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WEIGHTED COMPOSITION OPERATOR FROM BLOCH–TYPE SPACE TO H^∞ SPACE ON THE UNIT BALL

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Abstract. In this paper, we characterize those holomorphic symbols u on the unit ball $\mathbb{B}$ and holomorphic self-mappings φ of $\mathbb{B}$ for which the weighted composition operator uC_φ is bounded or compact from Bloch-type space to H^∞ space.

1. Introduction

Let $\mathbb{B}$ be the open unit ball in $\mathbb{C}^n$. When $n = 1$, the unit ball is just the unit disc $\mathbb{D}$. Let $H(\mathbb{B})$ be the family of all holomorphic functions on $\mathbb{B}$. We denote by H^∞ the space of holomorphic functions on $\mathbb{B}$, with the norm $\|f\|_\infty = \sup\limits_{z\in\mathbb{B}} |f(z)|$. And let $S(\mathbb{B})$ be the set of all holomorphic self-mappings on $\mathbb{B}$.

A positive continuous function ω on $[0,1)$ is called normal if there are two constants $b > a > 0$ such that

$$\frac{\omega(r)}{(1-r)^a} \downarrow 0, \qquad \frac{\omega(r)}{(1-r)^b} \uparrow \infty \tag{1.1}$$

as $r \to 1^-$. Given ω, we extend it to $\mathbb{B}$ by setting $\omega(z) = \omega(|z|)$. A function $f \in H(\mathbb{B})$ is said to belong to the Bloch-type space $\mathscr{B}_\omega$ if

$$\|f\|_{\mathscr{B},\omega} = \sup_{z\in\mathbb{B}} \omega(z)|\nabla f(z)| < \infty,$$

and it is said to belong to the little Bloch-type space $\mathscr{B}_{\omega,0}$ if

$$\lim_{|z|\to 1} \omega(z)|\nabla f(z)| = 0.$$

Here $\nabla f(z) = \left(\frac{\partial f}{\partial z_1}, \cdots, \frac{\partial f}{\partial z_n}\right)$ is the complex gradient of f. It is easy to check that both $\mathscr{B}_\omega$ and $\mathscr{B}_{\omega,0}$ are Banach spaces under the norm

$$\|f\|_\omega = |f(0)| + \|f\|_{\mathscr{B},\omega}$$

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and that $\mathscr{B}_{\omega,0}$ is a closed subspace of $\mathscr{B}_{\omega}$. When $\omega(r) = 1 - r^2$ and $\omega(r) = (1-r^2)^{1-\alpha}$ with $\alpha \in (0,1)$, two typical normal weights, the induced spaces $\mathscr{B}_{\omega}$ are the Bloch space and the Lipschitz space, respectively. And also, the space $\mathscr{B}_{(1-r^2)\log 1/(1-r^2)}$ is the logarithmic Bloch space which was studied in [1]. For $f \in H(\mathbb{B})$, let $\Re f(z) = \sum_{j=1}^{n} z_j \frac{\partial f}{\partial z_j}(z)$ be the radial derivative of f. From [2], we know that $f \in \mathscr{B}_{\omega}$ if and only if $\sup_{z\in\mathbb{B}} \omega(z)|\Re f(z)| < \infty$, and $f \in \mathscr{B}_{\omega,0}$ if and only if $\lim_{|z|\to 1} \omega(z)|\Re f(z)| = 0$. Furthermore, the norm $\|f\|_{\omega}$ is equivalent to $|f(0)| + \sup_{z\in\mathbb{B}} \omega(z)|\Re f(z)|$.

Given $u \in H(\mathbb{B})$ and $\varphi \in S(\mathbb{B})$, the weighted composition operator uC_{φ} is defined by

$$uC_{\varphi}(f)(z) = u(z)f(\varphi(z)), \quad f \in H(\mathbb{B}), \quad z \in \mathbb{B}.$$

It is interesting to provide a function theoretic characterization when u and φ induce a bounded or compact operator on some holomorphic function spaces. For some classical results, see, for example, [3, 4]. And very recently, the boundedness and compactness of the weighted composition operator between the Bloch space and H^{∞} was studied on the unit disc in [5]. The case of the unit polydisk was discussed in [6-8]. The same problem in the setting of the unit ball was investigated in [9, 10]. For related results see also [11-17] and the references therein.

The purpose of this work is to obtain the sufficient and necessary conditions on $u \in H(\mathbb{B})$ and $\varphi \in S(\mathbb{B})$, such that the operator uC_{φ} is bounded or compact from the Bloch-type space to H^{∞} space. Our work will generalize [5, 9, 10, 11].

In what follows, C will stand for positive constants whose value may change from line to line but not depend on the functions in $H(\mathbb{B})$. The expression $A \simeq B$ means $C^{-1}A \leqslant B \leqslant CA$.

2. Auxiliary results

Given a normal weight ω and $\delta \in [0,1)$, denote $k_0 = \max\left(0, [\log_2 \frac{1}{\omega(\delta)}]\right)$, $r_k = \omega^{-1}\left(\frac{1}{2^k}\right)$ and $n_k = [\frac{1}{1-r_k}]$ for $k > k_0$, where $[x]$ denotes the greatest integer not more than x. In what follows, we set the function h, as in [18], to be

$$h(z) = 1 + \sum_{k>k_0}^{\infty} 2^k z^{n_k}, \quad z \in \mathbb{D}.$$

The following lemma comes from [18].

LEMMA 2.1. ([18]) *Let ω be normal. Then there exists some function $h \in H(\mathbb{D})$ such that $|h(z)| \leqslant h(|z|)$, $h(r)$ is increasing with $r \in [0,1)$, and*

$$0 < \inf_{r\in[0,1)} \omega(r)h(r) \leqslant \sup_{r\in[0,1)} \omega(r)h(r) = \sup_{z\in\mathbb{D}} \omega(z)|h(z)| < \infty.$$

The following lemma are probably folklore but we will give a proof for the benefit of the reader.

LEMMA 2.2. ([2]) *Let ω be normal. The point evaluation functional Λ_z on $\mathscr{B}_\omega$, defined by*

$$\Lambda_z(f) = f(z) \quad \textit{for} \quad f \in \mathscr{B}_\omega,$$

is bounded with the norm estimate

$$\|\Lambda_z\| \simeq 1 + \int_0^{|z|} \frac{1}{\omega(t)} dt.$$

Proof. By the definition of $\mathscr{B}_\omega$ and $f(z) = f(0) + \int_0^1 \langle \nabla f(tz), \overline{z} \rangle dt$, we have

$$|f(z)| \leqslant |f(0)| + \int_0^1 |\nabla f(tz)||z| dt \leqslant \left\{ 1 + \int_0^{|z|} \frac{1}{\omega(t)} dt \right\} \|f\|_\omega.$$

That is

$$\|\Lambda_z\| \leqslant C \left(1 + \int_0^{|z|} \frac{1}{\omega(t)} dt \right). \tag{2.1}$$

On the other hand, given $w \in \mathbb{B}$, set

$$f_w(z) = 1 + \int_0^{\langle z, w \rangle} h(t) dt.$$

Then by Lemma 2.1, $\|f_w\|_\omega \leqslant C$. For $w \in \mathbb{B}$ with $|w| \geqslant \delta > 0$, by (1.1) and Lemma 2.1,

$$\begin{aligned} \int_{|w|^2}^{|w|} h(t) dt &\leqslant h(|w|)(1 - |w|) \leqslant C \frac{h(|w|^2)(1 - |w|)}{\omega(|w|) h(|w|^2)} \\ &\leqslant C h(|w|^2)(1 - |w|) \frac{\omega(|w|^2)}{\omega(|w|)} \\ &\leqslant C 2^b h(|w|^2)(1 - |w|). \end{aligned}$$

Similar to the above proof, for $w \in \mathbb{B}$ with $|w| \geqslant \delta > 0$, we get

$$\int_{|w|^4}^{|w|^2} h(t) dt \geqslant C h(|w|^2)(1 - |w|).$$

So, we obtain

$$\int_{|w|^2}^{|w|} h(t) dt \leqslant C h(|w|^2)(1 - |w|) \leqslant C \int_{|w|^4}^{|w|^2} h(t) dt$$

for $w \in \mathbb{B}$ with $|w| \geqslant \delta > 0$. Hence,

$$\int_0^{|w|^2} h(t) dt \leqslant \int_0^{|w|} h(t) dt \leqslant C \int_0^{|w|^2} h(t) dt.$$

This implies

$$|f_w(w)| = 1 + \int_0^{|w|^2} h(t)dt \geqslant 1 + C\int_0^{|w|} \frac{1}{\omega(t)}dt \geqslant C\left\{1 + \int_0^{|w|} \frac{1}{\omega(t)}dt\right\}.$$

This and (2.1) yield the conclusion of the lemma. The proof is completed. □

LEMMA 2.3. ([2]) *Let ω be normal and $\int_0^1 \frac{dt}{\omega(t)} < \infty$. If $\{f_j\}$ is a bounded sequence in $\mathscr{B}_\omega$ satisfying $f_j \to 0$ uniformly on any compact subset of $\mathbb{B}$, then $\lim_{j\to\infty} \sup_{z\in\mathbb{B}} |f_j(z)| = 0$.*

LEMMA 2.4. *Let ω be normal, and let $\varphi \in S(\mathbb{B})$, $u \in H(\mathbb{B})$. Suppose X is $\mathscr{B}_\omega$(or $\mathscr{B}_{\omega,0}$) and Y is H^∞, and suppose that $uC_\varphi : X \longrightarrow Y$ is bounded. Then $uC_\varphi : X \longrightarrow Y$ is compact if and only if for any bounded sequence $\{f_j\}$ in X which converges to 0 uniformly on any compact subset of $\mathbb{B}$, we have $\lim_{j\to\infty} \|uC_\varphi f_j\|_Y = 0$.*

Proof. It can be proved by Montel theorem and Lemma 2.2. The details are omitted here. □

3. Main results

In this section, we are going to characterize the boundedness and compactness of the operator uC_φ from the Bloch-type space to H^∞ space with the function theoretic properties on u and φ.

THEOREM 3.1. *Let ω be normal, and let $\varphi \in S(\mathbb{B})$, $u \in H(\mathbb{B})$. Then the following statements are equivalent.*

(1) *$uC_\varphi : \mathscr{B}_\omega \longrightarrow H^\infty$ is bounded.*

(2) *$uC_\varphi : \mathscr{B}_{\omega,0} \longrightarrow H^\infty$ is bounded.*

(3) $\sup_{z\in\mathbb{B}} |u(z)| \left\{1 + \int_0^{|\varphi(z)|} \frac{1}{\omega(t)}dt\right\} < \infty.$

Furthermore, the operator norm $\|uC_\varphi\|$ satisfies

$$\|uC_\varphi\| \simeq \sup_{z\in\mathbb{B}} |u(z)| \left\{1 + \int_0^{|\varphi(z)|} \frac{1}{\omega(t)}dt\right\}. \tag{3.1}$$

Proof. The implication $(1) \Longrightarrow (2)$ is trivial.

$(2) \Longrightarrow (3)$. Suppose $uC_\varphi : \mathscr{B}_{\omega,0} \longrightarrow H^\infty$ is bounded. Take $g(z) = 1 \in \mathscr{B}_{\omega,0}$, then

$$\sup_{z\in\mathbb{B}} |u(z)| = \|uC_\varphi g\|_\infty \leqslant C\|uC_\varphi\| < \infty. \tag{3.2}$$

Fixed $w \in \mathbb{B}$, set the test function f_w to be

$$f_w(z) = \int_0^{\langle z,\varphi(w)\rangle} h(t)dt. \tag{3.3}$$

Then by Lemma 2.1,

$$\omega(z)|\Re f_w(z)| = |\langle z,\varphi(w)\rangle|\omega(z)|h(\langle z,\varphi(w)\rangle)| \leqslant C < \infty$$

and $f_w \in \mathscr{B}_{\omega,0}$ from the fact that f_w is holomorphic on $\overline{\mathbb{B}}$. Similar to the proof of Lemma 2.2, we have

$$\int_{|\varphi(w)|^2}^{|\varphi(w)|} h(t)dt \leqslant Ch(|\varphi(w)|^2)(1-|\varphi(w)|) \leqslant C\int_{|\varphi(w)|^4}^{|\varphi(w)|^2} h(t)dt.$$

Then

$$\int_0^{|\varphi(w)|^2} h(t)dt \leqslant \int_0^{|\varphi(w)|} h(t)dt \leqslant C\int_0^{|\varphi(w)|^2} h(t)dt. \tag{3.4}$$

Hence,

$$\begin{aligned}|u(w)|\int_0^{|\varphi(w)|} h(t)dt \simeq |u(w)|\int_0^{|\varphi(w)|^2} h(t)dt &= |u(w)||f_w(\varphi(w))| \\ &\leqslant \sup_{z\in\mathbb{B}}|u(z)||f_w(\varphi(z))| = \|uC_\varphi f_w\|_\infty \\ &\leqslant C\|uC_\varphi\| < \infty.\end{aligned}$$

By Lemma 2.1,

$$|u(w)|\int_0^{|\varphi(w)|} \frac{1}{\omega(t)}dt \leqslant C\|uC_\varphi\| < \infty. \tag{3.5}$$

Combine (3.2) and (3.5) to yield

$$\sup_{w\in\mathbb{B}}|u(w)|\left\{1+\int_0^{|\varphi(w)|}\frac{1}{\omega(t)}dt\right\} \leqslant C\|uC_\varphi\| < \infty. \tag{3.6}$$

This gives the implication $(2) \Longrightarrow (3)$.

$(3) \Longrightarrow (1)$. Suppose the statement (3) is valid. By Lemma 2.2,

$$\begin{aligned}\|uC_\varphi f\|_\infty &= \sup_{z\in\mathbb{B}}|u(z)||f(\varphi(z))| \\ &\leqslant C\|f\|_\omega \sup_{z\in\mathbb{B}}|u(z)|\left\{1+\int_0^{|\varphi(z)|}\frac{1}{\omega(t)}dt\right\}\end{aligned}$$

for each $f \in \mathscr{B}_\omega$. Therefore, $uC_\varphi : \mathscr{B}_\omega \longrightarrow H^\infty$ is bounded. Furthermore,

$$\|uC_\varphi\| \leqslant C\sup_{z\in\mathbb{B}}|u(z)|\left\{1+\int_0^{|\varphi(z)|}\frac{1}{\omega(t)}dt\right\}. \tag{3.7}$$

The estimate (3.1) comes from (3.6) and (3.7). The proof is completed. □

THEOREM 3.2. *Let ω be normal with $\int_0^1 \frac{dt}{\omega(t)} < \infty$, and let $\varphi \in S(\mathbb{B})$, $u \in H(\mathbb{B})$. Then the following statements are equivalent.*

(1) *$uC_\varphi : \mathscr{B}_\omega \longrightarrow H^\infty$ is compact.*

(2) *$uC_\varphi : \mathscr{B}_{\omega,0} \longrightarrow H^\infty$ is compact.*

(3) *$u \in H^\infty$.*

Proof. The implications $(1) \Longrightarrow (2)$ and $(2) \Longrightarrow (3)$ are trivial. Now we prove $(3) \Longrightarrow (1)$. Suppose $u \in H^\infty$. By $\int_0^1 \frac{1}{\omega(t)}dt < \infty$, we have

$$\sup_{z\in\mathbb{B}} |u(z)| \left\{ 1 + \int_0^{|\varphi(z)|} \frac{1}{\omega(t)} dt \right\} < \infty.$$

By Theorem 3.1, $uC_\varphi : \mathscr{B}_\omega \longrightarrow H^\infty$ is bounded. Suppose $\{f_j\}$ is a bounded sequence in $\mathscr{B}_\omega$ and $f_j(z) \to 0$ uniformly on any compact subset of $\mathbb{B}$. Apply Lemma 2.3 to obtain

$$\|uC_\varphi f_j\|_\infty = \sup_{z\in\mathbb{B}} |u(z)||f_j(\varphi(z))| \leqslant \|u\|_\infty \sup_{z\in\mathbb{B}} |f_j(\varphi(z))| \to 0$$

as $j \to \infty$. By Lemma 2.4, $uC_\varphi : \mathscr{B}_\omega \longrightarrow H^\infty$ is compact. The proof is completed. □

THEOREM 3.3. *Let ω be normal with $\int_0^1 \frac{dt}{\omega(t)} = \infty$, and let $\varphi \in S(\mathbb{B})$, $u \in H(\mathbb{B})$. Then the following statements are equivalent.*

(1) *$uC_\varphi : \mathscr{B}_\omega \longrightarrow H^\infty$ is compact.*

(2) *$uC_\varphi : \mathscr{B}_{\omega,0} \longrightarrow H^\infty$ is compact.*

(3) *$u \in H^\infty$ and*

$$\lim_{|\varphi(z)|\to 1} |u(z)| \left\{ 1 + \int_0^{|\varphi(z)|} \frac{1}{\omega(t)} dt \right\} = 0. \tag{3.8}$$

Proof. $(1) \Longrightarrow (2)$ is obvious.

$(2) \Longrightarrow (3)$. Suppose $uC_\varphi : \mathscr{B}_{\omega,0} \longrightarrow H^\infty$ is compact. Then taking $f(z) = 1 \in \mathscr{B}_{\omega,0}$, we obtain $u \in H^\infty$. Let $\{\varphi(z^j)\}$ be a sequence in $\mathbb{B}$ such that $\lim_{j\to\infty} |\varphi(z^j)| = 1$. Define

$$f_j(z) = \frac{1}{\int_0^{|\varphi(z^j)|^2} h(t)dt} \left(\int_0^{\langle z, \varphi(z^j)\rangle} h(t)dt \right)^2, \quad z \in \mathbb{B}.$$

It is easy to check that $\{f_j\}$ is a bounded sequence in $\mathscr{B}_{\omega,0}$ and, by $\int_0^1 \frac{dt}{\omega(t)} = \infty$, $f_j(z) \to 0$ uniformly on any compact subset of $\mathbb{B}$ as $j \to \infty$. So, by Lemma 2.4 we get

$$\lim_{j\to\infty} \|uC_\varphi f_j\|_\infty = 0. \tag{3.9}$$

On the other hand, Lemma 2.1 and (3.4) imply

$$\begin{aligned} \|uC_\varphi f_j\|_\infty &= \sup_{z\in\mathbb{B}} |u(z)||f_j(\varphi(z))| \geqslant |u(z^j)||f_j(\varphi(z^j))| \\ &= |u(z^j)| \int_0^{|\varphi(z^j)|^2} h(t)dt \geqslant C|u(z^j)| \int_0^{|\varphi(z^j)|} \frac{dt}{\omega(t)}. \end{aligned}$$

This and (3.9) imply

$$\lim_{j\to\infty} |u(z^j)| \int_0^{|\varphi(z^j)|} \frac{dt}{\omega(t)} = 0. \tag{3.10}$$

Notice that

$$|u(z^j)| \leqslant C|u(z^j)| \int_0^{\frac{1}{2}} \frac{dt}{\omega(t)} \leqslant C|u(z^j)| \int_0^{|\varphi(z^j)|} \frac{dt}{\omega(t)}$$

when $|\varphi(z^j)| \geqslant \frac{1}{2}$. Thus, $\lim\limits_{j\to\infty} |u(z^j)| = 0$. From this and (3.10), we obtain (3.8).

$(3) \Longrightarrow (1)$. Suppose that statement (3) holds. Then for any $\varepsilon > 0$, there exists some $r \in (0,1)$ such that

$$|u(z)| \left\{ 1 + \int_0^{|\varphi(z)|} \frac{1}{\omega(t)} dt \right\} < \varepsilon \tag{3.11}$$

whenever $r < |\varphi(z)| < 1$. Let $\{f_j\}$ be a bounded sequence in $\mathscr{B}_\omega$ and $f_j(z) \to 0$ uniformly on any compact subsets of $\mathbb{B}$. Then for the above $\varepsilon > 0$, there is a positive integer J such that

$$|f_j(\varphi(z))| < \varepsilon \tag{3.12}$$

whenever $j > J$ and $|\varphi(z)| \leqslant r$. From the fact that $u \in H^\infty$, (3.11) and (3.12),

$$\begin{aligned}
\|uC_\varphi f_j\|_\infty &= \sup_{z\in\mathbb{B}} |u(z)||f_j(\varphi(z))| \\
&\leqslant \sup_{|\varphi(z)|\leqslant r} |u(z)||f_j(\varphi(z))| + \sup_{|\varphi(z)|>r} |u(z)||f_j(\varphi(z))| \\
&\leqslant \|u\|_\infty \cdot \varepsilon + C \sup_{|\varphi(z)|>r} |u(z)| \left\{ 1 + \int_0^{|\varphi(z)|} \frac{1}{\omega(t)} dt \right\} \|f_j\|_\omega \\
&= \|u\|_\infty \cdot \varepsilon + C\|f_j\|_\omega \cdot \varepsilon \\
&\leqslant (\|u\|_\infty + C \sup_{j=1,2,\cdots} \|f_j\|_\omega) \cdot \varepsilon
\end{aligned}$$

whenever $j > J$. Then, $\lim\limits_{j\to\infty} \|uC_\varphi f_j\|_\infty = 0$. Hence $uC_\varphi : \mathscr{B}_\omega \longrightarrow H^\infty$ is compact by Lemma 2.4. The proof is completed. □

Summarizing all above, we have the following corollary.

COROLLARY 3.1. *Let ω be normal with $\int_0^1 \frac{dt}{\omega(t)} < \infty$, and let $\varphi \in S(\mathbb{B})$, $u \in H(\mathbb{B})$. Then the following statements are equivalent.*

(1) *$uC_\varphi : \mathscr{B}_\omega \longrightarrow H^\infty$ is bounded.*
(2) *$uC_\varphi : \mathscr{B}_{\omega,0} \longrightarrow H^\infty$ is bounded.*
(3) *$uC_\varphi : \mathscr{B}_\omega \longrightarrow H^\infty$ is compact.*
(4) *$uC_\varphi : \mathscr{B}_{\omega,0} \longrightarrow H^\infty$ is compact.*
(5) *$u \in H^\infty$.*

For the normal weight μ, the weithted-type space H_μ^∞ is defined by

$$H_\mu^\infty = \left\{ f \in H(\mathbb{B}) : \|f\|_{H_\mu^\infty} = \sup_{z\in\mathbb{B}} \mu(z)|f(z)| < \infty \right\}.$$

We find that all these above results can be easily rewritten for the case when H^∞ is replaced by the space H_μ^∞. Here, we only exhibit these theorems, and omit the proof.

THEOREM 3.4. *Let ω and μ be both normal, and let $\varphi \in S(\mathbb{B})$, $u \in H(\mathbb{B})$. Then the following statements are equivalent.*

(1) $uC_\varphi : \mathscr{B}_\omega \longrightarrow H_\mu^\infty$ *is bounded.*

(2) $uC_\varphi : \mathscr{B}_{\omega,0} \longrightarrow H_\mu^\infty$ *is bounded.*

(3) $\sup_{z\in\mathbb{B}} \mu(z)|u(z)| \left\{1 + \int_0^{|\varphi(z)|} \frac{1}{\omega(t)}dt\right\} < \infty.$

Furthermore, the operator norm $\|uC_\varphi\|$ *satisfies*

$$\|uC_\varphi\| \simeq \sup_{z\in\mathbb{B}} \mu(z)|u(z)| \left\{1 + \int_0^{|\varphi(z)|} \frac{1}{\omega(t)}dt\right\}.$$

THEOREM 3.5. *Let ω and μ be both normal with $\int_0^1 \frac{dt}{\omega(t)} < \infty$, and let $\varphi \in S(\mathbb{B})$, $u \in H(\mathbb{B})$. Then the following statements are equivalent.*

(1) $uC_\varphi : \mathscr{B}_\omega \longrightarrow H_\mu^\infty$ *is compact.*

(2) $uC_\varphi : \mathscr{B}_{\omega,0} \longrightarrow H_\mu^\infty$ *is compact.*

(3) $u \in H_\mu^\infty$.

THEOREM 3.6. *Let ω and μ be both normal with $\int_0^1 \frac{dt}{\omega(t)} = \infty$, and let $\varphi \in S(\mathbb{B})$, $u \in H(\mathbb{B})$. Then the following statements are equivalent.*

(1) $uC_\varphi : \mathscr{B}_\omega \longrightarrow H_\mu^\infty$ *is compact.*

(2) $uC_\varphi : \mathscr{B}_{\omega,0} \longrightarrow H_\mu^\infty$ *is compact.*

(3) $u \in H_\mu^\infty$ *and*

$$\lim_{|\varphi(z)|\to 1} \mu(z)|u(z)| \left\{1 + \int_0^{|\varphi(z)|} \frac{1}{\omega(t)}dt\right\} = 0.$$

4. The cases of two special normal weights

Case 1. For $0 < \alpha < \infty$, take $\omega(r) = (1-r)^\alpha$. Then the Bloch-type space is just the α-Bloch space. By direct calculation we see

$$\int_0^x \frac{dt}{(1-r)^\alpha} \simeq \begin{cases} 1, & \text{if } 0 < \alpha < 1, \\ \ln \frac{1}{1-x}, & \text{if } \alpha = 1, \\ (1-x)^{1-\alpha}, & \text{if } \alpha > 1, \end{cases} \tag{4.1}$$

as $x \to 1^-$.

From Theorem 3.1-3.3 and (4.1), we obtain the main results Theorem 1-6 in [9]. On the other hand, for the case of $uC_\varphi : H^\infty \longrightarrow \mathscr{B}_\omega$, notice that if we replace α-Bloch space with the Bloch-type space, then the normal weight of image space does not almost affect the characterization of the weighted composition operator. Therefore, with the same approach in [9], we can obtain some necessary and sufficient conditions. Here, we only exhibit one of them as follows. And the proof will be omitted.

THEOREM 4.1. *Let ω be normal, and let $\varphi \in S(\mathbb{B})$, $u \in H(\mathbb{B})$. If $u \in \mathscr{B}_\omega$ and*

$$\sup_{z\in\mathbb{B}} \frac{\omega(z)|u(z)|}{1-|\varphi(z)|^2}|D\varphi(z)| < \infty,$$

then $uC_\varphi : H^\infty \longrightarrow \mathscr{B}_\omega$ is bounded.

Conversely, if $uC_\varphi : H^\infty \longrightarrow \mathscr{B}_\omega$ is bounded, then $u \in \mathscr{B}_\omega$ and

$$\sup_{z\in\mathbb{B}} \frac{\omega(z)|u(z)|}{1-|\varphi(z)|^2}|D\varphi(z)^T\overline{\varphi(z)^T}| < \infty.$$

Here,

$$D\varphi(z) = \begin{pmatrix} \frac{\partial\varphi_1(z)}{\partial z_1} & \cdots & \frac{\partial\varphi_1(z)}{\partial z_n} \\ \cdots & \cdots & \cdots \\ \frac{\partial\varphi_n(z)}{\partial z_1} & \cdots & \frac{\partial\varphi_n(z)}{\partial z_n} \end{pmatrix}, \quad |D\varphi(z)| = \left(\sum_{k,l=1}^{n} \left|\frac{\partial\varphi_l(z)}{\partial z_k}\right|^2\right)^{\frac{1}{2}}$$

and $D\varphi(z)^T$ is the transpose of the matrix $D\varphi(z)$ (see [19]).

In one variable case, Ohno obtained the sufficient and necessary condition on u and φ for which $uC_\varphi : H^\infty \longrightarrow \mathscr{B}$ (or $\mathscr{B}_0$) is bounded or compact in [5]. And Zhang characterized the boundedness of uC_φ from H^∞ to α-Bloch space on the unit disc in [11]. It is easy to check that those results of [5] and [11] still hold for uC_φ from H^∞ to $\mathscr{B}_\omega$ (or $\mathscr{B}_{\omega,0}$). We have a set of theorems parallel to the Theorem 2-4 of [5]. And the proof will be omitted.

THEOREM 4.2. *Let ω be normal, and let $\varphi \in S(\mathbb{D})$, $u \in H(\mathbb{D})$. Then $uC_\varphi : H^\infty \longrightarrow \mathscr{B}_\omega$ is bounded if and only if $u \in \mathscr{B}_\omega$ and*

$$\sup_{z\in\mathbb{D}} \frac{\omega(z)}{1-|\varphi(z)|^2}|u(z)\varphi'(z)| < \infty.$$

THEOREM 4.3. *Let ω be normal, and let $\varphi \in S(\mathbb{D})$, $u \in H(\mathbb{D})$. Then $uC_\varphi : H^\infty \longrightarrow \mathscr{B}_\omega$ is compact if and only if the following statements are all satisfied:*

(1) $u \in \mathscr{B}_\omega$;

(2) $\sup_{z\in\mathbb{D}} \omega(z)|u(z)\varphi'(z)| < \infty$;

(3) $\lim_{|\varphi(z)|\to 1} \omega(z)|u'(z)| = 0$;

(4) $\lim_{|\varphi(z)|\to 1} \frac{\omega(z)}{1-|\varphi(z)|^2}|u(z)\varphi'(z)| = 0.$

THEOREM 4.4. *Let ω be normal, and let $\varphi \in S(\mathbb{D})$, $u \in H(\mathbb{D})$. Then the following statements are equivalent.*

(1) $uC_\varphi : H^\infty \longrightarrow \mathscr{B}_{\omega,0}$ *is bounded.*

(2) $uC_\varphi : H^\infty \longrightarrow \mathscr{B}_{\omega,0}$ *is compact.*

(3) $u \in \mathscr{B}_{\omega,0}$ *and*

$$\lim_{|z|\to 1} \frac{\omega(z)}{1-|\varphi(z)|^2}|u(z)\varphi'(z)| = 0.$$

From this point of view, the property of uC_φ between H^∞ and $\mathscr{B}_\omega$ (or $\mathscr{B}_{\omega,0}$) is completely discussed. Our work will generalize those in [5, 11].

Case 2. For $0<\alpha<\infty$ and $0\leqslant\beta<\infty$, set

$$\omega(r)=(1-r)^\alpha\left(\ln\frac{e^{\frac{\beta}{\alpha}}}{1-r}\right)^\beta.$$

Then the Bloch-type space is called the logarithmic Bloch-type space (see [10]). When $\alpha=\beta=1$ it is just the logarithmic Bloch space. Our work will extend those theorems in [10]. By using the L'Hopital rule, as well as some simple estimates, we have

$$\int_0^x\frac{dt}{(1-r)^\alpha\left(\ln\frac{e^{\frac{\beta}{\alpha}}}{1-r}\right)^\beta}\simeq\begin{cases}1, & \text{if } 0<\alpha<1 \text{ or } \alpha=1,\beta>1,\\ \ln\ln\frac{e^{\frac{\beta}{\alpha}}}{1-x}, & \text{if } \alpha=1 \text{ and } \beta=1,\\ \left(\ln\frac{e^{\frac{\beta}{\alpha}}}{1-x}\right)^{1-\beta}, & \text{if } \alpha=1 \text{ and } \beta\in(0,1)\\ (1-x)^{1-\alpha}\left(\ln\frac{e^{\frac{\beta}{\alpha}}}{1-x}\right)^{-\beta}, & \text{if } \alpha>1 \text{ and } \beta\geqslant 0,\end{cases}$$

as $x\to 1^-$. This, together with Theorem 3.1-3.3, characterize the boundedness and compactness of uC_φ from logarithmic Bloch-type space to H^∞ space.

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