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UNCERTAINTY PRINCIPLE INEQUALITIES RELATED TO LAGUERRE–BESSEL TRANSFORM

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Abstract. In this paper, an analogous of Heisenberg inequality is established for Laguerre-Bessel transform. Also, a local uncertainty principle for this transform is investigated.

1. Introduction

The uncertainty principle states that a nonzero function and its Fourier transform cannot both be sharply localized. In the language of quantum mechanics, this principle says that an observer cannot simultaneously and precisely determines the values of position and momentum of a quantum particule. A mathematical formulation of this physical ideas is firstly developed by Heisenberg [3] in 1927. For $f \in L^2(\mathbb{R})$, a precise quantitative formulation of the uncertainty principle, usually called Heisenberg inequality, is the following

$$\int_{\mathbb{R}} x^2 |f(x)|^2 dx \cdot \int_{\mathbb{R}} \xi^2 \left|\widehat{f}(\xi)\right|^2 d\xi \geqslant \frac{1}{4}\left(\int_{\mathbb{R}} |f(x)|^2 dx\right)^2, \tag{1}$$

where $\hat{f}$ is the Fourier-Plancherel transform given for $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ by

$$\widehat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-ix\xi} dx.$$

This result does not appear in Heisenberg paper [3]. The relation (1) appears in Weyl paper [16] who credits the result to Pauli. In framework of Hankel transform, Bowie in [1] studied the Heisenberg uncertainty principle. Also for this transform, Rösler and Voit in [13] established a Heisenberg uncertainty inequality. They give, in their paper, a similar uncertainty inequality for Dunkl transform on the real line. On several variables, Rösler in [12] and Shimeno in [14] have proved, by different methods, a Heisenberg inequality for the Dunkl transform. Li and Liu established in [5] a Heisenberg inequality for Jacobi expansions. They generalized this result for Sturm-Liouville operators in

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[6]. Recently, Ma in [7] has provided a Heisenberg inequality for Jacobi transform. He obtained in [8] a Heisenberg uncertainty principle for Chébli-Trimèche hypergroups as a generalization of his previous statements in [7]. Since the 20's of last century, many works have been devoted to studying uncertainty principles in various forms. Among these, we can cite the works of Price [10, 11], whose aim is to establish local uncertainty inequality. In [9], Omri and Rachdi obtained a local uncertainty inequality in framework of the Riemann-Liouville operator.

In this paper, firstly we obtain an analogous of the Heisenberg inequality for the Laguerre-Bessel transform which will be defined below in section 2. Next, for this transform we develop further inequalities in the sharpest forms, which constitue the principle of local uncertainty. Throughout the paper, we denote $\mathbb{K}=[0,+\infty)\times[0,+\infty)$, $\widehat{\mathbb{K}}=[0,+\infty)\times\mathbb{N}$ and we designate by C a positive constant, which is not necessarily same at each occurrence.

2. Laguerre-Bessel transform

In this section, we collect some notations and results about the Laguerre-Bessel harmonics analysis. For more details, we refer the reader to [2].

For $\alpha\geqslant 0$, we consider the following system of partial differential operators

$$\begin{cases} D_1=\dfrac{\partial^2}{\partial t^2}+\dfrac{2\alpha}{t}\dfrac{\partial}{\partial t} \\[2ex] D_2=\dfrac{\partial^2}{\partial x^2}+\dfrac{2\alpha+1}{x}\dfrac{\partial}{\partial x}+x^2D_1, \quad (x,t)\in(0,+\infty)\times(0,+\infty). \end{cases}$$

For $(\lambda,m)\in\widehat{\mathbb{K}}$, the system

$$\begin{cases} D_1u &= -\lambda^2u \\[1ex] D_2u &= -2\lambda(2m+\alpha+1)u \\[1ex] u(0,0)=1, & \dfrac{\partial u}{\partial x}(0,0)=\dfrac{\partial u}{\partial t}(0,0)=0, \end{cases} \tag{2}$$

possesses a unique solution denoted $\varphi_{(\lambda,m)}$ and given by

$$\varphi_{(\lambda,m)}(x,t)=j_{\alpha-\frac{1}{2}}(\lambda t)\,\mathscr{L}_m^{\alpha}(\lambda x^2), \qquad (x,t)\in\mathbb{K},$$

where j_α is the normalized Bessel function given by

$$j_\alpha(x)=\Gamma(\alpha+1)\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\,\Gamma(\alpha+k+1)}\left(\frac{x}{2}\right)^{2k}$$

and $\mathscr{L}_m^{\alpha}$ is the Laguerre function defined on $[0,+\infty)$ by

$$\mathscr{L}_m^{\alpha}(x)=\frac{e^{-\frac{x}{2}}L_m^{\alpha}(x)}{L_m^{\alpha}(0)},$$

L_m^α being the Laguerre polynomial of degree m and order α given by

$$L_m^\alpha(x) = \sum_{j=0}^{m} \frac{\Gamma(m+\alpha+1)(-x)^j}{\Gamma(m-j+1)\Gamma(j+\alpha+1)j!} .$$

The Laguerre function satisfies the following relations (see [2])

$$\sup_{(x,t)\in\mathbb{K}} |\varphi_{(\lambda,m)}(x,t)| = 1, \quad \text{for all } (\lambda,m) \in \widehat{\mathbb{K}}. \tag{3}$$

Notations:

- $\mathscr{S}_*(\mathbb{K})$ the space of C^∞ functions on $\mathbb{R}^2$, even with respect to each variable and rapidly decreasing together with all their derivatives, that means

$$\forall k,p,q \in \mathbb{N},\ N_{k,p,q}(f) = \sup_{(x,t)\in\mathbb{K}} \left\{ (1+x^2+t^2)^k \left| \frac{\partial^{p+q}}{\partial x^p\, \partial t^q} f(x,t) \right| \right\} < +\infty.$$

- $L^p_\alpha(\mathbb{K})$, $p \in [1,+\infty]$, the spaces of measurable functions on $\mathbb{K}$ such that

$$\|f\|_{\alpha,p} = \left[\int_{\mathbb{K}} |f(x,t)|^p \, dm_\alpha(x,t) \right]^{\frac{1}{p}} < +\infty, \ \text{ if } p \in [1,+\infty),$$
$$\|f\|_{\alpha,\infty} = \operatorname*{ess\,sup}_{(x,t)\in\mathbb{K}} |f(x,t)| < +\infty,$$

where m_α is the positive measure defined on $\mathbb{K}$ by

$$dm_\alpha(x,t) = \frac{1}{\Gamma(\alpha+\frac{1}{2})\Gamma(\alpha+1)} x^{2\alpha+1}\, t^{2\alpha}\, dx\, dt .$$

- $L^p_{\gamma_\alpha}(\widehat{\mathbb{K}})$, $p \in [1,+\infty]$, the spaces of measurable functions on $\widehat{\mathbb{K}}$ such that

$$\|g\|_{\gamma_\alpha,p} = \left[\int_{\widehat{\mathbb{K}}} |g(\lambda,m)|^p \, d\gamma_\alpha(\lambda,m) \right]^{\frac{1}{p}} < +\infty, \ \text{ if } p \in [1,+\infty),$$
$$\|g\|_{\gamma_\alpha,\infty} = \operatorname*{ess\,sup}_{(\lambda,m)\in\widehat{\mathbb{K}}} |g(\lambda,m)| < +\infty,$$

where γ_α is the positive measure defined on $\widehat{\mathbb{K}}$ by

$$\int_{[0,+\infty)\times\mathbb{N}} g(\lambda,m)\, d\gamma_\alpha(\lambda,m) = \frac{1}{2^{2\alpha-1}\Gamma(\alpha+\frac{1}{2})} \sum_{m=0}^{\infty} L_m^\alpha(0) \int_0^{+\infty} g(\lambda,m)\ \lambda^{3\alpha+1} d\lambda.$$

We define the convolution product $f*g$ of two functions $f,g \in \mathscr{S}_*(\mathbb{K})$, by

$$(f*g)(x,t) = \int_{\mathbb{K}} T^{(\alpha)}_{(x,t)} f(y,s)\ g(y,s) dm_\alpha(y,s), \quad (x,t) \in \mathbb{K},$$

where $T^{(\alpha)}_{(x,t)}$, $(x,t) \in \mathbb{K}$, are the translation operators associated to the operators D_1 and D_2. For more details about these operators, we refer to [2].

LEMMA 2.1. *If $f \in L^p_\alpha(\mathbb{K})$, $g \in L^q_\alpha(\mathbb{K})$ such that $1 \leqslant p,q \leqslant \infty$ and $\frac{1}{p}+\frac{1}{q}-1=\frac{1}{r}$, then the function $f*g \in L^r_\alpha(\mathbb{K})$, and*

$$\|f*g\|_{\alpha,r} \leqslant \|f\|_{\alpha,p}\, \|g\|_{\alpha,q}.$$

We consider the dilations on $\mathbb{K}$ defined by

$$\delta_r(x,t) = (rx, r^2 t), \qquad r>0.$$

We also introduce a homogeneous norm, related to family $(\delta_r)_{r>0}$ defined by

$$|(x,t)| = (x^4+4t^2)^{\frac{1}{4}}.$$

We define the ball centered at $(0,0)$ of radius r by

$$B_r = \{(x,t)\in\mathbb{K} \;\; | \;\; |(x,t)|<r\}.$$

Let $f \in L^1_\alpha(\mathbb{K})$, the Laguerre-Bessel transform of f is defined by

$$\mathscr{F}_{LB}(f)(\lambda,m) = \int_{\mathbb{K}} f(x,t)\ \varphi_{(\lambda,m)}(x,t)dm_\alpha(x,t).$$

For f and $g \in L^1_\alpha(\mathbb{K})$, we have

$$\mathscr{F}_{LB}(f*g)(\lambda,m) = \mathscr{F}_{LB}(f)(\lambda,m)\ \mathscr{F}_{LB}(g)(\lambda,m).$$

The integral transform can be extended to an isometric isomorphism $L^2_\alpha(\mathbb{K})$ to $L^2_{\gamma_\alpha}(\widehat{\mathbb{K}})$ and we have the Plancherel formula

$$\|f\|_{\alpha,2} = \|\mathscr{F}_{LB}(f)\|_{\gamma_\alpha,2}, \qquad f\in L^1_\alpha(\mathbb{K})\cap L^2_\alpha(\mathbb{K}).$$

Let $L=-D_2$ and define L^b for $b\in\mathbb{R}$, as in [15, p.117]. Then by (2),

$$\mathscr{F}_{LB}\left(L^b f\right)(\lambda,m) = (2\lambda(2m+\alpha+1))^b\ \mathscr{F}_{LB}(f)(\lambda,m).$$

On the other hand, L is hypoelliptic on $\mathbb{K}$. Also, the heat operator $L+\partial_s$ is hypoelliptic on $\mathbb{K}\times(0,+\infty)$. Hence, similar arguments from the proof of Hunt's theorem [4, Theorem 3.4] give the following proposition

PROPOSITION 2.1. *There is a unique C^∞ function $h((x,t),s)=h_s(x,t)$ on $\mathbb{K}\times(0,+\infty)$ with the following properties*
i) $(L+\partial_s)h=0$ *on* $\mathbb{K}\times(0,+\infty)$,
ii) $h_s(x,t)\geqslant 0$ *and* $\int_{\mathbb{K}} h_s\, dm_\alpha = 1$,
iii) $h_{s_1}*h_{s_2} = h_{s_1+s_2}, \quad s_1,s_2>0.$

LEMMA 2.2. *For any $s>0$, $\mathscr{F}_{LB}(h_s)(\lambda,m) = e^{-2\lambda(2m+\alpha+1)s}$.*

Let $\{H^s \mid s>0\}$ be the heat semigroup. There is a unique smooth function $h((x,t),s) = h_s(x,t)$ on $\mathbb{K}\times(0,+\infty)$ such that $H^s f(x,t) = f * h_s(x,t)$.

h_s is called the heat kernel assocaited to L.

LEMMA 2.3.

$$\|h_s\|_{\alpha,2} \leqslant C\, s^{-\frac{3\alpha+2}{2}}. \tag{4}$$

Proof. By the Plancherel formula, we have $\|h_s\|_{\alpha,2} = \|\mathscr{F}_{LB}(h_s)\|_{\gamma_\alpha,2}$.

$$\|\mathscr{F}_{LB}(h_s)\|^2_{\gamma_\alpha,2} = \frac{1}{2^{2\alpha-1}\Gamma(\alpha+\frac{1}{2})}\int_0^{+\infty}\left(\sum_{m=0}^{+\infty} L_m^\alpha(0) e^{-8\lambda s m}\right) e^{-4\lambda s(\alpha+1)}\lambda^{3\alpha+1}\,d\lambda.$$

By the generating function identity for the Laguerre polynomials,

$$\sum_{m=0}^{+\infty} t^m L_m^\alpha(x) = \frac{1}{(1-t)^{\alpha+1}}\, e^{-\frac{xt}{1-t}}, \quad |t|<1,$$

we have

$$\|\mathscr{F}_{LB}(h_s)\|^2_{\gamma_\alpha,2} = \frac{s^{-3\alpha-2}}{2^{2\alpha-1}\Gamma(\alpha+\frac{1}{2})}\int_0^{+\infty}\left(\frac{1}{2\sinh(4u)}\right)^{\alpha+1} u^{3\alpha+1}\,du.$$

So, $\quad \|\mathscr{F}_{LB}(h_s)\|^2_{\gamma_\alpha,2} \leqslant C\, s^{-(3\alpha+2)}. \quad \square$

3. Heisenberg inequality for Laguerre-Bessel transform

LEMMA 3.1. *Let $0<a<3\alpha+2$, then for all $f\in L^2_\alpha(\mathbb{K})$, we have*

$$\|H^s f\|_{\alpha,2} \leqslant C\; s^{-\frac{a}{2}}\; \|\,|(x,t)|^a f\|_{\alpha,2}.$$

Proof. For $r>0$, let $f_r = f\,\chi_{B_r}$ and $f^r = f - f_r$.

Then, we have

$$|f^r(x,t)| \leqslant r^{-a}\,|(x,t)|^a\,|f(x,t)|.$$

So, we get

$$\|H^s f^r\|_{\alpha,2} \leqslant \|f^r\|_{\alpha,2} \leqslant r^{-a}\|\,|(x,t)|^a f\|_{\alpha,2}.$$

On the other hand, we have

$$\begin{aligned}\|H^s f_r\|_{\alpha,2} &= \|f_r * h_s\|_{\alpha,2}\\ &\leqslant \|f_r\|_{\alpha,1}\,\|h_s\|_{\alpha,2}\\ &\leqslant \|h_s\|_{\alpha,2}\,\big\|\,|(x,t)|^{-a}\chi_{B_r}\big\|_{\alpha,2}\,\|\,|(x,t)|^a f\|_{\alpha,2}.\end{aligned}$$

Since

$$\big\|\,|(x,t)|^{-a}\chi_{B_r}\big\|^2_{\alpha,2} = A_{\alpha,a}\, r^{6\alpha+4-2a},$$

where

$$A_{\alpha,a} = \frac{B_\alpha}{2^{2\alpha+3}\Gamma(\alpha+\frac{1}{2})\Gamma(\alpha+1)(3\alpha+2-a)}$$

and $B_\alpha = B(\frac{\alpha+1}{2}, \frac{2\alpha+1}{2})$, B is the beta function, we get

$$\begin{aligned} \|H^s f\|_{\alpha,2} &\leqslant \|H^s f_r\|_{\alpha,2} + \|H^s f^r\|_{\alpha,2} \\ &\leqslant r^{-a} \|\, |(x,t)|^a f\|_{\alpha,2} \left(1 + C\, \|h_s\|_{\alpha,2}\, r^{3\alpha+2}\right). \end{aligned}$$

By the relation (4), we obtain

$$\|H^s f\|_{\alpha,2} \leqslant r^{-a} \|\, |(x,t)|^a f\|_{\alpha,2} \left(1 + C\, s^{-\frac{3\alpha+2}{2}}\, r^{3\alpha+2}\right).$$

Choosing $r = s^{\frac{1}{2}}$, we obtain $\quad \|H^s f\|_{\alpha,2} \leqslant C\, s^{-\frac{a}{2}} \|\, |(x,t)|^a f\|_{\alpha,2}.$ $\square$

In the sequal, by proceeding as the same way of the papers of Ma [7, 8], we can derive the following result.

THEOREM 3.1. *Let $a,b > 0$, then for all $f \in L^2_\alpha(\mathbb{K})$, we have*

$$\|\, |(x,t)|^a f\|_{\alpha,2}^{\frac{2b}{a+2b}} \left\|(2(2m+\alpha+1)\lambda)^b \mathscr{F}_{LB}(f)\right\|_{\gamma_\alpha,2}^{\frac{a}{a+2b}} \geqslant C\, \|f\|_{\alpha,2}. \qquad (5)$$

Proof. $\mathscr{S}(\mathbb{K})$ is dense in $L^2_\alpha(\mathbb{K})$, so we need only to prove (5) for $\mathscr{S}(\mathbb{K})$.
Assume that $a < 3\alpha+2$.
If $b \leqslant 1$, then by Lemma 3.1 it follows that

$$\begin{aligned} \|f\|_{\alpha,2} &\leqslant \|H^s f\|_{\alpha,2} + \|(1-H^s)\, f\|_{\alpha,2} \\ &\leqslant C\, s^{-\frac{a}{2}} \|\, |(x,t)|^a f\|_{\alpha,2} + \left\|(1-H^s)\ (sL)^{-b}\ (sL)^b f\right\|_{\alpha,2}. \end{aligned}$$

Let $g = (sL)^b f$. Then

$$\left\|(1-H^s)\ (sL)^{-b}\ g\right\|_{\alpha,2} = \left\|\ (1-e^{-2\lambda(2m+\alpha+1)s})\ (2\lambda(2m+\alpha+1)s)^{-b}\ \ \mathscr{F}_{LB}(g)\right\|_{\gamma_\alpha,2}.$$

Since, if $b \leqslant 1$, the function $t \longmapsto (1-e^{-t})\, t^{-b}$ is bounded for $t \geqslant 0$ and therefore

$$\|f\|_{\alpha,2} \leqslant C\left(\ s^{-\frac{a}{2}} \|\, |(x,t)|^a f\|_{\alpha,2} + s^b \left\|L^b f\right\|_{\alpha,2}\right).$$

By optimizing in s as in [7], we can obtain

$$\|\, |(x,t)|^a f\|_{\alpha,2}^{\frac{2b}{a+2b}} \ \left\|L^b f\right\|_{\alpha,2}^{\frac{a}{a+2b}} \geqslant C\, \|f\|_{\alpha,2}.$$

Then the Plancherel formula yields the desired result.

If $b > 1$, then $u \leqslant 1 + u^b$ for $u \geqslant 0$. Hence

$$\frac{2(2m+\alpha+1)\lambda}{\varepsilon} \leqslant 1 + \left(\frac{2(2m+\alpha+1)\lambda}{\varepsilon}\right)^b, \quad \text{for } \varepsilon > 0.$$

It follows that

$$\|(2(2m+\alpha+1)\lambda)\mathscr{F}_{LB}(f)\|_{\gamma_\alpha,2} \leqslant \varepsilon \|f\|_{\alpha,2} + \varepsilon^{1-b} \left\|(2(2m+\alpha+1)\lambda)^b \mathscr{F}_{LB}(f)\right\|_{\gamma_\alpha,2}.$$

By optimizing in ε, we get

$$\|(2(2m+\alpha+1)\lambda)\mathscr{F}_{LB}(f)\|_{\gamma_\alpha,2} \leqslant C\|f\|_{\alpha,2}^{1-\frac{1}{b}} \left\|(2(2m+\alpha+1)\lambda)^b \mathscr{F}_{LB}(f)\right\|_{\gamma_\alpha,2}^{\frac{1}{b}}.$$

Together with (5) for $b = 1$, we get the result for $b > 1$.

If $a \geqslant 3\alpha + 2$ then we have

$$\frac{|(x,t)|}{\varepsilon} \leqslant 1 + \frac{|(x,t)|^a}{\varepsilon^a}, \quad \varepsilon > 0.$$

It follows that

$$\|\,|(x,t)|\, f\|_{\alpha,2} \leqslant \varepsilon \|f\|_{\alpha,2} + \varepsilon^{1-a} \|\,|(x,t)|^a f\|_{\alpha,2}.$$

Optimizing in ε, we get

$$\|\,|(x,t)|\, f\|_{\alpha,2} \leqslant C\|f\|_{\alpha,2}^{1-\frac{1}{a}} \|\,|(x,t)|^a f\|_{\alpha,2}^{\frac{1}{a}}.$$

Together with (5) for $a = 1$, we get the result for $a \geqslant 3\alpha + 2$. □

4. Local uncertainty inequalities

In this section, based on the ideas of Price [10, 11] and by similar techniques used on the paper [9] of Omri and Rachdi, we establish a local uncertainty inequalities related to Laguerre-Bessel transform.

THEOREM 4.1. *Let s be a real number such that $0 < s < 3\alpha + 2$. Then for all nonzero $f \in L^2_\alpha(\mathbb{K})$ and for all measurable subsets $E \subset \widehat{\mathbb{K}}$ such that $0 < \gamma_\alpha(E) < +\infty$, we have*

$$\|\mathscr{F}_{LB}(f)\chi_E\|_{\gamma_\alpha,2} \leqslant K_{\alpha,s}\, \gamma_\alpha(E)^{\frac{s}{2(3\alpha+2)}} \|\,|(x,t)|^s f\|_{\alpha,2}, \tag{6}$$

where

$$K_{\alpha,s} = \left(\frac{(3\alpha+2-s)B_\alpha}{2^{2\alpha+3}\Gamma(\alpha+\frac{1}{2})\Gamma(\alpha+1)s^2}\right)^{\frac{s}{2(3\alpha+2)}} \frac{3\alpha+2}{3\alpha+2-s}.$$

The equality in (6) does not hold.

Proof. Let $0 < s < 3\alpha+2$ and $f \in L^2_\alpha(\mathbb{K})$, by Minkowski's inequality, it follows

$$\|\mathscr{F}_{LB}(f)\chi_E\|_{\gamma_\alpha,2} \leqslant \|\mathscr{F}_{LB}(f\chi_{B_r})\chi_E\|_{\gamma_\alpha,2} + \left\|\mathscr{F}_{LB}(f\chi_{B_r^c})\chi_E\right\|_{\gamma_\alpha,2}.$$

Therefore

$$\|\mathscr{F}_{LB}(f)\chi_E\|_{\gamma_\alpha,2} \leqslant \gamma_\alpha(E)^{\frac{1}{2}} \|\mathscr{F}_{LB}(f\chi_{B_r})\chi_E\|_{\gamma_\alpha,\infty} + \left\|\mathscr{F}_{LB}(f\chi_{B_r^c})\right\|_{\gamma_\alpha,2}, \tag{7}$$

$$\|\mathscr{F}_{LB}(f)\chi_E\|_{\gamma_\alpha,2} \leqslant \gamma_\alpha(E)^{\frac{1}{2}} \|f\chi_{B_r}\|_{\alpha,1} + \left\|\mathscr{F}_{LB}(f\chi_{B_r^c})\right\|_{\gamma_\alpha,2}. \tag{8}$$

On the other hand, using the Schwartz inequality, we get

$$\|f\chi_{B_r}\|_{\alpha,1} \leqslant \|\,|(x,t)|^s\, f\chi_{B_r}\|_{\alpha,2}\, \left\|\,|(x,t)|^{-s}\chi_{B_r}\right\|_{\alpha,2}. \tag{9}$$

Therefore, we have

$$\begin{aligned}
\|f\chi_{B_r}\|_{\alpha,1} &\leqslant \|\,|(x,t)|^s f\chi_{B_r}\|_{\alpha,2}\, A_{\alpha,s}^{\frac{1}{2}}\, r^{3\alpha+2-s},\\
\left\|\mathscr{F}_{LB}(f\chi_{B_r^c})\right\|_{\gamma_\alpha,2} &\leqslant r^{-s} \|\,|(x,t)|^s\, f\|_{\alpha,2},\\
\|\mathscr{F}_{LB}(f)\chi_E\|_{\gamma_\alpha,2} &\leqslant g_{\alpha,s}(r)\ \ \|\,|(x,t)|^s\, f\|_{\alpha,2},
\end{aligned}$$

where $g_{\alpha,s}$ is the function defined on $(0,+\infty)$ by

$$g_{\alpha,s}(r) = r^{-s} + (A_{\alpha,s}\gamma_\alpha(E))^{\frac{1}{2}}\ r^{3\alpha+2-s}.$$

In particular, taking the minimal value of $g_{\alpha,s}$, we get the inequality

$$\|\mathscr{F}_{LB}(f)\chi_E\|_{\gamma_\alpha,2} \leqslant g_{\alpha,s}(r_0)\ \|\,|(x,t)|^s\, f\|_{\alpha,2},$$

where

$$r_0 = \left(\frac{s}{3\alpha+2-s}\right)^{\frac{1}{3\alpha+2}}\ (A_{\alpha,s}\gamma_\alpha(E))^{-\frac{1}{2(3\alpha+2)}}.$$

However $g_{\alpha,s}(r_0) = \gamma_\alpha(E)^{\frac{s}{2(3\alpha+2)}}\, K_{\alpha,s}$.

Let us prove that the equality in (6) does not hold. Suppose that there exists a nonzero function $f \in L^2_\alpha(\mathbb{K})$ such that

$$\|\mathscr{F}_{LB}(f)\chi_E\|_{\gamma_\alpha,2} = K_{\alpha,s}\ \gamma_\alpha(E)^{\frac{s}{2(3\alpha+2)}}\ \|\,|(x,t)|^s f\|_{\alpha,2}.$$

Combining the relations (7), (8) and (9), we get the following equalities

$$\left\|\mathscr{F}_{LB}(f\chi_{B_{r_0}})\chi_E\right\|_{\gamma_\alpha,2} = \gamma_\alpha(E)^{\frac{1}{2}}\ \left\|\mathscr{F}_{LB}(f\chi_{B_{r_0}})\right\|_{\gamma_\alpha,\infty}, \tag{10}$$

$$\left\|f\chi_{B_{r_0}}\right\|_{\alpha,1} = \left\|\mathscr{F}_{LB}(f\chi_{B_{r_0}})\right\|_{\gamma_\alpha,\infty}, \tag{11}$$

$$\left\|f\chi_{B_{r_0}}\right\|_{\alpha,1} = \|\,|(x,t)|^s\, f\|_{\alpha,2}\ \left\|\,|(x,t)|^{-s}\,\chi_{B_{r_0}}\right\|_{\alpha,2}. \tag{12}$$

From the equality condition of the Schwart inequality, it follows from the relation (12) that

$$|f(x,t)| = C\,|(x,t)|^{-2s}\chi_{B_{r_0}}(x,t).$$

By (3) and (11), it follows that there exists $(\lambda_0, m_0) \in \hat{\mathbb{K}}$ for which

$$\begin{aligned}\int_{B_{r_0}} |f(x,t)| dm_\alpha(x,t) &= \left|\int_{B_{r_0}} f(x,t)\varphi_{(\lambda_0,m_0)}(x,t)\, dm_\alpha(x,t)\right| \\ &\leqslant \int_{B_{r_0}} |f(x,t)|\, |\varphi_{(\lambda_0,m_0)}(x,t)|\, dm_\alpha(x,t) \\ &\leqslant \int_{B_{r_0}} |f(x,t)| dm_\alpha(x,t).\end{aligned}$$

Therefore, $|\varphi_{(\lambda_0,m_0)}(x,t)| = 1$ for $(x,t) \in B_{r_0}$, and thus, $\lambda_0 = 0$ and $f(x,t) = e^{i\theta_0}|f(x,t)|$, where $\theta_0 \in \mathbb{R}$.

On the other hand, by the relation (10), we get

$$\iint_E \left(\|\mathscr{F}_{LB}(f)\|^2_{\gamma_\alpha,\infty} - \left|\mathscr{F}_{LB}(f)(\lambda,m)\right|^2\right) d\gamma_\alpha(\lambda,m) = 0.$$

Then for almost every $(\lambda,m) \in E$, we have

$$|\mathscr{F}_{LB}(f)(\lambda,m)| = \|\mathscr{F}_{LB}(f)\|_{\gamma_\alpha,\infty}.$$

Thus, we deduce that,

$$|\mathscr{F}_{LB}(f)(\lambda,m)| = |\mathscr{F}_{LB}(f)(0,m_0)| \quad \text{a.e } (\lambda,m) \in E, \tag{13}$$

which implies that $\lambda = 0$. This contradicts the relation (13) because $\gamma_\alpha(E) > 0$. So the inequality (6) is stictly satisfied. □

LEMMA 4.1. *Let s be a real number such that $s > 3\alpha + 2$, then for all nonzero measurable function f on $\mathbb{K}$, we have*

$$\|f\|_{\alpha,1} \leqslant M^{\frac{1}{2}}_{\alpha,s}\, \|f\|^{1-\frac{3\alpha+2}{s}}_{\alpha,2}\, \|\,|(x,t)|^s f\|^{\frac{3\alpha+2}{s}}_{\alpha,2}, \tag{14}$$

where $M_{\alpha,s} = A_{\alpha,s}\, B\left(\frac{s-3\alpha-2}{s}, \frac{3\alpha+2}{s}\right)\left(\frac{s-3\alpha-2}{3\alpha+2}\right)^{\frac{3\alpha+2}{s}}$.

We have equality in (14) *if only if there exists $a > 0$ and $b > 0$ such that*

$$|f(x,t)| = \left(a + b\,|(x,t)|^{2s}\right)^{-1}.$$

Proof. The inequality (14) holds if $\|f\|_{\alpha,2} = +\infty$ or $\|\,|(x,t)|^s f\|_{\alpha,2} = +\infty$.

Assume that $\|f\|_{\alpha,2} + \|\,|(x,t)|^s f\|_{\alpha,2} < +\infty$.

From the hypothesis $s > 3\alpha + 2$, we deduce that for all $a > 0$ and $b > 0$, the function

$$(x,t) \longmapsto \left(a + b\,|(x,t)|^{2s}\right)^{-1}$$

belongs to $L^1_\alpha(\mathbb{K})$ and by Hölder's inequality, we have

$$\|f\|^2_{\alpha,1} \leqslant \left\| \left(1+ |(x,t)|^{2s}\right)^{\frac{1}{2}} f \right\|^2_{\alpha,2} \left\| \left(1+ |(x,t)|^{2s}\right)^{-\frac{1}{2}} \right\|^2_{\alpha,2}. \tag{15}$$

We have equality in (15) if and only if

$$|f(x,t)| = C\ \left(1+ |(x,t)|^{2s}\right)^{-1}. \tag{16}$$

But

$$\left\| \left(1+ |(x,t)|^{2s}\right)^{\frac{1}{2}} f \right\|^2_{\alpha,2} = \|f\|^2_{\alpha,2} + \|\ |(x,t)|^s f\|^2_{\alpha,2},$$

therefore

$$\|f\|^2_{\alpha,1} \leqslant N_{\alpha,s} \left(\|f\|^2_{\alpha,2} + \|\ |(x,t)|^s f\|^2_{\alpha,2}\right), \tag{17}$$

where

$$N_{\alpha,s} = \left\| \left(1+ |(x,t)|^{2s}\right)^{-\frac{1}{2}} \right\|^2_{\alpha,2}.$$

By straightforward calculus, we get

$$N_{\alpha,s} = A_{\alpha,s}\, B\left(\frac{s-3\alpha-2}{s}, \frac{3\alpha+2}{s}\right) \frac{3\alpha+2-s}{s}.$$

For $r>0$, we put

$$f_r(x,t) = r^{-(6\alpha+4)} f\left(\frac{x}{r}, \frac{t}{r^2}\right).$$

Then we have

$$\begin{aligned}
\|f_r\|_{\alpha,1} &= \|f\|_{\alpha,1}, \\
\|f_r\|^2_{\alpha,2} &= \frac{1}{r^{6\alpha+4}} \|f\|^2_{\alpha,2}, \\
\|\ |(x,t)|^s f_r\|^2_{\alpha,2} &= \frac{1}{r^{6\alpha+4-2s}} \|\ |(x,t)|^s f\|^2_{\alpha,2}.
\end{aligned}$$

Replacing f by f_r in the relation (17), we deduce that for all $r>0$, we have

$$\|f\|^2_{\alpha,1} \leqslant N_{\alpha,s} \left(r^{-(6\alpha+4)}\ \|f\|^2_{\alpha,2} + r^{2s-6\alpha-4}\ \|\ |(x,t)|^s f\|^2_{\alpha,2}\right).$$

In particular, for

$$r_0 = \left(\frac{(3\alpha+2)\,\|f\|^2_{\alpha,2}}{(s-3\alpha-2)\,\|\ |(x,t)|^s f\|^2_{\alpha,2}}\right)^{\frac{1}{2s}},$$

we get

$$\|f\|^2_{\alpha,1} \leqslant M_{\alpha,s}\ \|f\|^{2-\frac{6\alpha+4}{s}}_{\alpha,2}\ \|\ |(x,t)|^s f\|^{\frac{6\alpha+4}{s}}_{\alpha,2}. \tag{18}$$

Now suppose that we have equality in the relation (18). Then we have equality in (17) for f_{r_0} and by means of (16), we obtain

$$|f_{r_0}(x,t)| = C\,\left(1+\,|(x,t)|^{2s}\right)^{-1}$$

and then $|f(x,t)| = \left(a+b\,|(x,t)|^{2s}\right)^{-1}$. □

THEOREM 4.2. *Let s be a real number such that $s > 3\alpha+2$.*
Then for all nonzero $f \in L^2_\alpha(\mathbb{K})$ and for all measurable subset $E \subset \widehat{\mathbb{K}}$ such that $0 < \gamma_\alpha(E) < +\infty$, we have

$$\|\mathscr{F}_{LB}(f)\chi_E\|_{\gamma_\alpha,2} \leqslant M^{\frac{1}{2}}_{\alpha,s}\;\gamma_\alpha(E)^{\frac{1}{2}}\;\|f\|^{1-\frac{3\alpha+2}{s}}_{\alpha,2}\;\|\,|(x,t)|^s\,f\|^{\frac{3\alpha+2}{s}}_{\alpha,2}\,, \tag{19}$$

where $M_{\alpha,s}$ is the constant given by the relation (14). The equality in (19) does not hold.

Proof. Suppose that the right-hand side of (19) is finite. Then, according to Lemma 4.1, the function f belongs to $L^1_\alpha(\mathbb{K})$ and we have

$$\begin{aligned}\|\mathscr{F}_{LB}(f)\chi_E\|^2_{\gamma_\alpha,2} &\leqslant \gamma_\alpha(E)\,\|\mathscr{F}_{LB}(f)\|^2_{\gamma_\alpha,\infty}\\ &\leqslant \gamma_\alpha(E)\,\|f\|^2_{\alpha,1}\\ &\leqslant M_{\alpha,s}\,\gamma_\alpha(E)\;\|f\|^{2-\frac{6\alpha+4}{s}}_{\alpha,2}\;\|\,|(x,t)|^s\,f\|^{\frac{6\alpha+4}{s}}_{\alpha,2}\,,\end{aligned}$$

where $M_{\alpha,s}$ is the constant given by the relation (14).

Let us prove that the equality in (19) does not hold. Suppose that there exists a nonzero function $f \in L^2_\alpha(\mathbb{K})$ such that

$$\|\mathscr{F}_{LB}(f)\chi_E\|^2_{\gamma_\alpha,2} = M_{\alpha,s}\,\gamma_\alpha(E)\;\|f\|^{2-\frac{6\alpha+4}{s}}_{\alpha,2}\;\|\,|(x,t)|^s\,f\|^{\frac{6\alpha+4}{s}}_{\alpha,2}\,.$$

Consequently, we find

$$\begin{aligned}\|\mathscr{F}_{LB}(f)\chi_E\|^2_{\gamma_\alpha,2} &= \gamma_\alpha(E)\;\|\mathscr{F}_{LB}(f)\|^2_{\gamma_\alpha,\infty}\,,\\ \|f\|_{\alpha,1} &= \|\mathscr{F}_{LB}(f)\|_{\gamma_\alpha,\infty}\end{aligned}$$

and

$$\|f\|_{\alpha,1} = M^{\frac{1}{2}}_{\alpha,s}\;\|f\|^{1-\frac{3\alpha+2}{s}}_{\alpha,2}\;\|\,|(x,t)|^s\,f\|^{\frac{3\alpha+2}{s}}_{\alpha,2}\,. \tag{20}$$

Applying Lemma 4.1 and the relation (20), we deduce that

$$\forall (x,t) \in \mathbb{K},\; f(x,t) = \psi(x,t)\;\left(a+b\,|(x,t)|^{2s}\right)^{-1},$$

with $|\psi(x,t)| = 1$, $a > 0$ and $b > 0$.

By the same arguments of the proof of theorem 4.1, we show that the inequality (19) is stictly satisfied. □

THEOREM 4.3. *Let $s = 3\alpha+2$, then for all nonzero $f \in L^2_\alpha(\mathbb{K})$ and for all measurable set $E \subset \widehat{\mathbb{K}}$ such that $0 < \gamma_\alpha(E) < +\infty$, we have*

$$\|\mathscr{F}_{LB}(f)\chi_E\|_{\gamma_\alpha,2} < C_\alpha\ \gamma_\alpha(E)^{\frac{1}{2(3\alpha+2)}}\ \|f\|_{\alpha,2}^{\frac{3\alpha+1}{3\alpha+2}}\ \|\ |(x,t)|^s\ f\|_{\alpha,2}^{\frac{1}{3\alpha+2}},$$

with

$$C_\alpha = (3\alpha+2)^2\ (3\alpha+1)^{-\frac{6\alpha+1}{2(3\alpha+2)}-1}\ (A_{\alpha,s}(3\alpha+2-s))^{\frac{1}{2(3\alpha+2)}}.$$

Proof. We have $s = 3\alpha+2 > 1$, using the same manner as the end of the proof of theorem 3.1, we obtain

$$\|\ |(x,t)|\ f\|_{\alpha,2} \leqslant s\ (s-1)^{\frac{1}{s}-1}\ \|f\|_{\alpha,2}^{1-\frac{1}{s}}\ \|\ |(x,t)|^s\ f\|_{\alpha,2}^{\frac{1}{s}}.$$

By this inequality together with (6) taken for $s = 1$, we get the result for $s = 3\alpha+2$. □

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