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NEW SHARP ESTIMATES OF THE GENERALIZED EULER–MASCHERONI CONSTANT

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Abstract. The aim of this paper is to establish new sequences which converge towards the Euler-Mascheroni constant. Our results solve some open problems posed by Berinde [A new generalization of Euler's constant Creat. Math. Inform. 18 (2009) no. 2 123–128] and extend some results of DeTemple, [A quicker convergence to Euler's constant Amer. Math. Monthly 100 (1993) 468–470] and Sîntămărian [A generalization of Euler's constant, Numer. Algorithms 46 (2007), 141–151].

1. Introduction

One of the most important sequences in analysis of the form

$$\gamma_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \ln n,$$

considered by Leonhard Euler in 1735, is known to converge towards the limit $\gamma = 0.577215...$, which is now called the Euler-Mascheroni constant. First of all, we recall that the sequence $(\gamma_n)_{n\geqslant 1}$ converges to its limit like n^{-1}, since

$$\frac{1}{2n+1} < \gamma_n - \gamma < \frac{1}{2n}, \tag{1.1}$$

(see, *e.g.*, Alzer [1], or Young [14]). Tóth [13] proved

$$\frac{1}{2n+2/5} < \gamma_n - \gamma \leqslant \frac{1}{2n+1/3}, \quad n \geqslant 1, \tag{1.2}$$

then Qiu and Vuorinen [11] showed the double inequality

$$\frac{1}{2n} - \frac{1}{2n^2} < \gamma_n - \gamma \leqslant \frac{1}{2n} - \frac{\gamma - 1/2}{n^2}, \quad n \geqslant 1. \tag{1.3}$$

Questions on the fast approximations of the Euler-Mascheroni constant γ were also discussed by Karatsuba [4] and the following inequalities were obtained

$$\frac{1}{2n} - \frac{1}{12n^2} + \frac{1}{120n^4} - \frac{1}{126n^6} \leqslant \gamma_n - \gamma \leqslant \frac{1}{2n} - \frac{1}{12n^2} + \frac{1}{120n^4}. \tag{1.4}$$

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For every $a > 0$, the numbers of the form

$$\gamma(a) = \lim_{n\to\infty}\left(\frac{1}{a}+\frac{1}{a+1}+\ldots+\frac{1}{a+n-1}-\ln\frac{a+n-1}{a}\right)$$

were introduced in the monograph by Knopp [5]. There are known now as the generalized Euler-Mascheroni constant, since $\gamma(1)=\gamma$. In the recent past, many authors were preoccupied to give increasingly accurate estimates for $\gamma(a)$, similar to those given for γ, like (1.1)–(1.4).

In this sense, we mention the following sequences

$$x_n = \frac{1}{a}+\frac{1}{a+1}+\ldots+\frac{1}{a+n-1}-\ln\frac{a+n}{a}$$

and

$$y_n = \frac{1}{a}+\frac{1}{a+1}+\ldots+\frac{1}{a+n-1}-\ln\frac{a+n-1}{a}$$

which converge to $\gamma(a)$ like n^{-1}, since Sîntămărian [12] proved that for every integer $n \geqslant 1$,

$$\frac{1}{2(n+a)} < \gamma(a) - x_n < \frac{1}{2(n+a)-2}$$

and

$$\frac{1}{2(n+a)} < y_n - \gamma(a) < \frac{1}{2(n+a)-2}.$$

We give new better bounds for these sequences, showing the following

THEOREM 1. *For every $a > 0$, and integer $n \geqslant 2$, we have*

$$\frac{1}{2(n+a)-\frac{1}{4}} < \gamma(a) - x_n < \frac{1}{2(n+a)-\frac{1}{3}} \tag{1.5}$$

and

$$\frac{1}{2(n+a)-\frac{4}{3}} < y_n - \gamma(a) < \frac{1}{2(n+a)-\frac{5}{3}}. \tag{1.6}$$

In some sense, the constants $\frac{1}{3}$ and $\frac{5}{3}$ are sharp in (1.5)–(1.6), as we can see from the following:

THEOREM 2. *a) For every $a \geqslant \frac{13}{30}$ and every integer $n \geqslant 1$, we have*

$$\frac{1}{2(n+a)-\frac{1}{3}+\frac{1}{18n}} < \gamma(a) - x_n.$$

b) For every $a \geqslant \frac{17}{30}$ and every integer $n \geqslant 1$, we have

$$\frac{1}{2(n+a)-\frac{5}{3}+\frac{1}{18n}} < y_n - \gamma(a).$$

Very recently, Berinde [2, Theorem 2.2] introduced the sequences

$$z_n(a,b) = \frac{1}{a} + \frac{1}{a+1} + \ldots + \frac{1}{a+n-1} - \ln\left(\frac{a+n}{a} + b\right)$$

and

$$t_n(a,b) = \frac{1}{a} + \frac{1}{a+1} + \ldots + \frac{1}{a+n-1} - \ln\left(\frac{a+n-1}{a} + b\right)$$

convergent to $\gamma(a)$, and proved that for every integer $n \geqslant 1$,

$$z_n(a,b) < z_{n+1}(a,b) < \gamma(a) < t_n(a,b) < t_{n+1}(a,b)$$

and

$$0 < \frac{1}{a} - \ln\left(1 + b + \frac{1}{a}\right) < \gamma(a) < \frac{1}{a} - \ln b.$$

It is introduced in [7] the following general class of sequences

$$\mu_n(a,b,c) = \frac{1}{a} + \frac{1}{a+1} + \ldots + \frac{1}{a+n-2} + \frac{c}{a+n-1} - \ln\left(\frac{a+n-1}{a} + b\right),$$

depending on parameters a,b,c, with $a > 0$ and $b > -(a+1)/a$. Remark that $\mu_n(a,b,1) = t_n(a,b)$. A particular case of [7, Theorem 2.1] solves an open problem posed by Berinde [2] about the sequence $(t_n(a,b))_{n \geqslant 1}$. This answer is gathered in the following

THEOREM 3. *Let a, $b \in \mathbb{R}$ be given and satisfy $a > 0$ and $b > -(a+1)/a$.*
a) If $b \neq \frac{1}{2a}$, the speed of convergence of the sequence $(t_n(a,b))_{n \geqslant 2}$ is equal to n^{-1}, since

$$\lim_{n\to\infty} n(t_n(a,b) - \gamma(a)) = \frac{1}{2} - ab \neq 0.$$

b) If $b = \frac{1}{2a}$, the speed of convergence of the sequence

$$\beta_n = \frac{1}{a} + \frac{1}{a+1} + \ldots + \frac{1}{a+n-2} + \frac{1}{a+n-1} - \ln\left(\frac{a+n-1}{a} + \frac{1}{2a}\right)$$

equals n^{-2}, since

$$\lim_{n\to\infty} n^2(\beta_n - \gamma(a)) = \frac{1}{24}.$$

The proof of this Theorem 3 is based on the following result, which was first used in [6]–[10] to accelerate some convergences and to construct asymptotic expansions.

LEMMA 1. *If $(x_n)_{n \geqslant 1}$ is convergent to x and if there exists the limit*

$$\lim_{n\to\infty} n^k(x_n - x_{n+1}) = l \in \overline{\mathbb{R}},$$

with $k > 1$, then there exists the limit

$$\lim_{n\to\infty} n^{k-1}(x_n - x) = \frac{l}{k-1}.$$

For proof, see [9]. The following result gives a similar answer for the sequence $(z_n)_{n\geqslant 1}$.

THEOREM 4. *Let $a,\ b\in\mathbb{R}$ be given and satisfy $a>0$ and $b>-(a+1)/a$.*
a) If $b\neq -\frac{1}{2a}$, the speed of convergence of the sequence $(z_n(a,b))_{n\geqslant 2}$ is equal to n^{-1}, since

$$\lim_{n\to\infty} n\left(z_n(a,b)-\gamma(a)\right)=-\frac{1}{2}-ab\neq 0.$$

b) If $b=-\frac{1}{2a}$, the speed of convergence of the sequence

$$\delta_n=\frac{1}{a}+\frac{1}{a+1}+\ldots+\frac{1}{a+n-2}+\frac{1}{a+n-1}-\ln\left(\frac{a+n}{a}-\frac{1}{2a}\right)$$

equals n^{-2}, since

$$\lim_{n\to\infty} n^2\left(\delta_n-\gamma(a)\right)=\frac{1}{24}.$$

We have

$$z_n(a,b)-z_{n+1}(a,b)=-\frac{1}{a+n}-\ln\left(\frac{a+n}{a}+b\right)+\ln\left(\frac{a+n+1}{a}+b\right),$$

or, using a computer software, such as Maple,

$$z_n(a,b)-z_{n+1}(a,b)=\left(-\frac{1}{2}-ab\right)\frac{1}{n^2}+\left(a+ab+2a^2b+a^2b^2+\frac{1}{3}\right)\frac{1}{n^3}+O\left(\frac{1}{n^4}\right). \tag{1.7}$$

Now, we have

$$\lim_{n\to\infty} n^2\left(z_n(a,b)-z_{n+1}(a,b)\right)=-\frac{1}{2}-ab,$$

and if $ab=-\frac{1}{2}$, then

$$\lim_{n\to\infty} n^3\left(z_n(a,b)-z_{n+1}(a,b)\right)=\frac{1}{12}.$$

and Theorem 4 follows using Lemma 1.

Next we give some estimates of the sequences $(\beta_n)_{n\geqslant 1}$ and $(\delta_n)_{n\geqslant 1}$.

THEOREM 5. *a) For every integer $n\geqslant 1$, we have*

$$\frac{1}{24(n+a)^2}<\delta_n-\gamma(a)<\frac{1}{24(n+a-1)^2}$$

and

$$\frac{1}{24(n+a)^2}<\beta_n-\gamma(a)<\frac{1}{24(n+a-1)^2}.$$

This result is an extension of DeTemple's work [3] who defined the sequence

$$R_n = 1 + \frac{1}{2} + \dots + \frac{1}{n} - \ln\left(n + \frac{1}{2}\right)$$

and proved

$$\frac{1}{24(n+1)^2} < R_n - \gamma < \frac{1}{24n^2}.$$

(case $a = 1$). Also Theorem 5 responses to an open problem posed by Berinde [2].

2. The proofs

Proof of Theorem 1. The sequences

$$x_n' = x_n + \frac{1}{2(n+a) - \frac{1}{4}}, \quad x_n'' = x_n + \frac{1}{2(n+a) - \frac{1}{3}}$$

are convergent to $\gamma(a)$. Our inequalities (1.5) follows, if we prove that the sequence $(x_n')_{n\geqslant 1}$ is strictly increasing and the sequence $(x_n'')_{n\geqslant 1}$ is strictly decreasing. We have $x_{n+1}' - x_n' = c(n)$, where

$$c(x) = \frac{1}{a+x} - \ln\frac{a+x+1}{a} + \ln\frac{a+x}{a} + \frac{1}{2(x+1+a) - \frac{1}{4}} - \frac{1}{2(x+a) - \frac{1}{4}},$$

with the derivative

$$c'(x) = -\frac{P(x)}{(x+a+1)(x+a)^2(8x+8a-1)^2(8x+8a+7)^2},$$

where

$$P(x) = 512x^3 + (1536a - 128)x^2 + \left(1536a^2 - 256a - 672\right)x$$
$$+512a^3 - 128a^2 - 672a + 49.$$

As the polynomial $P(x+2)$ has all coefficients positive, it results that $P(n) > 0$, for every $n \geqslant 2$. Now, $c(x)$ is strictly decreasing, with $c(\infty) = 0$, so $c > 0$, on $[2, \infty)$. Thus $(x_n')_{n\geqslant 2}$ is strictly increasing and consequently, $x_n' < \gamma(a)$.

Let $x_{n+1}'' - x_n'' = d(n)$, where

$$d(x) = \frac{1}{a+x} - \ln\frac{a+x+1}{a} + \ln\frac{a+x}{a} + \frac{1}{2(x+1+a) - \frac{1}{3}} - \frac{1}{2(x+a) - \frac{1}{3}},$$

with the derivative

$$d'(x) = \frac{216x^2 + (432a + 240)x + 240a + 216a^2 - 25}{(x+a+1)(x+a)^2(6x+6a-1)^2(6x+6a+5)^2} > 0.$$

Now, $d(x)$ is strictly increasing, with $d(\infty) = 0$, so $d < 0$, on $[2, \infty)$. Thus $(x_n'')_{n\geqslant 2}$ is strictly decreasing and consequently, $x_n'' > \gamma(a)$.

The sequences

$$y_n' = y_n - \frac{1}{2(n+a) - \frac{4}{3}}, \quad y_n'' = y_n - \frac{1}{2(n+a) - \frac{5}{3}}$$

are convergent to $\gamma(a)$. Our inequalities (1.6) follows, if we prove that the sequence $(y_n')_{n\geqslant 1}$ is strictly decreasing and the sequence $(y_n'')_{n\geqslant 1}$ is strictly increasing. We have $y_{n+1}' - y_n' = e(n)$, where

$$e(x) = \frac{1}{a+x} - \ln\frac{a+x}{a} + \ln\frac{a+x-1}{a} - \frac{1}{2(x+1+a) - \frac{4}{3}} + \frac{1}{2(x+a) - \frac{4}{3}},$$

with the derivative

$$e'(x) = \frac{Q(x)}{2(x+a-1)(x+a)^2(3x+3a-2)^2(3x+3a+1)^2},$$

where

$$Q(x) = 81x^3 + (243a-81)x^2 + \left(243a^2 - 162a + 24\right)x + 24a - 81a^2 + 81a^3 + 8.$$

As the polynomial $Q(x+2)$ has all coefficients positive, it results that $Q(n) > 0$, for every $n \geqslant 2$. Now, $e(x)$ is strictly increasing, with $e(\infty) = 0$, so $e < 0$, on $[2,\infty)$. Thus $(y_n')_{n\geqslant 2}$ is strictly decreasing and consequently, $y_n' > \gamma(a)$.

Let $y_{n+1}'' - y_n'' = j(n)$, where

$$j(x) = \frac{1}{a+x} - \ln\frac{a+x}{a} + \ln\frac{a+x-1}{a} - \frac{1}{2(x+1+a) - \frac{5}{3}} + \frac{1}{2(x+a) - \frac{5}{3}},$$

with the derivative

$$j'(x) = -\frac{216x^2 + (432a-240)x + 216a^2 - 240a - 25}{(x+a-1)(x+a)^2(6x+6a-5)^2(6x+6a+1)^2} < 0.$$

Now, $j(x)$ is strictly decreasing, with $j(\infty) = 0$, so $j > 0$, on $[2,\infty)$. Thus $(y_n'')_{n\geqslant 2}$ is strictly increasing and consequently, $y_n'' < \gamma(a)$. □

Proof of Theorem 2. As in the proof of Theorem 1, we define the sequences

$$u_n = x_n + \frac{1}{2(n+a) - \frac{1}{3} + \frac{1}{18n}}, \quad v_n = y_n - \frac{1}{2(n+a) - \frac{5}{3} + \frac{1}{18n}}$$

and we prove that $(u_n)_{n\geqslant 1}$ is strictly increasing and $(v_n)_{n\geqslant 1}$ is strictly decreasing.

First, we have $u_{n+1} - u_n = k(n)$, and $v_{n+1} - v_n = l(n)$, where

$$k(x) = \frac{1}{x+a} - \ln\frac{a+x+1}{a} + \ln\frac{a+x}{a}$$

$$+\frac{1}{2(x+1+a)-\frac{1}{3}+\frac{1}{18(x+1)}}-\frac{1}{2(x+a)-\frac{1}{3}+\frac{1}{18x}}$$

and

$$l(x)=\frac{1}{x+a}-\ln\frac{a+x}{a}+\ln\frac{a+x-1}{a}$$
$$-\frac{1}{2(x+1+a)-\frac{5}{3}+\frac{1}{18(x+1)}}+\frac{1}{2(x+a)-\frac{5}{3}+\frac{1}{18x}},$$

with the derivatives

$$k'(x)=-\frac{R(x)}{(x+a+1)(x+a)^2(36x^2-6x+36ax+1)^2(36x^2+66x+36ax+36a+31)^2},$$

respective

$$l'(x)=\frac{S(x)}{(x+a-1)(x+a)^2(36x^2-30x+36ax+1)^2(36x^2+42x+36ax+36a+7)^2},$$

where

$$R(x)=15552(30a-13)x^5+\left(653\,184a+1166\,400a^2-250\,128\right)x^4$$
$$+\left(413\,424a+1842\,912a^2+979\,776a^3+37\,800\right)x^3$$
$$+\left(272\,808a+1131\,408a^2+1251\,936a^3+326\,592a^4+79\,200\right)x^2$$
$$+\left(85\,248a+349\,272a^2+498\,960a^3+287\,712a^4+46\,656a^5-7440\right)x$$
$$+2232a+19\,224a^2+58\,104a^3+63\,504a^4+23\,328a^5+961$$

and

$$S(x)=15\,552(30a-17)x^5+\left(1166\,400a^2-653\,184a+247\,536\right)x^4$$
$$+\left(979\,776a^3-116\,640a^2-239\,760a+360\,072\right)x^3$$
$$+\left(267\,624a-501\,552a^2+381\,024a^3+326\,592a^4+13\,104\right)x^2$$
$$+\left(132\,192a^4-26\,568a^2-60\,912a^3-26\,208a+46\,656a^5-2352\right)x$$
$$+504a-216a^2-7560a^3-14\,256a^4+23\,328a^5+49.$$

If we put $a=\frac{13}{30}+a'$, with $a'\geqslant 0$, then $R(n)$ becomes a polynomial with all coefficients positive. If we put $a=\frac{17}{30}+a''$, with $a''\geqslant 0$, then $S(n)$ becomes a polynomial with all coefficients positive. In consequence, $R>0$ and $S>0$, for every positive integer n and $a\geqslant\frac{13}{30}$, respective $a\geqslant\frac{17}{30}$.

Now, the function k is strictly decreasing, the function l is strictly increasing and the conclusion follows using the same arguments of Theorem 1. □

Proof of Theorem 5. Let us define the sequences

$$\delta_n' = \delta_n - \frac{1}{24\,(n+a-1)^2}, \quad \delta_n'' = \delta_n - \frac{1}{24\,(n+a)^2}.$$

It suffices to show that $(\delta_n')_{n\geqslant 1}$ is strictly increasing and $(\delta_n'')_{n\geqslant 1}$ is strictly decreasing. In this sense, let us put $\delta_{n+1}' - \delta_n' = m(n)$ and $\delta_{n+1}'' - \delta_n'' = p(n)$, where

$$m(x) = \frac{1}{a+x} - \ln\left(\frac{a+x+1}{a} - \frac{1}{2a}\right) + \ln\left(\frac{a+x}{a} - \frac{1}{2a}\right)$$

$$-\frac{1}{24\,(x+a)^2} + \frac{1}{24\,(x+a-1)^2}$$

and

$$p(x) = \frac{1}{a+x} - \ln\left(\frac{a+x+1}{a} - \frac{1}{2a}\right) + \ln\left(\frac{a+x}{a} - \frac{1}{2a}\right)$$

$$-\frac{1}{24\,(x+a+1)^2} + \frac{1}{24\,(x+a)^2},$$

with the derivatives

$$m'(x) = -\frac{24x^3 + (72a-35)x^2 + \left(72a^2 - 70a + 15\right)x + 15a - 35a^2 + 24a^3 - 1}{12\,(2x+2a+1)\,(2x+2a-1)\,(x+a)^3\,(x+a-1)^3},$$

respective

$$p'(x) = \frac{24x^3 + (72a+35)x^2 + \left(70a + 72a^2 + 15\right)x + \left(70a + 72a^2 + 15\right) + 1}{12\,(2x+2a+1)\,(2x+2a-1)\,(x+a)^3\,(x+a+1)^3}.$$

Now, the function m is strictly decreasing, p is strictly increasing, with $m(\infty) = p(\infty) = 0$, so $m > 0$ and $p < 0$. Consequently, $(\delta_n')_{n\geqslant 1}$ is strictly increasing and $(\delta_n'')_{n\geqslant 1}$ is strictly decreasing.

Let us define the sequences

$$\beta_n' = \beta_n - \frac{1}{24\,(n+a-1)^2}, \quad \beta_n'' = \beta_n - \frac{1}{24\,(n+a)^2}.$$

It suffices to show that $(\beta_n')_{n\geqslant 1}$ is strictly increasing and $(\beta_n'')_{n\geqslant 1}$ is strictly decreasing. In this sense, let us put $\beta_{n+1}' - \beta_n' = q(n)$ and $\delta_{n+1}'' - \delta_n'' = r(n)$, where

$$q(x) = \frac{1}{a+x} - \ln\left(\frac{a+x}{a} + \frac{1}{2a}\right) + \ln\left(\frac{a+x-1}{a} + \frac{1}{2a}\right)$$

$$-\frac{1}{24\,(x+a)^2} + \frac{1}{24\,(x+a-1)^2}$$

and

$$r(x) = \frac{1}{a+x} - \ln\left(\frac{a+x}{a} + \frac{1}{2a}\right) + \ln\left(\frac{a+x-1}{a} + \frac{1}{2a}\right) - \frac{1}{24(x+a+1)^2} + \frac{1}{24(x+a)^2},$$

with the derivatives

$$q'(x) = -\frac{24x^3 + (72a-35)x^2 + \left(72a^2 - 70a + 15\right)x + 15a - 35a^2 + 24a^3 + 24n^3 - 1}{12(2x+2a-1)(2x+2a+1)(x+a)^3(x+a-1)^3},$$

respective

$$r'(x) = \frac{24x^3 + (72a+35)x^2 + \left(70a + 72a^2 + 15\right)x + 15a + 35a^2 + 24a^3 + 1}{12(2x+2a-1)(2x+2a+1)(x+a)^3(x+a-1)^3}.$$

Now, the function q is strictly decreasing, r is strictly increasing, with $q(\infty) = r(\infty) = 0$, so $q > 0$ and $r < 0$. Consequently, $(\beta_n')_{n\geqslant 1}$ is strictly increasing and $(\beta_n'')_{n\geqslant 1}$ is strictly decreasing. $\square$

Finally, we are convinced that our new method is suitable for establishing other new estimates for the gamma and polygamma functions, or for the generalized harmonic sums.

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