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A HOFFMANN–JØRGENSEN INEQUALITY OF NA RANDOM VARIABLES WITH APPLICATIONS TO THE CONVERGENCE RATE

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Abstract. In this paper, we prove the Hoffmann-Jørgensen inequality for the negatively associated (NA) random variables. As an application, it is consequently used for the construction of the convergence rate for tail probabilities of partial sums of NA random variables. Our results extends the conclusions of Li and Spătaru's (2005).

1. Introduction

Li, Rao, Jiang and Wang (1995) gave a very general theorem about complete convergence for independent and identically distributed (i.i.d) random variables. The proofs of their theorems depended on the Hoffmann-Jørgensen inequality of independent random variables. Later, Ghosal and Chandra (1998) proved a Hoffmann-Jørgensen inequality for martingale differences sequence. And they also obtained a result similar to that of Li et al (1995). Recently, Li and Spătaru (2005) strengthened both the Hsu-Robbins-Erdos theorem on complete convergence and the results of Davis (1968), Lai (1974) and Gut (1980). Li and Spătaru's (2005) results are on the integrability of the following function:

$$f(x) = \sum_n a_n \mathsf{P}(|S_n| > xb_n), \quad x > 0,$$

where $a_n > 0$ and $\sum_n a_n = \infty$, and b_n is either $n^{1/p}$ $(0 < p < 2)$, $\sqrt{n \log n}$ or $\sqrt{n \log\log n}$. They studied the equivalent conditions on

$$\int_\varepsilon^\infty f(x^q)dx < \infty, \quad \varepsilon > 0. \tag{1.1}$$

It is well known that complete convergence implies almost sure convergence in view of the Borel-Cantelli Lemma. Bibliography and discussion on the convergence

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Paper MIA-16-24

rate for independent random variables were concerned for many years, and for a more comprehensive understanding, one can refer to Li et al.(1995) and Li and Spătaru (2005) for details. Particularly, in Li and Spătaru's (2005) work, a different approach for building the necessary and sufficient moment conditions for (1.1) was involved. Their main idea was based on the symmetrization and independent type Hoffmann-Jørgensen inequality to get the tail probabilities of the partial sum of the random variables. Since Hoffmann-Jørgensen inequality can be used as an important tool for deriving the complete rate, we are driven to establish the analogous results for negatively associated random variables; thus, we can extend the results of independent case to dependent one.

The random variables $X_1, X_2, \cdots, X_n$ are said to be *negatively associated* (NA) if for every pair of disjoint subset A_1, A_2 of $\{1,2,\cdots,n\}$,

$$\mathrm{Cov}(f(X_i, i \in A_1), g(X_j, j \in A_2)) \leqslant 0,$$

whenever f and g are increasing. NA is one qualitative version of negative dependence among random variables, and a great many basic theoretical properties have been derived for NA random variables. In this paper, we give a Hoffmann-Jørgensen inequality of NA random variables first. By applying the inequality, we establish several general results on the complete convergence for NA random variables, which extend the theorems in Li et al.(1995) and Li and Spătaru (2005). .

Here we give some notations. Throughout the paper, we will assume that $\{\overline{X}, \overline{X}_n;\ n \geqslant 1\}$ is an independent copy of $\{X, X_n;\ n \geqslant 1\}$, and we will consider the symmetrized random variables $X^s = X - \overline{X}$, $X_n^s = X_n - \overline{X_n}$, $S_n^s = X_1^s + \ldots + X_n^s$, $n \geqslant 1$. And let $\{X^*, X_n^*,\ n \geqslant 1\}$ denote a sequence of independent random variables, satisfying $X \stackrel{d}{=} X^*$, $X_n \stackrel{d}{=} X_n^*$, $n \geqslant 1$. In the sequel, $\log^+ x = \log(e \vee x)$, $x \geqslant 0$, $\sim$ between expressions will mean that the limit of their ratio is 1, and $[x]$ will denote the greatest integer less or equal than x.

This paper is organized as follows. In section 2, we will list some auxiliary lemmas and give the Hoffmann-Jørgensen inequality for NA random variables. Some general theorems on convergence rate and the proofs are collected in section 3. Comparing our results with that of independent random variable, we summarize our work in section 4.

2. Auxiliary Lemmas and the Hoffmann-Jørgensen inequality for NA Random Variables

In this section, we would state the Hoffmann-Jørgensen type inequality for NA random variables. In order to obtain the inequality and establish the general results on the complete convergence, some auxiliary lemmas will be given first. Lemma 2.1 is a useful version of the maximal inequality of Hoffmann-Jørgensen (see Li, Rao, Jiang and Wang (1995)) .

LEMMA 2.1. *Let $\{V_k : 1 \leqslant k \leqslant n\}$ be a finite sequence of independent symmetric real random variables and $S_n = \sum_{k=1}^n V_k$. Then for each integer $j \geqslant 1$, there exist* *positive numbers C_j and D_j depending only on j such that for all $t>0$,*

$$\mathsf{P}\left(|S_n|\geqslant 2jt\right)\leqslant C_j\mathsf{P}\left(\max_{1\leqslant k\leqslant n}|V_k|\geqslant t\right)+D_j(\mathsf{P}\left(|S_n|\geqslant t\right))^j.$$

Using Theorem 2.12 p. 71 in Hoffmann-Jørgensen (1977), we obtain the following contraction lemma (Lemma 2.2).

LEMMA 2.2. (Contraction Lemma) *Let $X=(X_1,\ X_2,\ \cdots,\ X_n)$ be a symmetric random vector and let $\phi_1,\cdots,\ \phi_n$: $\mathbf{R}\to\mathbf{R}$ be odd Borel functions satisfying $|\phi_k(x)|\leqslant x$ for all $x\in\mathbf{R}$ and all $k=1,\cdots,n$. Then we have*

$$\mathsf{P}\left(|\sum_{k=1}^{n}\phi_k(X_k)|>5c\right)\leqslant 6\mathsf{P}\left(|\sum_{k=1}^{n}X_k|>c\right),\forall c>0.$$

Proof. Let $c>0$ be given and let $(\varepsilon_1,\ \cdots,\ \varepsilon_n)$ be a Bernoulli vector which is independent of $(X_1,\ \cdots,\ X_n)$. By Theorem 2.15 p. 71 in Hoffmann-Jørgensen (1977), We have

$$\mathsf{P}\left(|\sum_{k=1}^{n}\varepsilon_k\phi_k(y_k)|>5c\right)\leqslant 6\mathsf{P}\left(|\sum_{k=1}^{n}\varepsilon_k y_k|>c\right),\quad \forall(y_1,\cdots,y_n)\in\mathbf{R}^n.$$

Since $(X_1,\ X_2,\ \cdots,\ X_n)$ is symmetric and independent of $(\varepsilon_1,\ \cdots,\ \varepsilon_n)$, we have that $(X_1,\ X_2,\ \cdots,\ X_n)$ and $(\varepsilon_1 X_1,\ \cdots,\ \varepsilon_n X_n)$ are equidistributied and since ϕ_k is odd, we have $\phi_k(\varepsilon_k X_k)=\varepsilon_k\phi_k(X_k)$. Hence, if P_X denote the distribution of X, we have

$$\begin{aligned}
&\mathsf{P}(|\sum_{k=1}^{n}\phi_k(X_k)|>5c)=\mathsf{P}(|\sum_{k=1}^{n}\phi_k(\varepsilon_k X_k)|>5c)=\mathsf{P}(|\sum_{k=1}^{n}\varepsilon_k\phi_k(X_k)|>5c)\\
&=\int_{\mathbf{R}^n}\mathsf{P}(|\sum_{k=1}^{n}\varepsilon_k\phi_k(y_k)|>5c)\mathsf{P}_X(dy)\\
&\leqslant 6\int_{\mathbf{R}^n}\mathsf{P}(|\sum_{k=1}^{n}\varepsilon_k y_k|>c)\mathsf{P}_X(dy)\\
&=6\mathsf{P}(|\sum_{k=1}^{n}\varepsilon_k X_k|>c)=6\mathsf{P}(|\sum_{k=1}^{n}X_k|>c). \quad \square
\end{aligned}$$

LEMMA 2.3. *For each $n\geqslant 1$, let $\{X_{nk},k\in Z\}$ be a sequence of NA real random variables, and $\{X_{nk}^*,k\in Z\}$ be a sequence of independent real random variables such that X_{nk}^* has the same distribution as X_{nk} for each $n\geqslant 1$, $k\in Z$, and the series $\sum_{k\in Z}X_k^*$ converges a.s. Suppose that*

$$S_n^*=\sum_{k\in Z}X_{nk}^*\xrightarrow{\mathsf{P}}0,$$

$$\sup_{k\in Z}|\mathrm{med}\,(X_{nk})|\to 0.$$

Then, we have

$$S_n = \sum_{k\in Z} X_{nk} \xrightarrow{\mathsf{P}} 0.$$

Proof. By the proof of Theorem 1 p. 326 of Chow and Teicher (1978) with some necessary modifications, we can get that $S_n^* = \sum_{k\in Z} X_{nk}^* \xrightarrow{\mathsf{P}} 0$ implies

$$\sum_{k\in Z} \mathsf{P}(|X_{nk}| \geqslant \varepsilon) \to 0,$$
$$\sum_{k\in Z} \mathsf{Var}(X_{nk} I\{|X_{nk}| < \varepsilon\}) \to 0,$$
$$\sum_{k\in Z} \mathsf{E} X_{nk} I\{|X_{nk}| < \varepsilon\} \to 0.$$

And by the classical truncated method, we can get $S_n = \sum_{k\in Z} X_{nk} \xrightarrow{\mathsf{P}} 0$. □

The next lemma comes from Li and Spătaru (2005).

LEMMA 2.4. *Let $\{U_n;\, n \geqslant 1\}$ be a sequence of random variables such that $U_n \xrightarrow{\mathsf{P}} 0$, $\{U_n';\, n \geqslant 1\}$ be an independent copy of $\{U_n;\, n \geqslant 1\}$, and let $\{a_n;\, n \geqslant 1\}$ be a sequence of nonnegative numbers, suppose $q > 0$ and $\delta \geqslant 0$. The following are equivalent:*

(i)

$$\int_\varepsilon^\infty \left(\sum_{n\geqslant 1} a_n \mathsf{P}(|U_n| > x^q) \right) dx < \infty, \quad \varepsilon > \delta^{1/q};$$

(ii) there exists $a > \delta^{1/q}$ such that

$$\int_a^\infty \left(\sum_{n\geqslant 1} a_n \mathsf{P}(|U_n - U_n'| > x^q) \right) dx < \infty,$$

and

$$\sum_{n\geqslant 1} a_n \mathsf{P}(|U_n| > \varepsilon) < \infty, \quad \varepsilon > \delta.$$

The following result on the approximation of sums of independent random variables is due to Sakhanenko (1980,1984,1985).

LEMMA 2.5. *For any sequence of independent random variables $\{\xi_n; n \geqslant 1\}$ with mean zero and finite variance, there exist a sequence of independent normal variables $\{\eta_n; n \geqslant 1\}$ with $\mathsf{E}\eta_n = 0$ and $\mathsf{E}\eta_n^2 = \mathsf{E}\xi_n^2$ such that, for all $Q > 2$ and $y > 0$,*

$$\mathsf{P}\left(\max_{k\leqslant n} \Big| \sum_{i=1}^k \xi_i - \sum_{i=1}^k \eta_i \Big| \geqslant y \right) \leqslant (AQ)^Q y^{-Q} \sum_{i=1}^n \mathsf{E}|\xi_i|^Q,$$

whenever $\mathsf{E}|\xi_i|^Q < \infty$, $i = 1, \dots, n$. Here, A is a universal constant.

The next lemma is the comparison theorem in Shao (2000), which will be used in the proof of Lemma 2.7.

LEMMA 2.6. (The Comparison Theorem) *Let* $\{X_i, 1 \leqslant i\}$ *be a negatively associated sequence, and let* $\{X_i^*, 1 \leqslant i\}$ *be a sequence of independent random variables such that* X_i^* *and* X_i *have the same distribution for each* $i = 1, 2, \cdots, n$*. Then*

$$\mathsf{E}f\left(\sum_{i=1}^{n} X_i\right) \leqslant \mathsf{E}f\left(\sum_{i=1}^{n} X_i^*\right) \tag{2.1}$$

for any convex function f *on* R^1*, whenever the expectation on the right hand side of (2.1) exists. If* f *is a non-decreasing convex function, then*

$$\mathsf{E}f\left(\max_{1\leqslant k\leqslant n}\sum_{i=1}^{k} X_i\right) \leqslant \mathsf{E}f\left(\max_{1\leqslant k\leqslant n}\sum_{i=1}^{k} X_i^*\right) \tag{2.2}$$

whenever the expectation on the right hand side of (2.2) exists.

To end this section, we state the following Hoffmann-Jørgensen type inequality for NA random variables which is one of our main tools.

LEMMA 2.7. (Hoffmann-Jørgensen inequality for NA random variables) *Let* $\{X_k, k \in Z\}$ *be a NA-sequence of symmetric random variables, and* $\{X_k^*, k \in Z\}$ *be a sequence of independent real random variables such that* X_k^* *has the same distribution as* X_k *for all* $k \in Z$ *and the series* $\sum_{k\in Z} X_k^*$ *converges a.s. Let* $\varepsilon > 0$ *and* $j \in N$ *be given and let* $Y_k^* = (-\frac{\varepsilon}{10j}) \vee (X_k^* \wedge \frac{\varepsilon}{10j})$ *denote the standard truncation of* X_k^* *at level* $\frac{\varepsilon}{10j}$*. Then the series* $\sum_{k\in Z} Y_k^*$ *and* $\sum_{k\in Z} X_k$ *converges a.s. and if we define* $S_k = \sum_{i\leqslant k} X_i$*, then we have*

$$\mathsf{P}(\sup_{k\in Z}|S_k| > 2\varepsilon) \leqslant \mathsf{P}\left(\sup_{k\in Z}|X_k| > \frac{\varepsilon}{10j}\right) + \frac{4^j}{\varepsilon}\int_{\varepsilon}^{\infty}\left(\mathsf{P}(|\sum_{k\in Z} Y_k^*| > \frac{x}{2j})\right)^j dx \tag{2.3}$$

$$\leqslant \mathsf{P}\left(\sup_{k\in Z}|X_k| > \frac{\varepsilon}{10j}\right) + \frac{24^j}{\varepsilon}\int_{\varepsilon}^{\infty}\left(\mathsf{P}(|\sum_{k\in Z} X_k^*| > \frac{x}{4j})\right)^j dx. \tag{2.4}$$

Proof. Let $Y_k = (-\varepsilon/(10j)) \vee (X_k \wedge (\varepsilon/(10j)))$. We assume the right of the inequality is finite. We have

$$\mathsf{P}(\sup_{l\in Z}|\sum_{k\leqslant l} X_k| > 2\varepsilon) \leqslant \mathsf{P}\left(\sup_{k\in Z}|X_k| > \frac{\varepsilon}{10j}\right) + \mathsf{P}(\sup_{l\in Z}|\sum_{k\leqslant l} Y_k| > 2\varepsilon).$$

Let $f(x) = (x-\varepsilon)_+ = (x-\varepsilon)\vee 0$, obviously, f is a non-decreasing convex function, then according to Lemma 2.6, we have

$$\begin{aligned}
&\mathsf{P}(\sup_{l\in Z}|\sum_{k\leqslant l} Y_k| > 2\varepsilon) = \lim_{m,t\to\infty}\mathsf{P}(\sup_{-t\leqslant l\leqslant m}|\sum_{-t\leqslant k\leqslant l} Y_k| > 2\varepsilon)\\
&\leqslant \lim_{m,t\to\infty}\Big\{\mathsf{P}(\sup_{-t\leqslant l\leqslant m}\sum_{-t\leqslant k\leqslant l} Y_k > \varepsilon) + \mathsf{P}(\sup_{-t\leqslant l\leqslant m}(-\sum_{-t\leqslant k\leqslant l} Y_k) > \varepsilon)\Big\}\\
&\leqslant \lim_{m,t\to\infty}\frac{1}{\varepsilon}\Big\{\mathsf{E}\{\max_{-t\leqslant l\leqslant m}\sum_{-t\leqslant k\leqslant l} Y_k - \varepsilon\}_+ + \mathsf{E}\{\max_{-t\leqslant l\leqslant m}(-\sum_{-t\leqslant k\leqslant l} Y_k) - \varepsilon\}_+\Big\}\\
&\leqslant \lim_{m,t\to\infty}\frac{1}{\varepsilon}\Big\{\mathsf{E}\{\max_{-t\leqslant l\leqslant m}\sum_{-t\leqslant k\leqslant l} Y_k^* - \varepsilon\}_+ + \mathsf{E}\{\max_{-t\leqslant l\leqslant m}(-\sum_{-t\leqslant k\leqslant l} Y_k^*) - \varepsilon\}_+\Big\}\\
&\leqslant \lim_{m,t\to\infty}\frac{2}{\varepsilon}\mathsf{E}\{\max_{-t\leqslant j\leqslant m}|\sum_{-t\leqslant k\leqslant l} Y_k^*| - \varepsilon\}_+\\
&= \lim_{m,t\to\infty}\frac{2}{\varepsilon}\int_\varepsilon^\infty \mathsf{P}(\max_{-t\leqslant l\leqslant m}|\sum_{-t\leqslant k\leqslant l} Y_k^*| > x)dx \leqslant \frac{4}{\varepsilon}\int_\varepsilon^\infty \mathsf{P}(|\sum_{k\in Z} Y_k^*| > x)dx,
\end{aligned}$$

where the last inequality follows from Lévy inequality. Since $|Y_k^*| \leqslant \varepsilon/(10j)$, we have $\mathsf{P}(|Y_k^*| \geqslant x/(2j)) = 0$ for $x \geqslant \varepsilon$. Applying Lemma 2.1 with $C_j := \frac{4^j-1}{3}$ and $D_j := 4^{j-1}$ we get

$$\mathsf{P}(\sup_{l\in Z}|\sum_{k\leqslant l} Y_k| > 2\varepsilon) \leqslant \frac{4^j}{\varepsilon}\int_\varepsilon^\infty \left(\mathsf{P}(|\sum_{k\in Z} Y_k^*| > \frac{x}{2j})\right)^j dx.$$

This proves (2.3). For (2.4), notice that Lemma 2.2 implies

$$\mathsf{P}\left(|\sum_{k\in Z} Y_k^*| > \frac{x}{2j}\right) \leqslant 6\mathsf{P}\left(|\sum_{k\in Z} X_k^*| > \frac{x}{10j}\right),$$

therefore, combining with (2.3) and the Lévy inequality, we fulfill the proof of Lemma 2.7. □

3. Some general convergence results

In this section, we give some general results on the convergence rate of the tail probabilities of sums of NA random variables, which extend those of Li and Sătaru (2005).

THEOREM 3.1. *Let $\{k_n\}$ be an increasing sequence. Suppose $\{X_{nk}, 1\leqslant k\leqslant k_n\}$ be a sequence of NA real random variables. Let $\{X_{nk}^*, 1\leqslant k\leqslant k_n\}$ be a sequence of independent r.v.'s, for which $X_{nk} \stackrel{d}{=} X_{nk}^*$, and $\{C_n\}$ be a sequence of constants. If for every $\varepsilon > 0$, we have*

$$\mathsf{P}\left(|\sum_{k=1}^{k_n} X_{nk}^*| \geqslant \varepsilon\right) = O(n^{-\alpha}) \quad \textit{as } n\to\infty, \textit{ for some } \alpha > 0 \textit{ chosen independently of } \varepsilon, \tag{3.1}$$

and

$$\mathsf{P}\left(|\sum_{k=1}^{k_n} X_{nk}| \geqslant \varepsilon\right) \to 0, \quad \text{as } n \to \infty \tag{3.2}$$

$$\sum_{n\geqslant 1} C_n \sum_{k=1}^{k_n} \mathsf{P}(|X_{nk}| \geqslant \varepsilon) < \infty, \tag{3.3}$$

$$\sum_{n\geqslant 1} \frac{C_n k_n}{n^r} < \infty \quad \text{for some } r > 0. \tag{3.4}$$

Then, for every $\varepsilon > 0$*,*

$$\sum_{n\geqslant 1} C_n \mathsf{P}\left(|\sum_{k=1}^{k_n} X_{nk}| \geqslant \varepsilon\right) < \infty. \tag{3.5}$$

Furthermore, if $\max_{1\leqslant k\leqslant k_n} |\mathsf{med}\,(X_{nk})| \to 0$ *as* $n \to 0$*, then condition (3.2) can be dropped.*

Proof. Let $\{\overline{X}_{nk}, 1 \leqslant k \leqslant k_n\}$ be an independent copy of $\{X_{nk}, 1 \leqslant k \leqslant k_n\}$ for each $n \geqslant 1$. It is easy to show that $\{X_{nk} - \overline{X}_{nk}, 1 \leqslant k \leqslant k_n\}$ is a sequence of NA random variables. Write $Z_{nk} = X_{nk} - \overline{X}_{nk}$. Let $\{\overline{X}^*_{nk}, 1 \leqslant k \leqslant k_n\}$ be an independent copy of $\{X^*_{nk}, 1 \leqslant k \leqslant k_n\}$, $Z^*_{nk} = X^*_{nk} - \overline{X}^*_{nk}$. Let $c = \varepsilon/(10j)$, where j is an integer to be specified later. Define

$$Y_{nk} := Z_{nk} I\{|Z_{nk}| \leqslant c\} + cI\{Z_{nk} > c\} - cI\{Z_{nk} < -c\} \quad \text{and} \quad S_n' = \sum_{k=1}^{k_n} Y_{nk},$$

$$Y^*_{nk} := Z^*_{nk} I\{|Z^*_{nk}| \leqslant c\} + cI\{Z^*_{nk} > c\} - cI\{Z^*_{nk} < -c\} \quad \text{and} \quad S_n^{*'} = \sum_{k=1}^{k_n} Y^*_{nk},$$

by (3.2), we have $\sum_{k=1}^{k_n} X_{nk} \xrightarrow{\mathsf{P}} 0$. Therefore, by the desymmetrization inequality (see Lemma 4.1 p. 81 in Hoffmann-Jørgensen (1977)), there exists an integer $n_\varepsilon \geqslant 1$ satisfying

$$\mathsf{P}\left(\left|\sum_{k=1}^{k_n} X_{nk}\right| > \varepsilon\right) \leqslant 2\mathsf{P}\left(\left|\sum_{k=1}^{k_n} Z_{nk}\right| > \varepsilon/2\right) \quad \forall n \geqslant n_\varepsilon$$

Since $|S_n^{'*}| \leqslant ck_n$ and $2jc = \frac{\varepsilon}{5}$, by the NA type Hoffmann-Jørgensen inequality (Lemma 2.7), we have,

$$\begin{aligned}\mathsf{P}\left(\left|\sum_{k=1}^{k_n} Z_{nk}\right| > \varepsilon/2\right) &\leqslant \mathsf{P}(\max_{1\leqslant k\leqslant k_n} |Z_{nk}| > c/4) + \frac{4^{j+1}}{\varepsilon}\int_{\varepsilon/4}^{\infty}\left(\mathsf{P}(|S_n^{*'}| \geqslant \frac{x}{2j})\right)^j \mathrm{d}x \\ &=: I_1 + I_2.\end{aligned}$$

Notice that

$$I_1 \leqslant \sum_{k=1}^{k_n} \mathsf{P}(|Z_{nk}| > c/4) \leqslant 2\sum_{k=1}^{k_n} \mathsf{P}(|X_{nk}| > c/8),$$

and

$$I_2 = \frac{4^{j+1}}{\varepsilon}\int_{\varepsilon/4}^{2jck_n}\left(\mathsf{P}(|S_n^{*'}| \geqslant \frac{x}{2j})\right)^j \mathrm{d}x \leqslant \frac{4^{j+1}}{\varepsilon}\cdot\frac{\varepsilon}{5}k_n\left(\mathsf{P}(|S_n^{*'}| \geqslant \frac{\varepsilon}{8j})\right)^j$$

Hence, it follows that

$$\begin{aligned}\mathsf{P}\left(\left|\sum_{k=1}^{k_n} X_{nk}\right| > \varepsilon\right) &\leqslant 2\mathsf{P}\left(\left|\sum_{k=1}^{k_n} Z_{nk}\right| > \varepsilon/2\right)\\ &\leqslant 4\sum_{k=1}^{k_n}\mathsf{P}(|X_{nk}| > c/8) + 4^j k_n\left(\mathsf{P}(|S_n^{*'}| \geqslant \frac{\varepsilon}{8j})\right)^j\end{aligned}$$

Applying the contraction lemma (Lemma 2.2), we get

$$\mathsf{P}\left(|S_n^{*'}| \geqslant \frac{x}{8j}\right) \leqslant 6\mathsf{P}\left(|S_n^{*'}| \geqslant \frac{\varepsilon}{40j}\right) \leqslant 12\mathsf{P}\left(\left|\sum_{k=1}^{k_n} X_{nk}\right| > \frac{\varepsilon}{80j}\right)$$

Therefore, by (3.1), we have

$$I_2 \leqslant 12\cdot 4^j k_n n^{-j\alpha}.$$

By choosing $j\alpha > r$, we get (3.5) from (3.3) and (3.4).

If $\max_{1\leqslant k\leqslant k_n}|\mathrm{med}\,(X_{nk})| \to 0$ as $n \to 0$, then by Lemma 2.3, we can obtain that $\sum_{k=1}^{k_n} X_{nk} \xrightarrow{\mathsf{P}} 0$. □

THEOREM 3.2. *For each $n \geqslant 1$, let $\{X_{nk}, k \in Z\}$ be a sequence of NA real random variables, and $\{X_{nk}^*, k \in Z\}$ be defined as before. If for every $\varepsilon > 0$ and some $j \geqslant 1$, we have*

$$\int_{\varepsilon}^{\infty}\mathsf{P}\left(|\sum_{k\in Z} X_{nk}^*| \geqslant x\right)^j dx \leqslant C, \quad C \text{ does not depend on } n \tag{3.6}$$

and

$$\mathsf{P}\left(|\sum_{k\in Z} X_{nk}^*| \geqslant \varepsilon\right) \to 0, \quad \sup_{k\in Z}|\mathrm{med}\,(X_{nk})| \to 0, \quad \text{as } n \to \infty, \tag{3.7}$$

then,

$$\sum_{n\geqslant 1} C_n \mathsf{P}\left(|\sum_{k\in Z} X_{nk}^*| \geqslant \varepsilon\right) < \infty \quad \text{for every } \varepsilon > 0 \tag{3.8}$$

implies

$$\sum_{n\geqslant 1} C_n \mathsf{P}\left(|\sum_{k\in Z} X_{nk}| \geqslant \varepsilon\right) < \infty \quad \textit{for every } \varepsilon > 0. \tag{3.9}$$

REMARK 3.1. Theorem 3.2 is very general convergence result. The conditions (3.6) and (3.7) are very easy to be checked by using Markov's inequality. For example, by checking (3.6) and (3.7), one can easily show that Theorem 2.2, Theorem 2.3, the "if part" of Theorem 2.4, Theorem 2.6 and Theorem 3.2 in Li, et. al. (1995) also hold for NA random variables. Moreover, by using Theorem 3.1 or Theorem 3.2 and a similar method of Li, et al. (1995) we can extend Theorem 2.1 of Li et al. (1995) to NA random variables. We do not restate these theorems here.

Proof. Let $S_n^* = \sum_{k\in Z} X_{nk}^*$, $S_{nk}^* = \sum_{i\leqslant k} X_{ni}^*$. By (3.7) we have $S_n^{*s} \xrightarrow{\mathsf{P}} 0$ and so $\mathsf{P}(\sup_{j\in Z}|S_{n,j}^{*s}| \geqslant \varepsilon) \to 0$ follows the Lévy inequality, where X^s denotes the symmetrization of a random variable X. Hence,

$$\mathsf{P}(\sup_{j\in Z}|X_{nj}^{*s}| \geqslant \varepsilon) \to 0 \quad \text{as } n\to\infty. \tag{3.10}$$

From (3.8) and the weak symmetric inequality, we have

$$\sum_{n\geqslant 1} C_n \mathsf{P}\left(|\sum_{k\in Z} X_{nk}^{*s}| \geqslant \varepsilon\right) < \infty \quad \text{for every } \varepsilon > 0.$$

By the Lévy inequality, we get

$$\sum_{n\geqslant 1} C_n \mathsf{P}\left(\sup_{j\in Z}|X_{nj}^{*s}| \geqslant \varepsilon\right) < \infty \quad \text{for every } \varepsilon > 0,$$

which implies

$$\sum_{n\geqslant 1} C_n \sum_{j\in Z} \mathsf{P}\left(|X_{nj}^{*s}| \geqslant \varepsilon\right) < \infty \quad \text{for every } \varepsilon > 0 \tag{3.11}$$

by (3.10). Moreover, by Lemma 2.3, we get $S_n \xrightarrow{\mathsf{P}} 0$. So

$$\begin{aligned}
&\sum_{n\geqslant 1} C_n \mathsf{P}\left(|\sum_{k\in Z} X_{nk}| \geqslant 4\varepsilon\right) \leqslant C\sum_{n\geqslant 1} C_n \mathsf{P}\left(|\sum_{k\in Z} X_{nk}^s| \geqslant \varepsilon\right)\\
&\leqslant C\sum_{n\geqslant 1} C_n \mathsf{P}(\sup_{j\in Z}|X_{nk}^s| \geqslant \frac{\varepsilon}{20j}) + C\sum_{n\geqslant 1} C_n \int_\varepsilon^\infty \left(\mathsf{P}(|\sum_{k\in Z} X_{nk}^{*s}| \geqslant \frac{x}{8j})\right)^{j+1} dx\\
&\leqslant C + C\sum_{n\geqslant 1} C_n \int_\varepsilon^\infty \left(\mathsf{P}(|\sum_{k\in Z} X_{nk}^{*}| \geqslant \frac{x}{16j})\right)^{j+1} dx \quad \text{(by (3.11))}\\
&\leqslant C + C\sum_{n\geqslant 1} C_n \mathsf{P}(|\sum_{k\in Z} X_{nk}^{*}| \geqslant \frac{\varepsilon}{16j}) \quad \text{(by (3.6))}\\
&< \infty \quad \text{(by (3.8))}.
\end{aligned}$$

This completes the proof. □

Next, we extend Theorem 1 of Li and Spătaru (2005) to NA random variables.

THEOREM 3.3. *For each $n \geqslant 1$, let $\{X_{nk};\ 1 \leqslant k \leqslant n\}$ be a finite sequence of NA symmetric random variables. Let $\{a_n;\ n \geqslant 1\}$ be a sequence of nonnegative numbers, and let $q > 0$ and $\delta \geqslant 0$. Assume that, for some $\rho > 0$,*

$$\sup_{n\geqslant 1} \mathsf{P}\left(\left|\sum_{k=1}^{n} X_{nk}^{*}\right| > x\right) = O(x^{-\rho}) \ \text{ as } x \to \infty. \tag{3.12}$$

The following are equivalent:

(i)

$$\int_{\varepsilon}^{\infty}\left(\sum_{n\geqslant 1} a_n \mathsf{P}\left(\max_{1\leqslant l\leqslant n}\left|\sum_{k=1}^{l} X_{nk}\right| > x^q\right)\right) dx < \infty, \quad \varepsilon > \delta^{1/q};$$

(ii) there exists $a > \delta^{1/q}$ such that

$$\int_{a}^{\infty}\left(\sum_{n\geqslant 1} a_n \mathsf{P}\left(\max_{1\leqslant k\leqslant n}|X_{nk}| > x^q\right)\right) dx < \infty, \tag{3.13}$$

and

$$\sum_{n\geqslant 1} a_n \mathsf{P}\left(\max_{1\leqslant l\leqslant n}\left|\sum_{k=1}^{l} X_{nk}\right| > \varepsilon\right) < \infty, \quad \varepsilon > \delta. \tag{3.14}$$

Proof. Clearly, (i)$\Longrightarrow$(3.13), and (i)$\Longrightarrow$(3.14). Thus we only need to show that (ii)$\Longrightarrow$(i). Obviously, we can assume $a > \varepsilon$. Set $j = [(\rho q)^{-1}] + 2$, $T = (20j)^{1/q}$, and

$$f(x) = \sum_{n\geqslant 1} a_n \mathsf{P}\left(\max_{1\leqslant l\leqslant n}\left|\sum_{k=1}^{l} X_{nk}\right| > x\right), \quad x > 0.$$

Then, by Lemma 2.7, for any $\varepsilon > \delta^{1/q}$, (3.12)-(3.14) imply

$$\begin{aligned}
\int_{\varepsilon}^{\infty} f(x^q)dx &= \int_{\varepsilon}^{Ta} f(x^q)dx + \int_{Ta}^{\infty} f(x^q)dx \leqslant Taf(\varepsilon^q) + T\int_{a}^{\infty} f(20jx^q)dx \\
&\leqslant Taf(\varepsilon^q) + TC_j \int_{a}^{\infty}\left(\sum_{n\geqslant 1} a_n \mathsf{P}\left(\max_{1\leqslant k\leqslant n}|X_{nk}| > x^q\right)\right) dx \\
&\quad + TC_j f(a^q)\int_{\varepsilon}^{\infty}\frac{1}{x^q}\int_{x^q}^{\infty}\left(\mathsf{P}(|\sum_{1\leqslant k\leqslant n} X_{nk}^{*}| \geqslant y)\right)^{j-1} dydx < \infty. \quad \square
\end{aligned}$$

In order to extend Theorem 4 and Theorem 6 of Li and Spătaru (2005). We first prove a general version of complete convergence theorem. Let $g(x) > 0$ be a nondecreasing function, which satisfies the following assumption.

ASSUMPTION 3.1. *For some $t \geqslant 1$, define $p(x) = g^{-1}(x)/x^t$, for some $c_0 > 0$*

$$p(x_2) \geqslant c_0 p(x_1), \quad \forall x_2 \geqslant x_1 > 0$$

Now set $X_k^{'} = (-g(n)) \vee X_k \wedge g(n) - \mathsf{E}(-g(n)) \vee X_k \wedge g(n)$, $1 \leqslant k \leqslant n$, $S_k^{'} = \sum_{i=1}^{k} X_k^{'}$, $\xi_i = \sum_{k=(i-1)m+1}^{im} X_k^{'}$, $1 \leqslant i \leqslant [n/m]$, $m \in N$, $T_k = \sum_{i=1}^{k} \xi_i$. Let $\{\xi_i^*\}$ is an independent copy of $\{\xi_i\}$, $T_k^* = \sum_{i=1}^{k} \xi_i^*$, $M_{[n/m]}^* = \max_{1 \leqslant k \leqslant [n/m]} |T_k^*|$.

THEOREM 3.4. *Let $\{X, X_n : n \geqslant 1\}$ be a zero mean strictly stationary NA sequence, and $S_n = \sum_{k=1}^{n} X_k$. Suppose $g(x)$ satisfies Assumption 3.1, $r \geqslant 1$, $rt > 1$. We assume the following conditions hold.*

(1) There exist $\varepsilon_0 \geqslant 0$ and $m \in N$ such that for every $\varepsilon > \varepsilon_0$,

$$\sum_{n=1}^{\infty} \frac{n^{r-2}\sqrt{[n/m]\mathsf{E}\xi_1^2}}{g(n)} \exp\Big(-\frac{\varepsilon^2 g^2(n)}{2[n/m]\mathsf{E}\xi_1^2}\Big) < \infty; \tag{3.15}$$

(2) $\mathsf{E}(g^{-1}(|X|))^r < \infty$, $M_{[n/m]}^*/g(n) \xrightarrow{\mathsf{P}} 0$ for m in (1) and

$$\sup_{n \geqslant 1} \mathsf{P}\Big(M_{[n/m]}^* \geqslant xg(n)/(12j)\Big) = O(x^{-\delta}) \quad \text{as } x \to \infty \text{ for some } \delta > 0;$$

(3) $\sum_{n=1}^{k} n^{r-1}/g(n) \leqslant Ck^r/g(k)$.

Then, we have

$$\sum_{n=1}^{\infty} n^{r-2} \mathsf{P}(\max_{1 \leqslant k \leqslant n} |S_k| \geqslant \varepsilon g(n)) < \infty \quad \text{for every } \varepsilon > \varepsilon_0. \tag{3.16}$$

Proof. By Assumption 3.1, we can get

$$\frac{\mathsf{E}|X|I\{|X| \geqslant g(n)\}}{g(n)} \leqslant \frac{\mathsf{E}g^{-1}(|X|)I\{|X| \geqslant g(n)\}}{n}.$$

Hence,

$$\frac{n\mathsf{E}(-g(n)) \vee X_k \wedge g(n)}{g(n)} \to 0 \quad \text{as } n \to \infty.$$

Therefore, we have

$$\begin{aligned} I &:= \sum_{n=1}^{\infty} n^{r-2} \mathsf{P}\Big(\max_{1 \leqslant k \leqslant n} |S_k| \geqslant \varepsilon g(n)\Big) \\ &\leqslant \sum_{n=1}^{\infty} n^{r-1} \mathsf{P}(|X| \geqslant g(n)) + C \sum_{n=1}^{\infty} n^{r-2} \mathsf{P}\Big(\max_{1 \leqslant k \leqslant n} |S_k^{'}| \geqslant (1-a)\varepsilon g(n)\Big) \\ &=: I_1 + I_2. \end{aligned}$$

It is easy to see that $I_1 < \infty$ since $\mathsf{E}(g^{-1}(|X|))^r < \infty$. And for every $0 < a < 1$,

$$\begin{aligned} I_2 &\leqslant C\frac{1}{a}\sum_{n=1}^{\infty}\frac{n^{r-2}}{g(n)}\mathsf{E}\{\max_{1\leqslant k\leqslant n}|S_k^{'}| - \frac{1-a}{1+a}\varepsilon g(n)\}_+ \\ &\leqslant C\frac{1}{a}\sum_{n=1}^{\infty}\frac{n^{r-2}}{g(n)}\mathsf{E}\{\max_{1\leqslant k\leqslant [n/m]}|T_k| - (1-a^{'})\varepsilon g(n)\}_+ \\ &\quad + C\frac{1}{a}\sum_{n=1}^{\infty}\frac{n^{r-2}}{g(n)}\mathsf{E}\{\max_{1\leqslant i\leqslant [n/m]+1}\max_{(i-1)m+1\leqslant l\leqslant im}|\sum_{k=(i-1)m+1}^{l}X_k^{'}| - a\varepsilon g(n)\}_+ \\ &=: I_3 + I_4, \end{aligned}$$

where $1 - a^{'} = (1-2a-a^2)/(1+a)$. Using the same argument for proving Lemma 2.7, by Theorem 1 of Shao (2000) we have

$$\begin{aligned} I_3 &\leqslant C\frac{1}{a}\sum_{n=1}^{\infty}\frac{n^{r-2}}{g(n)}\mathsf{E}\{\max_{1\leqslant k\leqslant [n/m]}|T_k^*| - (1-a^{'})\varepsilon g(n)\}_+ \\ &= C\frac{1}{a}\sum_{n=1}^{\infty}\frac{n^{r-2}}{g(n)}\int_{(1-a^{'})\varepsilon g(n)}^{\infty}\mathsf{P}\Big(M_{[n/m]}^* \geqslant x\Big)dx \\ &\leqslant C\sum_{n=1}^{\infty}n^{r-2}\mathsf{P}\Big(M_{[n/m]}^* \geqslant (1-a^{'})\varepsilon g(n)\Big) + C\frac{1}{a}\sum_{n=1}^{\infty}\frac{n^{r-2}}{g(n)}\int_{\Delta}^{\infty}\mathsf{P}\Big(M_{[n/m]}^* \geqslant x\Big)dx \\ &=: I_5 + I_6, \end{aligned}$$

where $\Delta = Q\varepsilon g(n)$, Q will be specified later. By noting condition (2) and using Lemma 2.1, we get

$$\begin{aligned} I_6 &\leqslant C\frac{1}{a}\sum_{n=1}^{\infty}\frac{n^{r-2}}{g(n)}\int_{\Delta}^{\infty}\mathsf{P}\Big(M_{[n/m]}^{*s} \geqslant x/2\Big)dx \\ &\leqslant C\frac{1}{a}\sum_{n=1}^{\infty}\frac{n^{r-2}}{g(n)}\int_{\Delta}^{\infty}\mathsf{P}\Big(M_{[n/m]}^{*s} \geqslant x/(6j)\Big)^j dx \\ &\leqslant C\frac{1}{a}\sum_{n=1}^{\infty}n^{r-2}\mathsf{P}\Big(M_{[n/m]}^* \geqslant Q\varepsilon g(n)/(12j)\Big)\int_{Q\varepsilon}^{\infty}\sup_{n\geqslant 1}\mathsf{P}\Big(M_{[n/m]}^* \geqslant xg(n)/(12j)\Big)^{j-1}dx \\ &\leqslant CI_5, \end{aligned}$$

where we let Q be large enough such that $\Delta > 12jmg(n)$. So we only need to estimate I_4, I_5. For I_5, by applying Lemma 2.4, we have

$$\begin{aligned} I_5 &\leqslant C\sum_{n=1}^{\infty}n^{r-2}\mathsf{P}\Big(\max_{1\leqslant k\leqslant [n/m]}|\sum_{i=1}^{k}Y_i| \geqslant (1-2a^{'})\varepsilon g(n)\Big) \\ &\quad + C\sum_{n=1}^{\infty}n^{r-2}\mathsf{P}\Big(\max_{1\leqslant k\leqslant [n/m]}|\sum_{i=1}^{k}(\xi_i^* - Y_i)| \geqslant a^{'}\varepsilon g(n)\Big) \\ &=: I_7 + I_8, \end{aligned}$$

where $Y_i \sim N(0, \mathsf{E}\xi_i^2)$. Since $\mathsf{P}(N \geqslant x) \sim (\sqrt{2\pi}x)^{-1}\exp(-x^2/2)$ as $x \to \infty$, we have

$$I_7 \leqslant C\sum_{n=1}^{\infty} \frac{n^{r-2}\sqrt{[n/m]\mathsf{E}\xi_1^2}}{g(n)} \exp\Big(-\frac{(1-2a^{'})^2\varepsilon^2 g^2(n)}{2[n/m]\mathsf{E}\xi_1^2}\Big) < \infty$$

by letting a be small enough such that $(1-2a^{'})\varepsilon > \varepsilon_0$. And by Assumption 3.1, we get that for $p \geqslant t$, $x^{1/p}/g(x) \leqslant Cy^{1/p}/g(y)$ for every $x > y$. Hence

$$\begin{aligned}
I_8 &\leqslant C\sum_{n=1}^{\infty} n^{r-1}\frac{\mathsf{E}|X|^q I\{|X| \leqslant g(n)\}}{g^q(n)} \\
&\leqslant C\sum_{n=1}^{\infty} \frac{n^{r-1}}{g^q(n)} \sum_{j=1}^{n} \mathsf{E}|X|^q I\{j-1 \leqslant g^{-1}(|X|) \leqslant j\} \\
&\leqslant C\sum_{j=1}^{\infty} \mathsf{E}|X|^q I\{j-1 \leqslant g^{-1}(|X|) \leqslant j\} \sum_{n=j}^{\infty} n^{r-1-q/p} n^{q/p}/g^q(n) \\
&\leqslant C\sum_{j=1}^{\infty} \mathsf{E}|X|^q I\{j-1 \leqslant g^{-1}(|X|) \leqslant j\} \sum_{n=j}^{\infty} n^{r-1-q/p} j^{q/p}/g^q(j) \\
&\leqslant C\sum_{j=1}^{\infty} g^q(j)\mathsf{P}(j-1 \leqslant g^{-1}(|X|) < j)\frac{j^r}{g^q(j)} \\
&\leqslant C\mathsf{E}(g^{-1}(|X|))^r < \infty.
\end{aligned}$$

For I_4, by (3) we have

$$\begin{aligned}
I_4 &\leqslant C\sum_{n=1}^{\infty} \frac{n^{r-1}}{g(n)}\mathsf{E}|X|I\{|X| \geqslant a\varepsilon g(n)\} \\
&\leqslant C\sum_{n=1}^{\infty} \frac{n^{r-1}}{g(n)} \sum_{n=k}^{\infty} g(k)\mathsf{P}(a\varepsilon g(k) \leqslant |X| \leqslant a\varepsilon g(k+1)) \\
&= \sum_{k=1}^{\infty}\sum_{n=1}^{k} \frac{n^{r-1}}{g(n)} g(k)\mathsf{P}(a\varepsilon g(k) \leqslant |X| \leqslant a\varepsilon g(k+1)) \\
&\leqslant C\sum_{k=1}^{\infty} k^r\mathsf{P}(a\varepsilon g(k) \leqslant |X| \leqslant a\varepsilon g(k+1)) \leqslant C\mathsf{E}(g^{-1}(|X|))^r < \infty.
\end{aligned}$$

This completes the proof. □

Theorem 3.5 and Theorem 3.6 below extend Theorem 4 and Theorem 6 of Li and Spătaru (2005) respectively.

THEOREM 3.5. *Let $r > 1$ and $q > 0$, and put*

$$f(x) = \sum_{n \geqslant 1} n^{r-2}\mathsf{P}(\max_{1 \leqslant k \leqslant n} |S_k| > x\sqrt{n\log n}), \quad x > 0.$$

(i) If $\int_\varepsilon^\infty f(x^q)dx < \infty$ *holds for some* $\varepsilon > 0$*, then*

$$\begin{cases} \mathsf{E}|X|^{1/q} < \infty & \text{if} \quad q \leqslant 1/2r, \\ \mathsf{E}[|X|^{2r}(\log^+|X|)^{-r}] < \infty & \text{if} \quad q > 1/2r. \end{cases} \tag{3.17}$$

(ii) If (3.17) *holds and* $0 < \sigma^2 := \mathsf{E}X^2 + \sum_{k=1}^\infty \mathsf{E}X_1X_{k+1} < \infty$*, then* $\int_\varepsilon^\infty f(x^q)dx < \infty$*,* $\varepsilon > (\sigma\sqrt{2r-2})^{1/q}$.

Proof. We prove (ii) first. Let $f^{(s)}(x)$ be defined in the same way as $f(x)$ with S_k^s taking the place of S_k. By Theorem 3.4, we can get

$$f^{(s)}(x) < \infty \;\; \varepsilon > 2\sigma\sqrt{r-1} \quad \text{and} \quad f(x) < \infty \;\; \varepsilon > \sigma\sqrt{2(r-1)}$$

(3.17) implies that

$$\int_a^\infty \Big(\sum_{n=2}^\infty n^{r-2}\mathsf{P}\Big(\max_{1\leqslant k\leqslant n}|X_k^s| > x^q\sqrt{n\log n}\Big)\Big)dx < \infty, \quad a > 0.$$

By Markov's inequality, we have

$$\mathsf{P}(\max_{1\leqslant k\leqslant n}|S_k^s| > x\sqrt{n\log n}) \leqslant \frac{\mathsf{E}(X^s)^2}{x^2}, \quad n \geqslant 2.$$

Then, by Theorem 3.3 it follows that

$$\int_\varepsilon^\infty f^{(s)}(x^q)dx < \infty, \;\; \varepsilon > 2\sigma\sqrt{r-1}.$$

Now the proof is completed by Lemma 2.4.

The proof for (i) is standard and we omit it here. □

THEOREM 3.6. *Let* $q > 0$*, and put*

$$f(x) = \sum_{n\geqslant 3}\frac{1}{n}\mathsf{P}(\max_{1\leqslant k\leqslant n}|S_k| > x\sqrt{n\log\log n}), \quad x > 0.$$

(i) If $\int_\varepsilon^\infty f(x^q)dx < \infty$ *for some* $\varepsilon > 0$*, then*

$$\begin{cases} \mathsf{E}|X|^{1/q} < \infty & \text{if} \quad q < 1/2 \\ \mathsf{E}[|X|^2(\log^+|X|)/\log^+\log^+|X|] < \infty & \text{if} \quad q = 1/2\;. \\ \mathsf{E}X^2 < \infty & \text{if} \quad q > 1/2 \end{cases} \tag{3.18}$$

(ii) If (3.18) *holds and* $0 < \sigma^2 := \mathsf{E}X^2 + \sum_{k=1}^\infty \mathsf{E}X_1X_{k+1} < \infty$*, then* $\int_\varepsilon^\infty f(x^q)dx < \infty$*,* $\varepsilon > (\sigma\sqrt{2})^{1/q}$.

Proof. Since the proof of (i) is standard. We only prove (ii) here. By Theorem 3.1, we can get

$$\sum_{n\geqslant 1}\frac{1}{n}\mathsf{P}(\max_{1\leqslant k\leqslant n}|S_k^s| > \varepsilon\sqrt{n\log\log n}), \;\; \varepsilon > 2\sigma \;\; \text{and } f(x) < \infty, \;\; \varepsilon > \sigma\sqrt{2}.$$

(3.18) implies that

$$\int_a^{\infty}\Big(\sum_{n=2}^{\infty}\frac{1}{n}\mathsf{P}\Big(\max_{1\leqslant k\leqslant n}|X_k^s|>x^q\sqrt{n\log\log n}\Big)\Big)dx<\infty,\quad a>0.$$

By Markov's inequality, we have

$$\mathsf{P}(\max_{1\leqslant k\leqslant n}|S_k^s|>x\sqrt{n\log\log n})\leqslant\frac{\mathsf{E}(X^s)^2}{x^2},\quad n\geqslant 2.$$

Then, by Theorem 3.3 and Lemma 2.4, we complete the proof. □

4. Conclusion

Now we will end this paper with the following summary. This article establishes an NA type Hoffmann-Jørgensen inequality first. As an application, the inequality is used to get some general theorems on complete convergence for NA random variables. For independent cases, Li et al. (1995) provided some relaxed conditions to guarantee the convergence rate. Li and Spătaru (2005) employed the symmetrization method and the independent Hoffmann-Jørgensen inequality to derive the necessary and sufficient moment condition for (1.1). By using the NA type Hoffmann-Jørgensen inequality, we obtain the analogous results. With (3.6) and (3.7) being checked, one can easily show that Theorem 2.2, Theorem 2.3, the "if part" of Theorem 2.4, Theorem 2.6 and Theorem 3.2 in Li, et. al. (1995) also hold for NA random variables. By using Theorem 3.1 or Theorem 3.2 and a similar method of Li, et al. (1995) we can extend Theorem 2.1 of Li et al. (1995) to NA random variables. Moreover, our Theorem 3.3, Theorem 3.5 and Theorem 3.6 are the extensions of Theorem 1, Theorem 4 and Theorem 6 of Li and Spătaru (2005). Therefore, our work can be regarded as the extensions of results of Li et al. (1995) and Li and Spătaru (2005).

REFERENCES

[1] L. E. BAUM AND M. KATZ, *Convergence rates in the law of large numbers*, Trans. Amer. Math. Soc. **120** (1965), 108–123.

[2] Y. S. CHOW AND H. TEICHER, *Probability Theory*, Springer-Verlag, 1978.

[3] Y. S. CHOW AND T. L. LAI, *Paley-type inequalities and convergence rates ralated to the law of large numbers and extended renewal theory*, Z. Wahrsch. Verw. Geb. **45** (1978), 1–19.

[4] J. A. DAVIS, *Convergence rates for the law of the iterated logarithm*, Ann. Math. Statist. **39** (1968a), 1479–1485.

[5] J. A. DAVIS, *Convergence rates for probabilities of moderate deviations*, Ann. Math. Statist. **39** (1968b), 2016–2028.

[6] S. GHOSAL AND T. K. CHANDRA, *Complete convergence of martingale arrays*, J. Theor. Probab **11** (1998), 621–631.

[7] A. GUT, *Convergence rates for probabilities of moderate deviations for sums of random variables with multidimensional indices*, Ann. Probab. **8** (1980), 298–313.

[8] J. HOFFMANN-JØRGENSEN, *Probability in Banach Spaces*, Ecole d'Eté de Probabilités de Saint Flour, VI 1976, Lecture Notes in Mathematics No 598, Springer Verlag, 1977, pp. 1–186.

[9] P. L. HSU AND H. ROBBINS, *Complete convergence and the law of large numbers*, Proc. Nat. Acad. Sci. U.S.A. **33** (1947), 25–31.

[10] T. C. HU AND N. C. WEBER, *On the rate of convergence in the strong law of large numbers for arrays*, Bull. Austral. Math. Soc. **45** (1992), 479–482.
[11] N. C. JAIN, *Tail probabilities for sums of independent Banach space valued random variables*, Z. Wahrsh. verw. Gebiete **33** (1975),155–166.
[12] O. KLESOV AND A. ROSALSKY, *A nonclassical law of the iterated logarithm for i.i.d. square integrable random variables*, Stochast. Anal. Appl. **19** (2001), 627–641.
[13] T. L. LAI, *Limit theorems for delayed sums*, Ann. Probab. **2** (1974), 432–440.
[14] D. LI, M. B. RAO, T. JIANG, AND X. WANG, *Complete convergence and almost sure convergence of weighted sums of random variables*, J. Theor. Probab. **8** (1995), 49–76.
[15] D. LI, A. SPĂTARU, *Refinement fo convergence rates for tail probabilities*, J. Theor. Probab. **18** (2005), (4) 933–947.
[16] Y. C. QI, *On the strong convergence of arrays*, Bull. Austral. Math. Soc. **50** (1994), 219–223.
[17] A. I. SAKHANENKO, *On unimprovable estimates of the rate of convergence in the invariance principle*, In Colloquia Math. Soci. János Bolyai, **32** (1980), pp. 779–783, Nonparametric Statistical Inference, Budapesp (Hungary).
[18] A. I. SAKHANENKO, *On estimates of the rate of convergence in the invariance principle*, In Advances in Probab. Theory: Limit Theorems and Related Problems (A.A. Borovkov, Ed.) pp. 124–135, Springer, New York, 1984.
[19] A. I. SAKHANENKO, *Convergence rate in the invariance principle for nonidentically distributed variables with exponential moments*, In Advances in Probab. Theory: Limit Theorems for Sums of Random Variables, (A.A. Borovkov, Ed.) pp. 2–73, Springer, New York, 1985.
[20] Q. M. SHAO, *A comparison theorem on moment inequalities between negatively associated and independent random variables*, J. Theor. Probab. **13** (2000),343–356.
[21] C. SU, T. C. HU, AND H. Y. LIANG, *On the logarithm law for strictly stationary and negatively associated arrays*, Theory Probab. Appl. **46** (2003), 369–380.

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