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STRICT VERIFICATION OF APPROXIMATE MIDCONVEXITY ON NON–CONVEX SETS

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Abstract. Let V be a subset of an Abelian group G and let $\omega : V \times V \to [0,\infty]$ be given. We say that a function $f : V \to \mathbb{R}$ is $\omega(\cdot,\cdot)$*-midconvex* if

$$f(x) \leqslant \frac{f(x-\delta)+f(x+\delta)}{2} + \omega(x-\delta, x+\delta)$$

for $x \in V, \delta \in G$ such that $x-\delta, x+\delta \in V$. Our aim is to provide a computer assisted method to estimate

$$\sup\{f \in V \to \mathbb{R} \colon f \in \mathscr{B}(V;W), f \text{ is } \omega(\cdot,\cdot)\text{-midconvex}\},$$

where $\mathscr{B}(V;W)$ denotes the set of real-valued, bounded from above functions on V which are zero on W $(W \subset V)$. We present an algorithm which for given $\varepsilon > 0$ enables us, under reasonable assumptions, to find the above supremum with accuracy ε. We test our results for $V = \{0, \frac{1}{N}, \dots, \frac{N-1}{N}, 1\}$ and $W = \{0,1\}$, where $N \in \mathbb{N}$ is fixed.

1. Introduction

The main idea of our investigation lies in joining together the notions of approximate convexity and convexity on non-convex sets.

Let us first recall some basic information concerning approximate convexity. The term “approximate convexity” was introduced by D. H. Hyers and S. M. Ulam [5] in 1952. Its variation adapted to Jensen convexity can be stated as follows:

DEFINITION 1.1. ([13]) Let X be a normed space, V be a convex subset of X, and ε be a nonnegative constant. A function $f : V \to \mathbb{R}$ is said to be ε*-midconvex* (or ε*-Jensen convex*) if

$$Jf(x,y) = f\left(\frac{x+y}{2}\right) - \frac{f(x)+f(y)}{2} \leqslant \varepsilon \quad \text{for } x,y \in V \colon \frac{x+y}{2} \in V.$$

A natural generalization of this definition for normed spaces lies in replacing the constant ε by a function ω which depends on the norm of the difference $\|x-y\|$:

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DEFINITION 1.2. Let V be a convex subset of a normed space X and let $\omega\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a given function. We say that $f\colon V \to \mathbb{R}$ is *$\omega(\cdot)$-midconvex* (or *$\omega(\cdot)$-Jensen convex*) if

$$f\left(\frac{x+y}{2}\right) \leqslant \frac{f(x)+f(y)}{2} + \omega(\|x-y\|) \quad \text{for } x,y \in V : \frac{x+y}{2} \in V.$$

For some recent results we refer the reader to [14, 16]. The general research question lies in verifying how far from convex functions are $\omega(\cdot)$-approximately convex functions. To measure this we will the convexity difference operator defined by

$$Cf(x,y;t) := f(tx+(1-t)y) - tf(x) - (1-t)f(y) \quad \text{for } x,y \in V, t \in [0,1]$$

will be useful. The method of attack of this problem in many cases is based on the reduction to one dimensional case, which is stated in the following trivial observation:

OBSERVATION 1.3. Let V be a convex subset of a Banach space and let $f\colon V \to \mathbb{R}$ be given. Then f is $\omega(\cdot)$-midconvex iff for every $x,y \in V$, the function $\varphi_{x,y} : [0,1] \to \mathbb{R}$ defined by

$$\varphi_{x,y}(t)\colon [0,1] \ni t \to Cf(x,y;t) \in \mathbb{R},$$

is $\omega_{x,y}(\cdot)$-midconvex, where $\omega_{x,y}(r) := \omega(\|x-y\|r)$.

Observe that the above mentioned function $\varphi_{x,y}$ satisfies $\varphi_{x,y}(0) = \varphi_{x,y}(1) = 0$. As in general case to obtain convexity from Jensen convexity we need (local) boundedness, we see that the study of $\omega(\cdot)$-approximately convex functions can be reduced to investigation of the set

$$J_\omega([0,1],\{0,1\}) := \{f \in \mathscr{B}([0,1];\{0,1\})\colon f \text{ is } \omega(\cdot)\text{-midconvex}\},$$

where $\omega\colon [0,1] \to \mathbb{R}_+$ is given and $\mathscr{B}(V;W)$ denotes the set of all real-valued bounded from above functions on set V which are zero on W. It occurs that the optimal bound of this set, defined by

$$f_\omega([0,1],\{0,1\}) := \sup\{f \in J_\omega([0,1],\{0,1\})\}$$

is usually fractal Takagi-like functions, see [1, 2, 9, 10, 13, 17].

Our second motivation lies in the recent generalization of (Jensen) convexity to non-convex sets (or in general arbitrary subsets of groups) proposed and studied by W. Jarczyk and M. Laczkovich [7, 8]:

DEFINITION 1.4. ([8]) Let G be an Abelian group and let V be a subset of G. We say that $f\colon V \to \mathbb{R}$ is *convex* if the following inequality holds

$$f(x) \leqslant \frac{f(x-\delta)+f(x+\delta)}{2} \quad \text{for } x \in V, \delta \in G \text{ such that } x-\delta, x+\delta \in V.$$

In our paper we generalize the definition of approximate convexity in the spirit of the previous definition:

DEFINITION 1.5. Let V be a subset of an Abelian group G and let $\omega : V \times V \to [0,\infty]$ such that $\omega(x,x) = 0$ for $x \in V$ be given.

We say that a function $f : V \to \mathbb{R}$ is $\omega(\cdot,\cdot)$*-midconvex* (or $\omega(\cdot,\cdot)$*-Jensen convex*) if

$$f(x) \leqslant \frac{f(x-\delta)+f(x+\delta)}{2} + \omega(x-\delta,x+\delta) \quad \text{for } x \in V, \delta \in G\colon\, x-\delta, x+\delta \in V.$$

Observe that for $\omega \equiv 0$ we directly obtain Definition 1.4.

Similarly to the standard case, the study of $\omega(\cdot,\cdot)$-midconvex functions and their understanding can be often deduced from the properties of the set

$$J_\omega(V;W) := \{f \in \mathscr{B}(V;W)\colon\, f \text{ is } \omega(\cdot,\cdot)\text{-midconvex}\}.$$

Our aim in this paper is to present a computer assisted approach which given a finite set V can find within a specified error bound the optimal estimation from above of $J_\omega(V,W)$, that is

$$f_\omega(V,W) := \sup\{f \in J_\omega(V,W)\}.$$

We illustrate our approach in the simplest case when $V = \{0, 1/N, \ldots,\ (N-1)/N, 1\}$ and $W = \{0,1\}$.

2. Estimate of optimal $\omega(\cdot,\cdot)$-midconvex functions

In this section we discuss the construction of optimal ω-Jensen convex functions.

Let V be a given subset of an Abelian group G. From now on we assume that $\omega : V \times V \to [0,\infty]$ such that $\omega(x,x) = 0$ for $x \in V$ is fixed.

For $x \in V$ and $\delta \in G$ such that $x-\delta, x+\delta \in V$ we define

$$Rf(x,\delta) := \min\{f(x), \frac{f(x-\delta)+f(x+\delta)}{2} + \omega(x-\delta,x+\delta)\}$$

for $f \in [-\infty,\infty)^V$.

Now, we introduce the operation $\mathscr{P} : [-\infty,\infty)^V \to [-\infty,\infty)^V$ as follows

$$\mathscr{P}f(x) := \inf\{Rf(x,\delta) | \delta \in G\colon\, x-\delta, x+\delta \in V\} \quad \text{for } f \in [-\infty,\infty)^V.$$

PROPOSITION 2.1. Let $f, g \in [-\infty,\infty)^V$ be arbitrary functions. Then operation $\mathscr{P}$ has following properties:

(i) $\mathscr{P}g \leqslant g$,

(ii) if $g \geqslant f$, then $\mathscr{P}g \geqslant \mathscr{P}f$,

(iii) $\mathscr{P}(0) \equiv 0$,

(iv) $\mathscr{P}g \geqslant 0$ for $g \geqslant 0$.

Proof. Properties (i) and (ii) are obvious. Assumption $\omega \geqslant 0$ implies property (iii). Last property is a simple consequence of properties (ii) and (iii). $\square$

Furthermore, because $\mathscr{P}$ is decreasing the operation $\mathscr{P}^\infty : [-\infty,\infty)^V \to [-\infty,\infty)^V$

$$\mathscr{P}^\infty f := \lim_{n\to\infty} \mathscr{P}^n f$$

is well-defined. By Proposition 2.1 we get that $\mathscr{P}^\infty g \geqslant 0$ for $g \geqslant 0$.

As a consequence of Proposition 2.1 we get:

LEMMA 2.2. Let $f \in [-\infty,\infty)^V$ be $\omega(\cdot,\cdot)$-midconvex. Then $f = \mathscr{P}f = \mathscr{P}^\infty f$.

Proof. Since f is $\omega(\cdot,\cdot)$-midconvex:

$$f(x) \leqslant \frac{f(x-\delta)+f(x+\delta)}{2} + \omega(x-\delta,x+\delta) \quad \text{for } x \in V, \delta \in G\colon x-\delta, x+\delta \in V.$$

Thus

$$f(x) = \min\{f(x), \frac{f(x-\delta)+f(x+\delta)}{2} + \omega(x-\delta,x+\delta)\} = Rf(x,\delta)$$

for $x \in V, \delta \in G$ such that $x-\delta, x+\delta \in V$. Consequently $f = \mathscr{P}f$, which trivially implies that $f = \mathscr{P}^\infty f$. $\square$

LEMMA 2.3. Let $f \in [-\infty,\infty)^V$ be $\omega(\cdot,\cdot)$-midconvex and let $g \in [-\infty,\infty)^V$ be such that $g \geqslant f$. Then $f \leqslant \mathscr{P}^\infty g$.

Proof. By Lemma 2.2 and Proposition 2.1 (iii) we obtain $f = \mathscr{P}f \leqslant \mathscr{P}g$. Consequently by induction we have $f \leqslant \mathscr{P}^n g$ for $n \in \mathbb{N}$ and therefore $f = \mathscr{P}^\infty g$. $\square$

Let $W \subset V$ be fixed. We are interested in the class of approximately convex functions which are zero on W. We want to find the optimal estimation (from above) of elements of this class. We put

$$f_\omega(V;W) := \sup\{f \in J_\omega(V;W)\}.$$

There appears a question how to compute the function $f_\omega(V;W)$.

In many cases, the estimation and properties of the $\omega(\cdot,\cdot)$-Jensen convex functions can be deduced from the knowledge of $f_\omega(V;W)$. For example, if we want to find an estimate of f (which we assume to be bounded form above and ω-Jensen convex) on the interval $[a,b]$, by subtracting the respective affine function (namely $x \to f(a) + \frac{x-a}{b-a}[f(b)-f(a)]$) we can reduce to the case when $f(a) = f(b) = 0$. Thus we can restrict to investigation of bounded approximately Jensen convex functions on the interval $[0,1]$, which are zero at 0 and 1 (so $V = [0,1]$ and $W = \{0,1\}$).

Next theorem gives us the procedure how to find the upper bound of $f_\omega(V;W)$. By $\mathbb{1}_{V;W} : V \to \mathbb{R}$ we denote

$$\mathbb{1}_{V;W} : V \ni v \to \begin{cases} 1 \text{ for } v \in V \setminus W, \\ 0 \text{ for } v \in W. \end{cases}$$

THEOREM 2.4. *Let V and $W \subset V$ be given subsets of an Abelian group G. We assume that*

$$\exists\, A \geqslant 0\ \forall\, f \in J_\omega(V;W)\colon\ f \leqslant A. \tag{1}$$

Then

$$f_\omega(V;W) = \mathscr{P}^\infty(A\mathbb{1}_{V;W})$$

and f_ω is $\omega(\cdot,\cdot)$-midconvex.

Proof. Since $f_\omega(V;W) = \sup\{f \in J_\omega(V;W)\}$ by (1) we trivially obtain

$$f_\omega(V;W) \leqslant A\mathbb{1}_{V;W}$$

and consequently, by Lemma 2.3, the inequality

$$f_\omega(V;W) \leqslant \mathscr{P}^\infty(A\mathbb{1}_{V;W})$$

holds.

We prove the opposite inequality. For $n \in \mathbb{N} \cup \{\infty\}$ we put

$$g_n := \mathscr{P}^n(A\mathbb{1}_{V;W}).$$

Clearly, g_n converges pointwise, as $n \to \infty$, to $g_\infty := \lim_{n\to\infty} g_n$. On the other hand directly from the definition of operation $\mathscr{P}$ we know that

$$g_{n+1}(v) \leqslant \frac{g_n(v-\delta)+g_n(v+\delta)}{2} + \omega(v-\delta, v+\delta) \quad \text{for } \delta \in G\colon\ v-\delta, v+\delta \in V.$$

By taking the limit we get

$$g_\infty(v) \leqslant \frac{g_\infty(v-\delta)+g_\infty(v+\delta)}{2} + \omega(v-\delta, v+\delta) \quad \text{for } \delta \in G\colon\ v-\delta, v+\delta \in V,$$

which implies that g_∞ is $\omega(\cdot,\cdot)$-Jensen convex, and consequently $g_\infty \in J_\omega(V;W)$. □

The assumption (1) does not always hold. Let us consider following examples.

EXAMPLE 2.5. Let $V \subset [0,1]$ be such that

$$x \in [0,1/2] \cap V \Rightarrow 2x \in V,\ x \in [1/2,1] \cap V \Rightarrow 2x-1 \in V$$

and $W = \{0,1\} \subset V$. Then by [16, Proposition 2.1] condition (1) is satisfied.

EXAMPLE 2.6. Let $V = \{0, \frac{1}{3}, 1\}$ and $W = \{0,1\}$. Then an arbitrary function $f\colon V \to \mathbb{R}$ such that $f|_W \equiv 0$ is $\omega(\cdot,\cdot)$-midconvex. Consequently condition (1) is not satisfy for the pair (V,W).

An interesting problem is to characterize properties of sets satisfies condition (1).

Now we can easily obtain lower bound of optimal $\omega(\cdot,\cdot)$-midconvex function.

THEOREM 2.7. *Let V and $W \subset V$ be given subsets of an Abelian group G. We assume that condition* (1) *holds. Let $h\colon V \to \mathbb{R}$ be such that*

$$h \geqslant \mathscr{P}^{\infty}(A\mathbb{1}_{V;W}).$$

If $(1-\varepsilon)h$ is $\omega(\cdot,\cdot)$-midconvex for some $\varepsilon \in (0,1)$, then

$$(1-\varepsilon)h \leqslant f_{\omega}(V;W) \leqslant h. \tag{2}$$

Proof. According to Theorem 2.4 we get $f_{\omega} \leqslant h$, because function $\mathscr{P}^{\infty}(A\mathbb{1}_{V;W})$ is $\omega(\cdot,\cdot)$-midconvex. Lower bound of f_{ω} in (2) is a consequence of definition f_w as a supremum of set $J_{\omega}(V;W)$ while directly from the assumptions $(1-\varepsilon)h \in J_{\omega}(V;W)$. □

3. Strict numerical verification

In this section we give two algorithms which help us to encode the results obtained in the previous section and create application which founds bounds of $f_{\omega}(V;W)$ for V and $W \subset V$ finite subsets of an Abelian group G.

We introduce algorithm that summarizes results obtained in Theorem 2.4 and Theorem 2.7 which give us that outcome function from our construction is $\omega(\cdot,\cdot)$-midconvex:

choose
  $A \geqslant 0$ such that $\forall\, f \in J_{\omega}(V;W)\colon\, f \leqslant A$
  $\varepsilon \in (0,1)$ (precision)
  $n \leftarrow 1$
repeat
    $h_n \leftarrow$ upper bound for $\mathscr{P}^{n}(A\mathbb{1}_{V;W})$
    $n \leftarrow n+1$
until $(1-\varepsilon)h_n$ is not $\omega(\cdot,\cdot)$-midconvex
return we get estimation $(1-\varepsilon)h_n \leqslant f_{\omega}(V;W) \leqslant h_n$

As it occurs the above algorithm is inconvenient for implementation because states *calculate* h_n and *check that* $(1-\varepsilon)h_n$ *is* $\omega(\cdot,\cdot)$*-midconvex* slow it down. Hence we try to modify those calculations to make it faster.

But first we have to answer the question: how we can find upper bound for $\mathscr{P}^{n}(A\mathbb{1}_{V;W})$ for fixed $n \in \mathbb{N}$? To solve this problem we prepared all calculations using interval aritmetics which allows us to deal with finite precision of computer calculations and control error value [3, 15] (for implementation see [6]). When we work with interval arithmetic, instead of considering real number (ex. $\sqrt{3}$) we work with the interval (ex. $[1.7320;1.7321]$) which contains our number lies between lower and upper bound of this interval.

Let us start with useful notations:

$$K(V) = \{(v,\delta) | v \in V, \delta \in G : v-\delta, v+\delta \in V\},$$

where V is given finite subset of Abelian group G. Observe that then $\operatorname{card} K(V)$ is also finite as $\operatorname{card} K(V) \leqslant (\operatorname{card} V)^2$.

Now we introduce operator Q to speed up numerical computations.

DEFINITION 3.1. Let V be given finite subset of an Abelian group G and let $(v,\delta) \in K(V)$. We define operator $Q_{(v,\delta)} : [-\infty,\infty)^V \to [-\infty,\infty)^V$ as follows:

$$Q_{(v,\delta)} f \colon V \ni x \to \begin{cases} Rf(x,\delta) & \text{if } x = v, \\ f(x) & \text{otherwise,} \end{cases}$$

for $f \in [-\infty,\infty)^V$.

As we see for every $f \in [-\infty,\infty)^V$ the operator $Q_{(v,\delta)}$ modifies the function f most at the point v. Also we get that $Q_{(v,\delta)} f \leqslant f$.

From now on, we fix the sequence $s = (s_1,\dots,s_n)$ of elements of $K(V)$ such that $K(V) = \bigcup_{i=1}^n \{s_i\}$, where $n = \operatorname{card} K(V)$. We define

$$Q_s := Q_{s_n} \circ \dots \circ Q_{s_1}.$$

To simplify notation from now on we use the letter Q instead of Q_s.

As we show, we can apply the operator Q for function $h_A \colon V \ni v \to A\mathbb{1}_{V;W} \in \mathbb{R}_+$ and obtain upper bound for $\mathscr{P}(A\mathbb{1}_{V;W})$.

LEMMA 3.2. Let V be a finite subset of Abelian group G. We have

$$\mathscr{P}^{\operatorname{card} K(V)} f \leqslant Qf \leqslant \mathscr{P} f \quad \text{for } f \in [-\infty,+\infty)^V.$$

Proof. Let $f \in [-\infty,+\infty)^V$. According to Definition 3.1 we have that $\mathscr{P} f \leqslant Q_{(v,\delta)} f$ for all $(v,\delta) \in K(V)$, which implies $\mathscr{P}^{\operatorname{card} K(V)} f \leqslant Qf$.

We check now second inequality, so we want to show that for every $v \in V$: $Qf(v) \leqslant \mathscr{P} f(v)$. Let us choose arbitrary $v \in V$. We have that

$$\mathscr{P} f(v) = \inf \left\{ \frac{f(v-\delta)+f(v+\delta)}{2} + \omega(v-\delta, v+\delta) | \delta \in G \colon v-\delta, v+\delta \in V \right\}.$$

Because V is finite there exists such $\delta \in G$ realize those infimum. Thus we obtain $Q_{(v,h)}$ shuch that $Q_{(v,h)} f(v) \leqslant \mathscr{P} f(v)$. This finishes the proof, because v was arbitraty choosen. □

We see that the operator Q converges faster then $\mathscr{P}$, but from the perspective of computer calculations these operators are similar.

It is left to verify that condition (1) holds. However in the case where $V = [0,1]_N = \{0, 1/N, \dots, (N-1)/N, 1\}$ and $W = \{0,1\}$ we can put (see. [16, Corollary 2.1])

$$A = 2 \sup_{x,y \in [0,1]_N} \omega(x,y).$$

Thus we obtain the following observation (special case of Theorem 2.7).

THEOREM 3.3. *Let* $\omega : [0,1]_N \times [0,1]_N \to \mathbb{R}_+$ *and* $A \geqslant 2\sup\omega$ *be given. Let* $h\colon [0,1]_N \to \mathbb{R}$ *be such that*

$$h \geqslant Q^k(A\mathbb{1}_{[0,1]_N;\{0,1\}})$$

for some $k \in \mathbb{N}$. *If* $(1-\varepsilon)h$ *is* $\omega(\cdot,\cdot)$*-midconvex for some* $\varepsilon \in (0,1)$, *then*

$$(1-\varepsilon)h \leqslant f_\omega([0,1]_N;\{0,1\}) \leqslant h.$$

We can conclude by presenting full algorithm for finding estimation of $f_\omega([0,1]_N;\{0,1\})$:

choose
$A \geqslant 0$ such that for fixed $\omega\colon V \times V \to \mathbb{R}_+, A \geqslant 2\sup\omega$
$h_A\colon V \ni v \to A\mathbb{1}_{V;W} \in \mathbb{R}_+$
for $n \in \{1,2,\dots,N_{MAX}\}$ **do**
$h_A \leftarrow Qh_A$
end for
return h_A – upper bound of $\mathscr{P}^\infty(A\mathbb{1}_{V;W})$

Using the operator Q we can get function h_A – upper bound of $f_\omega([0,1]_N;\{0,1\})$. To obtain lower bound we calculate the error considered in Theorem 3.3 by choosing $\varepsilon \in (0,1)$ such that

$$\frac{1}{1-\varepsilon} \geqslant \sup\left\{\frac{h_A(x) - \frac{h_A(x-\delta)+h_A(x+\delta)}{2}}{\omega(x-\delta,x+\delta)} : x-\delta, x, x+\delta \in [0,1]_N, \delta \in \mathbb{R}, \delta \neq 0\right\}. \quad (3)$$

We created application (using Java programming language and following libraries: Interval Arithmetics Library [6], NetBeans Pack for OpenGL Java Development [12]) which applied operator Q to specified function ω and construct obtained function plot.

This application is available to download from:

http://www.ii.uj.edu.pl/~misztalk/index.php?page=convex

Plots prepared in this program are presented on Figures 1 and 3. All this pictures presents not one but two functions – lower and upper bound of $J_\omega([0,1]_N;\ \{0,1\})$, however the distance between them is so small that we cannot separate them from each other.

REMARK 3.4. (Numerical Experiments) We investigate how many iteration of the operator Q we need to obtain small ε. So let us fix $\omega(x,y) = |x-y|$ for $x,y \in [0,1]_{1024}$. We apply operator Q and then calculate ε according to equation (3). The results are presented on Figure 2. Surprising is that we need such few iterations to get high precision level – in this case it is sufficient to take 10 iterations to obtain $\varepsilon = 5.684 \cdot 10^{-14}$.

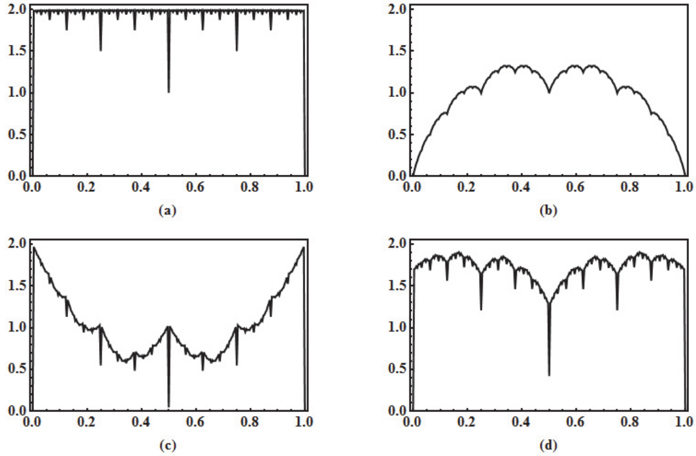

Figure 1: *Iteration of operator Q for different functions* ω*: (a)* $\omega(x,y) = |x-y|^{0.001}$, $x,y \in [0,1]_{1024}$. *We obtain* $\varepsilon = 2.22 \cdot 10^{-16}$. *(Compare with* [13]*). (b)* $\omega(x,y) = |x-y|$, $x,y \in [0,1]_{1024}$, $\varepsilon = 4.663 \cdot 10^{-15}$. *For this* ω *we have Takagi-like function* [2]. *(c)* $\omega(x,y) = (\cos|x-y|)^5$, $x \in [0,1]_{1024}$, $\varepsilon = 8.882 \cdot 10^{-16}$. *(d)* $\omega(x,y) = \sin(\exp|x-y|)$, $x \in [0,1]_{1024}$, $\varepsilon = 2.22 \cdot 10^{-16}$.

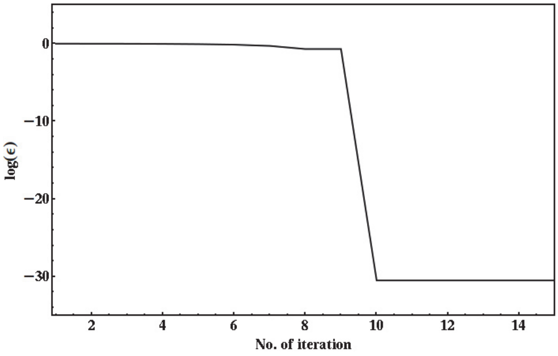

Figure 2: *Error* ε *as a function of iteration the operator Q for* $\omega(x,y) = |x-y|$ *under interval* $[0,1]_{1024}$.

4. Estimation of optimal midconvexity on $[0,1]_N$

In this section we recall two estimations for locally bounded $\alpha(\cdot)$-midconvex functions on $[0,1]_N$. We put $d(x) := 2\mathrm{dist}(x,\mathbb{Z})$ for $x \in \mathbb{R}$. Then the estimations can be stated as follows:

THEOREM 4.1. ([16, Corollary 2.1, Proposition 3.1]) *Let* $N = 2^k$ *for a certain* $k \in \mathbb{N}$. *Let* $h : [0,1]_N \to \mathbb{R}$, $h(0) = h(1) = 0$ *be an* $\alpha(\cdot)$*-midconvex function. Then*

$$h(q) \leqslant \min\Big\{ \underbrace{\sum_{k=0}^{\infty} \frac{1}{2^k}\alpha(d(2^k q))}_{E_1(q)}, \underbrace{\sum_{k=0}^{\infty} \alpha(1/2^k) d(2^k q)}_{E_2(q)} \Big\} \text{ for } q \in [0,1]_N. \tag{4}$$

We estimate RHS of (4) using interval arithmetics. Certainly for fixed $N = 2^k$, where $k \in \mathbb{N}$ we have for $q \in [0,1]_N$

$$E_1(q) = \sum_{k=0}^{N-1} \frac{1}{2^k}\alpha(d(2^k q)) + \sum_{k=N}^{\infty} \frac{1}{2^k}\alpha(d(2^k q)) = \sum_{k=0}^{N-1} \frac{1}{2^k}\alpha(d(2^k q)) + \frac{1}{2^{N-1}}\alpha(0), \tag{5}$$

$$E_2(q) = \sum_{k=0}^{N-1} \alpha(\frac{1}{2^k}) d(2^k q) + \sum_{k=N}^{\infty} \alpha(\frac{1}{2^k}) d(2^k q) = \sum_{k=0}^{N-1} \alpha(\frac{1}{2^k}) d(2^k q). \tag{6}$$

As you can see by doing a simple transformation we reduce the infinite sums to finite expressions.

OBSERVATION 4.2. Let V and $W \subset V$ be given subsets of an Abelian group G. If $V \subset \widehat{V}$, then $f_\omega(\widehat{V};W)|_V \leqslant f_\omega(V;W)$.

We prove that in some cases the estimation given by (4) is not optimal.

THEOREM 4.3. *Let $V = [0,1]_N$ for $N = 2^k$, $k \in \mathbb{N}$, $k \geqslant 3$ and $W = \{0,1\}$. For $\omega(x,y) = \sin(\cos(|x-y|))$ approximations of $f_\omega([0,1]_N, \{0,1\})$ obtained by (4) are not optimal (see Figure 3).*

Proof. First we consider $k = 3$. Then $N = 2^3 = 8$. Using interval arithmetics for estimations obtained in (5) and (6) we obtain

$$E_1(3/8) \in [1.594665139738596 - \varepsilon_1, 1.594665139738596 + \varepsilon_1],$$
$$E_2(3/8) \in [1.391691873308314 - \varepsilon_2, 1.391691873308314 + \varepsilon_2],$$

where $\varepsilon_1 = 7.327471962526033 \times 10^{-15}$, $\varepsilon_2 = 7.105427357601002 \times 10^{-15}$.

On the other hand, according to estimation calculated using our application we have

$$\mathscr{P}^\infty h_A(3/8) \leqslant Q^{25} h_A(3/8) \leqslant 1.247341841544101,$$

where

$$h_A = A\mathbb{1}_{[0,1]_8;\{0,1\}},$$
$$A > 1.682941969615793 \geqslant 2\sin(1) = 2\sup\{\sin(\cos(|x-y|)) : x,y \in [0,1]_8\}.$$

Thus

$$\mathscr{P}^\infty h_A(3/8) \leqslant \min\{E_1(3/8), E_2(3/8)\}. \tag{7}$$

Now we consider an arbitrary $k \geqslant 3$. Clearly by (4) we get that $E_1(3/8)$ and $E_2(3/8)$ are independent of the choice of k. By Observation 4.2, we conclude that for $k \geqslant 3$ by (7)

$$f_\omega([0,1]_{2^k},\{0,1\})(3/8) \leqslant f_\omega([0,1]_{2^3},\{0,1\})(3/8) = \mathscr{P}^\infty h_A(3/8) \leqslant \min\{E_1(3/8),E_2(3/8)\}. \qquad \square$$

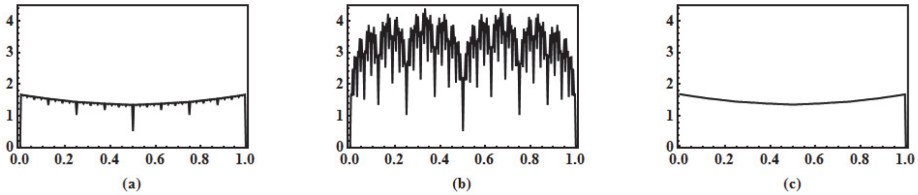

Figure 3: *Graph of comparison of three estimators: (a)* $\mathscr{P}^\infty$*, (b)* E_1*, (c)* E_2 *for* $\omega(x,y) = \sin(\cos(|x-y|))$ *on the set* $[0,1]_{256}$.

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