}_\mu]d\mathbb{P} \\ &\leqslant \frac{1}{4f_r^{s*}\left(\frac{t}{2}\right)} \int_{\{\mu<\tau\}} (\mathbb{E}[((s(f))^2 - (s_\mu(f))^2)^{\frac{r}{2}} \mid \mathscr{F}_\mu])^{\frac{1}{r}} d\mathbb{P} \\ &\leqslant \frac{1}{4}\mathbb{P}(\mu < \infty) \leqslant \frac{t}{2}.\end{aligned}$$

Thus we get

$$\begin{aligned}\mathbb{P}\left(s(f) > 4f_r^{s*}\left(\frac{t}{2}\right) + s(f)^*(2t)\right) &\leqslant \mathbb{P}(\tau < \infty) + \mathbb{P}(\nu < \infty, \mu < \tau) \\ &\leqslant \frac{t}{2} + \frac{t}{2} \leqslant t.\end{aligned}$$

It follows that

$$s(f)^*(t) \leqslant 4f_r^{s*}\left(\frac{t}{2}\right) + s(f)^*(2t), \quad t > 0.$$

The proof is completed. □

REMARK 1. (1) Set $\eta_n = (\mathbb{E}_n[S^2(f) - S_{n-1}^2(f) \,|^{\frac{r}{2}})^{\frac{1}{r}}$, define stopping times:

$$\mu = \inf\{n : S_n(f) > S(f)^*(2t)\},$$

$$\nu = \inf\left\{n : S_n(f) > 4f_r^{S*}\left(\frac{t}{2}\right) + S(f)^*(2t)\right\},$$

$$\tau = \inf\left\{n : \eta_n > f_r^{S*}\left(\frac{t}{2}\right)\right\}.$$

Similar to the proof of Lemma 1, we can prove that for $1 \leqslant r < \infty$,

$$S(f)^*(t) \leqslant 4f_r^{S*}\left(\frac{t}{2}\right) + S(f)^*(2t), \quad t > 0.$$

(2) It was proved by Long [2,3] that

$$(M(f))^*(t) \leqslant 4f^{\sharp*}\left(\frac{t}{2}\right) + (M(f))^*(2t), \quad t > 0.$$

LEMMA 2. [2,4] *Let (F,G) be a pair of nonnegative measurable functions on $(\Omega,\mathscr{F},\mathbb{P})$. If (F,G) satisfies the rearrangement inequality :*

$$F^*(t) \leqslant F^*(2t) + CG^*\left(\frac{t}{2}\right), \quad \forall t > 0.$$

Then with the same C, we have

$$F^*(t) \leqslant 2CG^*\left(\frac{t}{2}\right) + \frac{C}{\log 2}\int_t^\infty \frac{G^*(s)}{s}ds, \quad \forall t > 0.$$

LEMMA 3. [5] (Hardy's inequality) *If $1 \leqslant q < \infty$, $r > 0$ and f is a nonnegative function defined on $(0,\infty)$, then*

$$\left(\int_0^\infty \left(\int_t^\infty f(u)du\right)^q t^r \frac{dt}{t}\right)^{\frac{1}{q}} \leqslant \frac{q}{r}\left(\int_0^\infty (tf(t))^q t^r \frac{dt}{t}\right)^{\frac{1}{q}}.$$

Now we are in the position to state our main results.

THEOREM 1. *Let $0 < p < \infty, 1 \leqslant q, r < \infty$. Then for any martingale $f = (f_n)_{n\geqslant 0}$ we have*

(1) $\| M(f) \|_{p,q} \leqslant C_{p,q} \| f^\sharp \|_{p,q}$;
(2) $\| S(f) \|_{p,q} \leqslant C_{p,q} \| f_r^S \|_{p,q}$;
(3) $\| s(f) \|_{p,q} \leqslant C_{p,q} \| f_r^s \|_{p,q}$.

Proof. We only prove (1), since the proofs of (2) and (3) are similar.

It follows from (2) in Remark 1, Lemma 2 and 3 that

$$\begin{aligned}
\| M(f) \|_{p,q} &= \left(\int_0^\infty (M(f)^*(t))^q t^{\frac{q}{p}} \frac{dt}{t}\right)^{\frac{1}{q}} \\
&\leqslant C_q \left(\left(\int_0^\infty \left(f^{\sharp *}\left(\frac{t}{2}\right)\right)^q t^{\frac{q}{p}} \frac{dt}{t}\right)^{\frac{1}{q}} + \left(\int_0^\infty \left(\int_t^\infty \frac{f^{\sharp *}(s)}{s} ds\right)^q t^{\frac{q}{p}} \frac{dt}{t}\right)^{\frac{1}{q}} \right) \\
&\leqslant C_{p,q} \left(\int_0^\infty (f^{\sharp *}(t))^q t^{\frac{q}{p}} \frac{dt}{t}\right)^{\frac{1}{q}} \\
&= C_{p,q} \| f^\sharp \|_{p,q} .
\end{aligned}$$

The proof is completed. □

Let $1 \leqslant p < \infty$ and $p = q$, it follows from Theorem 1 that

COROLLARY 1. *Let* $1 \leqslant p, r < \infty$. *Then for any martingale* $f = (f_n)_{n \geqslant 0}$ *we have*
(1) $\| M(f) \|_p \leqslant C_p \| f^\sharp \|_p$;
(2) $\| S(f) \|_p \leqslant C_p \| f_r^S \|_p$;
(3) $\| s(f) \|_p \leqslant C_p \| f_r^s \|_p$.

REMARK 2. In F. Weisz [1], it was proved that the inequality (1) in Corollary 1 holds for $1 < p < \infty$, and that the inequalities (2) and (3) in Corollary 1 hold for $0 < r < p < \infty$. Here we show that the inequality (1) also holds for $p = 1$, and that there is no connection between p and r for the inequalities (2) and (3) to hold when $1 \leqslant p, r < \infty$.

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