Mathematical
Inequalities
& Applications
Volume 16, Number 2 (2013), 477–481 doi:10.7153/mia-16-34

CORACH–PORTA–RECHT INEQUALITY FOR CLOSED RANGE OPERATORS

MARYAM KHOSRAVI

(*Communicated by I. Perić*)

Abstract. By $\mathbb{B}(\mathscr{H})$ we denote the space of all bounded linear operators on a Hilbert space $\mathscr{H}$. In 2001, Seddik characterized all invertible self-adjoint operators using the Corach-Porta-Recht inequality

$$\|SXS^{-1}+S^{-1}XS\| \geqslant 2\|X\|.$$

In this paper, we find a characterization of closed range self-adjoint operators using a version of this inequality for closed range operators.

1. Introduction and preliminaries

In [1], Corach, Porta and Recht proved that if S is a self-adjoint invertible operator on a Hilbert space $\mathscr{H}$, then for all $X \in \mathbb{B}(\mathscr{H})$ the following (C-P-R) inequality holds:

$$\|SXS^{-1}+S^{-1}XS\| \geqslant 2\|X\|.$$

They used this inequality as a key factor in their study of differential geometry. J. I. Fujii, M. Fujii, Furuta and Nakamoto [2], showed that this inequality is equivalent to Heinz inequality which is one of the most essential inequalities in operator theory.

Seddik [5] could find a characterization of non-zero scalars of invertible self-adjoint operators base on this inequality.

In this paper we discuss about a version of this inequality for Moore-Penrose invertible operators.

DEFINITION 1.1. Let $\mathscr{A}$ be an algebra with involution and $a \in \mathscr{A}$. If there exists an element $x \in \mathscr{A}$ satisfied the following four equations

$$\begin{aligned} axa &= a & xax &= x \\ (ax)^* &= ax & (xa)^* &= xa, \end{aligned} \tag{$*$}$$

then x is called a Moore-Penrose inverse of a and denoted by $a^\dagger$.

Mathematics subject classification (2010): Primary: 47A30, 15A09, secondary: 47B15.
Keywords and phrases: Moore-Penrose inverse, operator inequality, closed range operators, self-adjoint operators.

Paper MIA-16-34

It is easy to show that the Moore-Penrose inverse of an element a is unique.

If $a \in \mathscr{A}$ is Moore-Penrose invertible, then

(a) $a^{\dagger\dagger} = a$,

(b) a^* is Moore-Penrose invertible and $a^{*\dagger} = a^{\dagger *}$,

(c) If a is invertible, then $a^\dagger = a^{-1}$,

(d) $(aa^*)^\dagger = a^{*\dagger} a^\dagger$.

Harte and Mbekhta in [3] proved that if $\mathscr{H}$ is a Hilbert space and $T \in \mathbb{B}(\mathscr{H})$ then the following conditions are equivalent:

i) T has a generalized inverse (that is there exists an operator $S \in \mathbb{B}(\mathscr{H})$ for which $TST = T$ and $STS = S$).

ii) $\mathscr{R}(T)$ is closed.

iii) T has a Moore-Penrose inverse.

In this case $TT^\dagger$ is the projection on $\mathscr{R}(T)$ and $T^\dagger T$ is the projection on $\mathscr{R}(T^*)$.

In this note, we present a version of C-P-R inequality for Moore-Penrose invertible operators. In addition, for a close subspace $\mathscr{K}$ of Hilbert space $\mathscr{H}$, we give a characterization of all hermitian operators with range $\mathscr{K}$.

2. Main results

In [4], McIntosh proved that

$$\|A^*AX + XBB^*\| \geqslant 2\|AXB\|,$$

for all operators $A, X, B \in \mathbb{B}(\mathscr{H})$. Using this inequality we can state the following result.

THEOREM 2.1. *Let S be a hermitian operator on Hilbert space $\mathscr{H}$ such that $\mathscr{R}(S)$ is closed. Then*

$$\|SXS^\dagger + S^\dagger XS\| \geqslant 2\|PXP\|,$$

where $P = SS^\dagger$.

Proof. From McIntosh inequality and the relation $SS^\dagger = (SS^\dagger)^* = S^\dagger S$, we have

$$\|SXS^\dagger + S^\dagger XS\| = \|SS(S^\dagger XS^\dagger) + (S^\dagger XS^\dagger)SS\| \geqslant 2\|SS^\dagger XS^\dagger S\| = 2\|PXP\|. \qquad \square$$

In general, it is not true that $(ab)^\dagger = b^\dagger a^\dagger$. However, in polar decomposition of an operator we can deduced the next result.

LEMMA 2.2. *Let S be an operator with close range and $S = U|S|$ be the polar decomposition of S. Then*

$$S^{\dagger} = |S|^{\dagger}U^{*}, \qquad \& \qquad |S|^{\dagger} = S^{\dagger}U.$$

Proof. First note that as a result of polar decomposition, we have $\mathscr{R}(|S|) = \mathscr{R}(S^{*})$ and therefore is closed. Since the Moore-Penrose inverse is unique, the following relations lead to the first equation:

1. $S(|S|^{\dagger}U^{*})S = U|S|(|S|^{\dagger}U^{*})S = U|S||S|^{\dagger}|S| = U|S| = S$.
2. $(|S|^{\dagger}U^{*})S(|S|^{\dagger}U^{*}) = |S|^{\dagger}|S||S|^{\dagger}U^{*} = |S|^{\dagger}U^{*}$.
3. $S(|S|^{\dagger}U^{*}) = U|S||S|^{\dagger}U^{*}$ which is hermitian, because $|S||S|^{\dagger}$ is hermitian.
4. $(|S|^{\dagger}U^{*}S) = |S|^{\dagger}|S|$ which is hermitian.

The second equality is proved similarly. □

REMARK 2.3. By the previous lemma it is seen that

$$|S||S|^{\dagger} = |S|^{\dagger}|S| = S^{\dagger}U|S| = S^{\dagger}S.$$

In addition

$$|S|^{\dagger}(H) = |S|^{\dagger}|S|(H) = S^{\dagger}S(H) = S^{\dagger}(H) = S^{*}(H) = |S|(H).$$

So if $S = U|S|$ is the polar decomposition of S, then U is isometry on $|S|^{\dagger}(H)$.

Using Lemma 2.2, we can deduce the following version of Theorem 2.1, similarly to [2]:

THEOREM 2.4. *Let S,T be operators on Hilbert space $\mathscr{H}$ such that $\mathscr{R}(S)$ and $\mathscr{R}(T)$ are closed. Then*

$$\|S^{*}XT^{\dagger} + S^{\dagger}XT^{*}\| \geqslant 2\|PXQ\|,$$

where $P = SS^{\dagger}$ and $Q = T^{\dagger}T$.

Proof. First we proved the inequality for the case that $T = S$.
Let $S = U|S|$ be the polar decomposition of S. Then

$$\begin{aligned}
\|S^{*}XS^{\dagger} + S^{\dagger}XS^{*}\| &= \||S|U^{*}X|S|^{\dagger}U^{*} + |S|^{\dagger}U^{*}X|S|U^{*}\| \\
&= \|U(|S|^{\dagger}X^{*}U|S| + |S|X^{*}U|S|^{\dagger})\| \\
&= \||S|^{\dagger}X^{*}U|S| + |S|X^{*}U|S|^{\dagger}\| && \text{(By Remark 2.3)} \\
&\geqslant 2\||S||S|^{\dagger}X^{*}U|S||S|^{\dagger}\| \\
&= 2\||S||S|^{\dagger}U^{*}X|S||S|^{\dagger}\| \\
&= 2\|U^{*}SS^{\dagger}XS^{\dagger}S\| \\
&= 2\|SS^{\dagger}XS^{\dagger}S\| = 2\|PXQ\|. && (U^{*} \text{ is isometry on } S(H))
\end{aligned}$$

Now let S,T be two arbitrary operators for which $\mathscr{R}(S)$ and $\mathscr{R}(T)$ is closed. Using the previous part, for closed range operator $\begin{bmatrix} S & 0 \\ 0 & T \end{bmatrix}$ and all operators of the form $\begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix}$ on the Hilbert space $\mathscr{H} \oplus \mathscr{H}$, we have

$$\left\| \begin{bmatrix} S^* & 0 \\ 0 & T^* \end{bmatrix} \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix} \begin{bmatrix} S^\dagger & 0 \\ 0 & T^\dagger \end{bmatrix} + \begin{bmatrix} S^\dagger & 0 \\ 0 & T^\dagger \end{bmatrix} \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix} \begin{bmatrix} S^* & 0 \\ 0 & T^* \end{bmatrix} \right\|$$
$$\geqslant 2 \left\| \begin{bmatrix} SS^\dagger & 0 \\ 0 & TT^\dagger \end{bmatrix} \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix} \begin{bmatrix} S^\dagger S & 0 \\ 0 & T^\dagger T \end{bmatrix} \right\|,$$

That is

$$\|S^* X T^\dagger + S^\dagger X T^*\| \geqslant 2\|PXQ\|. \qquad \square$$

In [5], Seddik obtained the following characterization of the invertible operators which satisfy the C-P-R inequality for all $X \in \mathbb{B}(\mathscr{H})$.

THEOREM 2.5. [5] *The set of all invertible operators S, for which*

$$\forall X \in \mathscr{H}, \quad \|SXS^{-1} + S^{-1}XS\| \geqslant 2\|X\|.$$

is the set $\{\lambda M : \lambda \in \mathbb{C}^*,\ M$ is an invertible self-adjoint operator$\}$.

Now we prove a version of this theorem for Moore-Penrose invertible operators.

THEOREM 2.6. *Let $\mathscr{K}$ be a closed subspace of $\mathscr{H}$ and P be the projection on $\mathscr{K}$. If $S \in \mathbb{B}(\mathscr{H})$ with $\mathscr{R}(S) = \mathscr{R}(S^*) = \mathscr{K}$ and*

$$\forall X \in \mathbb{B}(\mathscr{H}), \quad \|SXS^\dagger + S^\dagger XS\| \geqslant 2\|PXP\|,$$

then $S = \lambda T$, for some non-zero $\lambda \in \mathbb{C}$ and some self-adjoint operator T with $\mathscr{R}(T) = \mathscr{K}$.

Proof. From the hypothesis, we can simply write

$$\|S(PXP)S^\dagger + S^\dagger(PXP)S\| = \|SXS^\dagger + S^\dagger XS\| \geqslant 2\|PXP\|,$$

So from Theorem 2.5, we have $S = \lambda T$ as operators on $\mathbb{B}(\mathscr{K})$. Since $S = 0$ on $\mathscr{K}^\perp$, we can get the result. $\square$

From Theorem 2.1 and 2.6, the following theorem is immediately follows.

THEOREM 2.7. *Let $\mathscr{K}$ be a closed subspace of $\mathscr{H}$ and P be the projection on $\mathscr{K}$. If $S \in \mathbb{B}(\mathscr{H})$ with $\mathscr{R}(S) = \mathscr{R}(S^*) = \mathscr{K}$, then the following conditions are equivalent:*

- $\forall X \in \mathbb{B}(\mathscr{H}), \quad \|SXS^\dagger + S^\dagger XS\| \geqslant 2\|PXP\|,$
- *$S = \lambda T$ for some self-adjoint operator T with $\mathscr{R}(T) = \mathscr{K}$, and non-zero scalar λ.*

Acknowledgment

The author would like to express her thanks to the referee for useful and heart-warming suggestions.

REFERENCES

[1] G. CORACH, R. PORTA AND L. RECHT, *An operator inequality*, Linear Algebra Appl. **142** (1990), 153–158.
[2] J. I. FUJII, M. FUJII, T. FURUTA AND R. NAKAMOTO, *Norm inequalities equivalent to Heinz inequality*, Proc. Amer. Math. Soc. **118** (1993), 827–830.
[3] R. HARTE AND M. MBEKHTA, *On generalized inverses in C^*-algebras*, Studia Mathematics (1992), no. 103, 71–77.
[4] A. MCINTOSH, *Heinz inequalities and perturbation of spectral families*, Macquarie Mathematical Reports, Macquarie Univ., 1979.
[5] A. SEDDIK, *Some results related to the Corach-Porta-Recht inequality*, Proc. Amer. Math. Soc. **129**, 10 (1987), 3009–3015.

(Received October 28, 2011)

Maryam Khosravi
Faculty of Mathematics and Computer
& Mahani Mathematical Research Center
Shahid Bahonar University of Kerman
Kerman, Iran
e-mail: khosravi_m@mail.uk.ac.ir