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UNIFORM BOUNDEDNESS OF CONDITIONAL EXPECTATION OPERATORS ON A BANACH FUNCTION SPACE

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Abstract. Let X be a Banach function space over a nonatomic probability space. The quasi-Banach space weak–X is defined in a natural way. We give some necessary and sufficient conditions on X for all the conditional expectation operators to be uniformly bounded operators from X into weak–X.

1. Introduction

Let (Ω,Σ,μ) be a *nonatomic* probability space, that is, a probability space such that there is no μ-atom in Σ. We let L_0 denote the real linear space of all measurable functions f on Ω such that $|f|<\infty$ a.e. As usual, we identify two functions in L_0 if they are equal a.e.

Given Banach spaces X and Y, we write $X \hookrightarrow Y$ to mean that X is continuously embedded in Y, that is, $X \subset Y$ and the inclusion map is continuous.

DEFINITION 1. A Banach space X of functions in L_0 is called a *Banach function space* if it satisfies the following conditions:

(B1) $L_\infty \hookrightarrow X \hookrightarrow L_1$.

(B2) If $|x| \leqslant |y|$ a.e. and $y \in X$, then $x \in X$ and $\|x\|_X \leqslant \|y\|_X$.

(B3) If $x_n \in X$ for all $n \in \mathbb{N}$, $0 \leqslant x_n \uparrow x$ a.e., and $\sup_{n\in\mathbb{N}} \|x_n\|_X < \infty$, then $x \in X$ and $\|x\|_X = \sup_{n\in\mathbb{N}} \|x_n\|_X$.

We adopt the convention that if $x \in L_0 \setminus X$, then $\|x\|_X = \infty$.

From (B2) it follows that $x \in X$ if and only if $|x| \in X$, and that the norms of x and $|x|$ are equal.

For example, Lebesgue spaces, Orlicz spaces, and Lorentz spaces are Banach function spaces. An important feature of these spaces is that the norm of a function depends only on the distribution of the function.

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DEFINITION 2. A Banach function space X is said to be *rearrangement-invariant* if it has the property that whenever x and y have the same distribution and $y \in X$, then $x \in X$ and $\|x\|_X = \|y\|_X$.

DEFINITION 3. Let X be a Banach function space. For $x \in L_0$, let

$$\|x\|_{\text{w-}X} = \sup_{0<\lambda<\infty} \lambda \left\| 1_{\{\omega\in\Omega:\, |x(\omega)|>\lambda\}} \right\|_X, \tag{1}$$

where 1_A denotes the indicator function of $A \in \Sigma$. The linear space weak–X, which is denoted by w-X, consists of all $x \in L_0$ such that $\|x\|_{\text{w-}X} < \infty$.

For example, w-$L_p = L_{p,\infty}$ for all $p \in [1,\infty]$ (see [15, Lemma 3.8, p. 191]).

Let X be a Banach function space. It is clear that $\|x\|_{\text{w-}X} = 0$ if and only if $x = 0$ a.e., and that $\|\alpha x\|_{\text{w-}X} = |\alpha|\,\|x\|_{\text{w-}X}$ for all $x \in$ w-X and all $\alpha \in \mathbb{R}$. Moreover,

$$\|x+y\|_{\text{w-}X} \leqslant 2\big(\|x\|_{\text{w-}X} + \|y\|_{\text{w-}X}\big)$$

for all $x, y \in$ w-X. Thus w-X is a quasi-normed space. In fact, w-X is a maximal quasi-Banach function space in the sense of [5]. In this paper, however, the structure of w-X as a quasi-Banach function space will not be discussed. It is clear that $X \subset$ w-X and

$$\|x\|_{\text{w-}X} \leqslant \|x\|_X$$

for all $x \in X$. It is also clear that

$$\|1_A\|_{\text{w-}X} = \|1_A\|_X$$

for all $A \in \Sigma$.

We let $\mathbb{E}[x]$ denote the expectation of $x \in L_1$ and $\mathbb{E}[x|\mathscr{A}]$ denote the conditional expectation of $x \in L_1$ given a sub-σ-algebra $\mathscr{A}$ of Σ. It is well known that for every sub-σ-algebra $\mathscr{A}$, the operator $\mathbb{E}[\,\cdot\,|\mathscr{A}]$ (restricted to L_p) is a linear contraction on L_p. However $\mathbb{E}[\,\cdot\,|\mathscr{A}]$ is not necessarily a bounded linear operator on a Banach function space X into itself. In fact, we know that there is a constant $C > 0$ such that

$$\big\|\mathbb{E}[x|\mathscr{A}]\big\|_X \leqslant C\,\|x\|_X$$

for all $x \in X$ and all sub-σ-algebras $\mathscr{A}$ if and only if X can be equivalently renormed so as to be rearrangement-invariant (see [6, Lemma 2] and [9, Proposition 1]). Of course, if this is the case, then

$$\big\|\mathbb{E}[x|\mathscr{A}]\big\|_{\text{w-}X} \leqslant C\,\|x\|_X, \tag{2}$$

for all $x \in X$ and all sub-σ-algebras $\mathscr{A}$.

The main result of this paper, Theorem 1, gives necessary and sufficient conditions on X for (2) to hold for all x and all $\mathscr{A}$, and Example 2 shows that a Banach function space X may not be equivalently renormed so as to be rearrangement-invariant even if (2) holds for all x and all $\mathscr{A}$. On the other hand, Theorem 2 shows that if X

is a suitable weighted Orlicz space and if (2) holds for all x and all $\mathscr{A}$, then X can be equivalently renormed so as to be rearrangement-invariant. In addition, Theorem 3 shows that (2) holds for all x and all $\mathscr{A}$ if and only if the inequality

$$\|\mathrm{M}f\|_{\text{w-}X} \leqslant C\|f_\infty\|_X$$

holds for all uniformly integrable martingales $f = (f_n)_{n\in\mathbb{Z}_+}$, where

$$\mathrm{M}f = \sup_{n\in\mathbb{Z}_+} |f_n| \quad \text{and} \quad f_\infty = \lim_{n\to\infty} f_n \text{ a.e.}$$

2. Preliminaries

Let X be a Banach function space. We denote by B_X the closed unit ball in X, and define X' to be the set of all $y \in L_0$ such that

$$\|y\|_{X'} := \sup\big\{\mathbb{E}\big[|xy|\big] : x \in B_X\big\} < \infty.$$

It is easily checked that X' forms a Banach function space; X' is called the *associate space* of X. For example, $(L_p)' = L_{p'}$ for all $p \in [1,\infty]$, where p' is the conjugate exponent of p. It is clear that if $x \in X$ and $y \in X'$, then $xy \in L_1$ and

$$\mathbb{E}\big[|xy|\big] \leqslant \|x\|_X\,\|y\|_{X'}. \tag{3}$$

We call this *Hölder's inequality*. If we let X'' denote the associate space of X', then $X = X''$ and $\|x\|_X = \|x\|_{X''}$ for all $x \in X$ (see [1, Theorem 2.7, p. 10]). In particular,

$$\|1_A\|_X = \|1_A\|_{X''} = \sup\big\{\mathbb{E}\big[|y|1_A\big] : y \in B_{X'}\big\} \tag{4}$$

for all $A \in \Sigma$.

Given $x \in L_0$, we define a function $x^*\colon [0,1] \to [0,\infty]$ by

$$x^*(t) = \inf\big\{\lambda > 0 : \mu\{\omega \in \Omega : |x(\omega)| > \lambda\} \leqslant t\big\}, \quad t \in [0,1],$$

with the convention that $\inf\emptyset = \infty$. Then x^* is the unique nonincreasing right-continuous function whose distribution (with respect to Lebesgue measure) is the same as that of $|x|$; x^* is called the *nonincreasing rearrangement* of x. Thus, nonnegative functions x and y in L_0 have the same distribution if and only if $x^* = y^*$ on $[0,1]$.

A function $\varphi\colon [0,1] \to [0,\infty)$ is said to be *quasi-concave* if it satisfies the following conditions:

(i) $\varphi(t) = 0$ if and only if $t = 0$.

(ii) $\varphi(t)$ is nondecreasing on $[0,1]$.

(iii) $t^{-1}\varphi(t)$ is nonincreasing on $(0,1]$.

Given a quasi-concave function φ, define $M(\varphi)$ to be the set of all $x \in L_0$ such that

$$\|x\|_{M(\varphi)} := \sup_{t\in(0,1]} \frac{\varphi(t)}{t} \int_0^t x^*(s)\,ds < \infty,$$

and define $M^*(\varphi)$ to be the set of all $x \in L_0$ such that

$$\|x\|_{M^*(\varphi)} := \sup_{t\in(0,1]} \varphi(t)x^*(t) < \infty.$$

Then $M(\varphi)$ is a rearrangement-invariant Banach function space, while $M^*(\varphi)$ is a rearrangement-invariant quasi-Banach function space (see [1, Proposition 5.8, p. 69]; see also [10, p. 114]). Both of these spaces are called *Marcinkiewicz spaces*. Clearly $M(\varphi) \subset M^*(\varphi)$ and

$$\|x\|_{M^*(\varphi)} \leqslant \|x\|_{M(\varphi)} \quad \text{for } x \in M(\varphi).$$

For example, if $1 < p < \infty$ and $\varphi_p(t) = t^{1/p}$, then $M(\varphi_p) = M^*(\varphi_p) = L_{p,\infty}$, and if $\varphi_1(t) = t$, then $M(\varphi_1) = L_1$ and $M^*(\varphi_1) = L_{1,\infty}$ (see [15, p. 191 and p. 204]; see also [12, p. 164]). Although Marcinkiewicz spaces are now classical spaces, they are still investigated as listed in [12, pp. 165–166].

Note that every quasi-concave function on $[0,1]$ is continuous on $(0,1]$ (see [10, p. 49]). From this fact it follows that

$$\|x\|_{M^*(\varphi)} = \sup_{t\in(0,1]} \varphi(t)x^*(t-), \tag{5}$$

where $x^*(t-) = \inf_{s<t} x^*(s)$.

Suppose that X is a rearrangement-invariant Banach function space. It is clear that if $A, B \in \Sigma$ and $\mu(A) = \mu(B)$, then $\|1_A\|_X = \|1_B\|_X$. Since (Ω, Σ, μ) is nonatomic, the range of μ is equal to $[0,1]$, and hence there is a unique function $\varphi_X\colon [0,1] \to [0,\infty)$ such that

$$\varphi_X(\mu(A)) = \|1_A\|_X$$

for all $A \in \Sigma$. We call φ_X the *fundamental function* of X. Note that φ_X is quasi-concave (see [1, Corollary 5.3, p. 67]). Note also that

$$\text{w-}X = M^*(\varphi_X) \quad \text{and} \quad \|x\|_{\text{w-}X} = \|x\|_{M^*(\varphi_X)} \quad \text{for } x \in \text{w-}X, \tag{6}$$

provided X is a rearrangement-invariant Banach function space (see Lemma 3 below).

It is easy to see that the fundamental function of $M(\varphi_X)$ is equal to φ_X. In fact, $M(\varphi_X)$ is the largest rearrangement-invariant Banach function space whose fundamental function is equal to that of X (see [1, Proposition 5.9, p. 70] or [14, Theorem 4]).

3. The main result

Let X be a Banach function space. We begin by defining two functions associated with X. Let

$$\Sigma(t) = \{A \in \Sigma\colon \mu(A) = t\}$$

for each $t \in [0,1]$, and define $\overline{\varphi}_X : [0,1] \to [0,\infty)$ and $\underline{\varphi}_X : [0,1] \to [0,\infty)$ by letting

$$\overline{\varphi}_X(t) = \sup\{\|1_A\|_X : A \in \Sigma(t)\} \quad \text{and} \quad \underline{\varphi}_X(t) = \inf\{\|1_A\|_X : A \in \Sigma(t)\}$$

for each $t \in [0,1]$. Then by Hölder's inequality (3), we have

$$t \leqslant \overline{\varphi}_X(t)\,\overline{\varphi}_{X'}(t) \tag{7}$$

for all $t \in [0,1]$. Moreover, if X is rearrangement-invariant, then both $\overline{\varphi}_X$ and $\underline{\varphi}_X$ coincide with φ_X, and the equality holds in (7) (see [1, Theorem 5.2, p. 66]).

The following lemma shows that we can associate to each Banach function space X the Marcinkiewicz space $M(\overline{\varphi}_X)$.

LEMMA 1. *If X is a Banach function space, then the function $\overline{\varphi}_X$ is quasi-concave.*

Proof. Obviously $\overline{\varphi}_X$ is nondecreasing on $[0,1]$, and $\overline{\varphi}_X(t) = 0$ if and only if $t = 0$. We need only show that $t^{-1}\overline{\varphi}_X(t)$ is nonincreasing on $(0,1]$. Since (Ω,Σ,μ) is nonatomic, we have

$$\int_0^t y^*(s)\,ds = \max\{\mathbb{E}[|y|1_A] : A \in \Sigma(t)\} \tag{8}$$

for all $y \in L_1$ and all $t \in [0,1]$ (see [3, (5.8)] or [15, Lemma 3.17, p. 201]). From (4) and (8), we obtain

$$\overline{\varphi}_X(t) = \sup\{\mathbb{E}[|y|1_A] : y \in B_{X'},\, A \in \Sigma(t)\} = \sup\left\{\int_0^t y^*(s)\,ds : y \in B_{X'}\right\}. \tag{9}$$

Since the function

$$(0,1] \ni t \longmapsto \frac{1}{t}\int_0^t y^*(s)\,ds \in \mathbb{R}$$

is nonincreasing, it follows from (9) that $t^{-1}\overline{\varphi}_X(t)$ is nonincreasing on $(0,1]$. This completes the proof. □

Let $\mathscr{A}$ be a sub-σ-algebra of Σ. If we let $Tx = \mathbb{E}[x|\mathscr{A}]$ for $x \in L_1$, then the linear operator T is a contraction on L_1 and the restriction of T to L_∞ is a contraction on L_∞. We call such a linear operator T an *L_1–L_∞-contraction.*

THEOREM 1. *Let X be a Banach function space. Then the following are equivalent:*

(i) *There is a constant $C > 0$ such that for all $x \in X$ and all L_1–L_∞-contractions T,*

$$\|Tx\|_{\text{w-}X} \leqslant C\,\|x\|_X.$$

(ii) *There is a constant $C > 0$ such that for all $x \in X$ and all sub-σ-algebras $\mathscr{A}$ of Σ,*

$$\big\|\mathbb{E}[x|\mathscr{A}]\big\|_{\text{w-}X} \leqslant C\,\|x\|_X. \tag{10}$$

(iii) *There is a constant* $C>0$ *such that for all* $t\in[0,1]$*,*

$$\overline{\varphi}_X(t)\,\overline{\varphi}_{X'}(t)\leqslant Ct. \tag{11}$$

(iv) $X\hookrightarrow M(\overline{\varphi}_X)$.

Moreover if X *satisfies these equivalent conditions, then* w-$X=M^*(\overline{\varphi}_X)$ *and there is a constant* $c>0$ *such that for all* $x\in$ w-X*,*

$$\|x\|_{\text{w-}X}\leqslant\|x\|_{M^*(\overline{\varphi}_X)}\leqslant c\,\|x\|_{\text{w-}X}.$$

Note that (11) can be rewritten as $\overline{\varphi}_{X'}(t)\,\overline{\varphi}_{X''}(t)\leqslant Ct$. It follows that (10) holds for all $x\in X$ and all sub-σ-algebras $\mathscr{A}$ if and only if the inequality

$$\left\|\mathbb{E}[x|\mathscr{A}]\right\|_{\text{w-}X'}\leqslant C\,\|x\|_{X'}$$

holds for all $x\in X'$ and all sub-σ-algebras $\mathscr{A}$.

If X can be equivalently renormed so as to be rearrangement-invariant, then the equivalent conditions of Theorem 1 hold; see [6, Lemma 2] or [1, Proposition 5.9, p. 70]. However, as Example 2 in Section 5 shows, the converse is not true in general.

For the proof of Theorem 1, we will need three lemmas.

LEMMA 2. *Let* X *be a Banach function space. Then* $M^*(\overline{\varphi}_X)\subset$ w-X *and*

$$\|x\|_{\text{w-}X}\leqslant\|x\|_{M^*(\overline{\varphi}_X)}$$

for all $x\in M^*(\overline{\varphi}_X)$.

Proof. Let $x\in M^*(\overline{\varphi}_X)$ and let $\lambda>0$. It suffices to show that

$$\lambda\left\|1_{\{\omega\in\Omega:\,|x(\omega)|>\lambda\}}\right\|_X\leqslant\|x\|_{M^*(\overline{\varphi}_X)}.$$

We may assume $\mu\{\omega\in\Omega:|x(\omega)|>\lambda\}>0$. Let

$$t_\lambda=\inf\{s\in[0,1]:x^*(s)\leqslant\lambda\}.$$

Then $t_\lambda>0$ and $[0,t_\lambda)=\{t\in[0,1]:x^*(t)>\lambda\}$; hence $\lambda\leqslant x^*(t_\lambda-)$. Since x^* and $|x|$ have the same distribution, we have $\mu\{\omega\in\Omega:|x(\omega)|>\lambda\}=t_\lambda$. Therefore

$$\lambda\left\|1_{\{\omega\in\Omega:\,|x(\omega)|>\lambda\}}\right\|_X\leqslant x^*(t_\lambda-)\,\overline{\varphi}_X(t_\lambda)\leqslant\|x\|_{\text{w-}X},$$

where we have used (5) with φ replaced by $\overline{\varphi}_X$. Thus the proof is complete. □

LEMMA 3. *Let* X *be a Banach function space. Suppose that* (ii) *of Theorem* 1 *holds. Then* X *satisfies the following conditions*:

(i) *There is a constant* $c>0$ *such that for all* $t\in[0,1]$*,*

$$\overline{\varphi}_X(t)\leqslant c\underline{\varphi}_X(t). \tag{12}$$

(ii) $\text{w-}X = M^*(\overline{\varphi}_X)$ *and there is a constant* $c > 0$ *such that for all* $x \in \text{w-}X$,

$$\|x\|_{\text{w-}X} \leqslant \|x\|_{M^*(\overline{\varphi}_X)} \leqslant c\,\|x\|_{\text{w-}X}\,. \tag{13}$$

Moreover conditions (i) *and* (ii) *are equivalent, and the constants in* (i) *and* (ii) *can be chosen to be the same.*

Proof. Let C be the constant in (10). We show that if $t \in (0,1]$ and $A, B \in \Sigma(t)$, then

$$\|1_B\|_X \leqslant (2C+1)\,\|1_A\|_X\,,$$

which implies (i). To this end, let $\mathscr{A}$ be the sub-σ-algebra generated by the single set $A \triangle B = (A\setminus B)\cup(B\setminus A)$. Then we have $1_{A\triangle B} = 2\mathbb{E}[1_{A\setminus B}|\mathscr{A}]$ a.e., and hence by (ii) of Theorem 1,

$$\|1_{B\setminus A}\|_X = \|1_{B\setminus A}\|_{\text{w-}X} \leqslant \left\|1_{A\triangle B}\right\|_{\text{w-}X} = 2\left\|\mathbb{E}[1_{A\setminus B}|\mathscr{A}]\right\|_{\text{w-}X} \leqslant 2C\,\|1_{A\setminus B}\|_X\,.$$

Therefore

$$\|1_B\|_X \leqslant \|1_{A\cap B}\|_X + \|1_{B\setminus A}\|_X \leqslant \|1_A\|_X + 2C\,\|1_{A\setminus B}\|_X \leqslant (1+2C)\,\|1_A\|_X\,.$$

as required.

We now show that (i) and (ii) are equivalent. Suppose first that (i) holds. The first inequality of (13) has already been established (Lemma 2). To prove the second inequality of (13), let $x \in \text{w-}X$, $t \in (0,1]$, and $\lambda < x^*(t)$. Then, since x^* and $|x|$ have the same distribution, we have $\mu\{\omega \in \Omega : |x(\omega)| > \lambda\} \geqslant t$. Hence by (12),

$$\lambda\overline{\varphi}_X(t) \leqslant c\lambda\underline{\varphi}_X(t) \leqslant c\lambda\left\|1_{\{\omega\in\Omega:\,|x(\omega)|>\lambda\}}\right\|_X \leqslant c\,\|x\|_{\text{w-}X}\,.$$

Letting $\lambda \uparrow x^*(t)$, we have $x^*(t)\,\overline{\varphi}_X(t) \leqslant c\,\|x\|_{\text{w-}X}$, which implies the second inequality of (13).

Suppose now that (ii) holds. Let $t \in (0,1]$ and let $\varepsilon > 0$. Then there exists $A \in \Sigma(t)$ such that $\underline{\varphi}_X(t) + \varepsilon > \|1_A\|_X$. From (13) it follows that

$$c\underline{\varphi}_X(t) + c\varepsilon > c\,\|1_A\|_X = c\,\|1_A\|_{\text{w-}X} \geqslant \|1_A\|_{M^*(\overline{\varphi}_X)} = \overline{\varphi}_X(t).$$

Letting $\varepsilon \downarrow 0$, we obtain (12). This completes the proof. $\square$

From Lemma 3 we see that if X is a rearrangement-invariant Banach function space, then (6) holds.

LEMMA 4. *If* T *be an* L_1–L_∞*-contraction, then for all* $x \in L_1$ *and all* $t \in [0,1]$,

$$\int_0^t (Tx)^*(s)\,ds \leqslant \int_0^t x^*(s)\,ds.$$

Proof. Recall from [1, Theorem 6.2, p. 74] that for all $x \in L_1$,

$$\int_0^t x^*(s)\,ds = \inf\big\{ \|y\|_{L_1} + t\,\|z\|_{L_\infty} : y \in L_1,\ z \in L_\infty,\ x = y + z \big\}.$$

The lemma immediately follows from this formula. □

We are now ready to prove Theorem 1.

Proof of Theorem 1. Since the operator $\mathbb{E}[\,\cdot\,|\mathscr{A}]$ is an $L_1 - L_\infty$-contraction, (i) implies (ii).

(ii)⇒(iii) Let $t \in [0,1]$, let $A \in \Sigma(t)$, and let $\mathscr{A}$ be the sub-σ-algebra generated by the single set A. Then for all $x \in B_X$, we have $\mathbb{E}\big[|x|1_A\big]1_A \leqslant t\,\mathbb{E}\big[|x|\big|\mathscr{A}\big]$ a.e. Hence by (ii),

$$\begin{aligned}\mathbb{E}\big[|x|1_A\big]\,\underline{\varphi}_X(t) \leqslant \mathbb{E}\big[|x|1_A\big]\,\|1_A\|_X &= \big\|\mathbb{E}\big[|x|1_A\big]1_A\big\|_{\text{w-}X} \\ &\leqslant t\,\big\|\mathbb{E}\big[|x|\big|\mathscr{A}\big]\big\|_{\text{w-}X} \leqslant Ct\,\|x\|_X \leqslant Ct.\end{aligned}$$

By the definition of the norm on X', we have $\|1_A\|_{X'}\,\underline{\varphi}_X(t) \leqslant Ct$, which implies

$$\overline{\varphi}_{X'}(t)\,\underline{\varphi}_X(t) \leqslant Ct.$$

From this inequality and (12), we conclude that

$$\overline{\varphi}_X(t)\,\overline{\varphi}_{X'}(t) \leqslant cCt,$$

Thus (iii) holds.

(iii)⇒(iv) Let $x \in X$ and $t \in (0,1]$. By (8) there exists $A \in \Sigma(t)$ such that

$$\int_0^t x^*(s)\,ds = \mathbb{E}\big[|x|1_A\big].$$

Hence, by Hölder's inequality (3) and (iii),

$$\frac{\overline{\varphi}_X(t)}{t}\int_0^t x^*(s)\,ds \leqslant \frac{\overline{\varphi}_X(t)}{t}\,\|1_A\|_{X'}\,\|x\|_X \leqslant \frac{\overline{\varphi}_X(t)\,\overline{\varphi}_{X'}(t)}{t}\,\|x\|_X \leqslant C\,\|x\|_X\,.$$

Since $t \in (0,1]$ is arbitrary, it follows that $x \in M(\overline{\varphi}_X)$ and

$$\|x\|_{M(\overline{\varphi}_X)} \leqslant C\,\|x\|_X\,.$$

Thus $X \hookrightarrow M(\overline{\varphi}_X)$, as required.

(iv)⇒(i) Let T be an $L_1 - L_\infty$-contraction and let $x \in X$. From (iv) and Lemma 4 we see that $Tx \in M(\overline{\varphi}_X)$ and

$$\|Tx\|_{M(\overline{\varphi}_X)} \leqslant \|x\|_{M(\overline{\varphi}_X)} \leqslant C\,\|x\|_X\,,$$

where C is a constant which is independent of $x \in X$. On the other hand, by Lemma 2, $Tx \in \text{w-}X$ and

$$\|Tx\|_{\text{w-}X} \leqslant \|Tx\|_{M^*(\overline{\varphi}_X)} \leqslant \|Tx\|_{M(\overline{\varphi}_X)}\,.$$

Thus $\|Tx\|_{\text{w-}X} \leqslant C\,\|x\|_X$, as required.

The last statement of Theorem 1 is an immediate consequence of Lemma 3. Thus the proof is complete. □

4. The case of a weighted Orlicz space

Let $\Phi\colon [0,\infty) \to [0,\infty]$ be a *Young function*, that is, a left-continuous convex function such that

$$\lim_{u\to 0+} \Phi(u) = \Phi(0) = 0 \quad \text{and} \quad \lim_{u\to\infty} \Phi(u) = \infty.$$

In what follows, we will assume for simplicity that Φ is strictly increasing. Note that Φ is strictly increasing if and only if $0 < \Phi(t) < \infty$ for all $t \in (0,\infty)$. We say that Φ satisfies the Δ_2*-condition* if there are constants $k > 0$ and $u_0 \geqslant 0$ such that

$$\Phi(2u) \leqslant k\Phi(u) < \infty \quad \text{for all } u \in [u_0,\infty). \tag{14}$$

Recall that the *complementary function* $\Psi\colon [0,\infty) \to [0,\infty)$ is defined by

$$\Psi(v) = \sup\{uv - \Phi(u) : 0 \leqslant u < \infty\}, \quad v \in [0,\infty).$$

Let $w \in L_0$ be a strictly positive function such that $\mathbb{E}[w] = 1$. We call such a w a *weight function*. We let μ_w denote the probability measure defined by

$$\mu_w(A) = \mathbb{E}[w1_A], \quad A \in \Sigma.$$

Recall that the *weighted Orlicz space* $L_{\Phi,w}$ is a Banach space consisting of all $x \in L_0$ such that $\mathbb{E}\big[\Phi(\lambda^{-1}|x|)w\big] < \infty$ for some $\lambda > 0$. Recall also that the norm of $x \in L_{\Phi,w}$ is given by

$$\|x\|_{\Phi,w} = \inf\{\lambda > 0 : \mathbb{E}\big[\Phi(\lambda^{-1}|x|)w\big] \leqslant 1\}.$$

If $w = 1$ a.e., we write L_Φ for $L_{\Phi,w}$, and $\|\cdot\|_{L_\Phi}$ for $\|\cdot\|_{L_{\Phi,w}}$.

Note that $L_{\Phi,w}$ is a rearrangement-invariant Banach function space over (Ω,Σ,μ_w) (see [1, Definition 8.10, p. 270]). In particular, (B2) and (B3) of Definition 1 hold with X replaced by $L_{\Phi,w}$. Moreover $L_\infty \hookrightarrow L_{\Phi,w}$. However $L_{\Phi,w}$ is not necessarily embedded in L_1; one can show that if Ψ is the complementary function of Φ and if $\mathbb{E}\big[\Psi(w^{-1})w\big] < \infty$, then $L_{\Phi,w} \hookrightarrow L_1$ and $L_{\Phi,w}$ is a Banach function space over (Ω,Σ,μ). See [8, Section 4] for details.

Although $L_{\Phi,w}$ may not be a Banach function space over (Ω,Σ,μ), we can define the quasi-norm $\|\cdot\|_{\text{w-}L_{\Phi,w}}$ by replacing X with $L_{\Phi,w}$ in (1).

THEOREM 2. *Let Φ be a strictly increasing Young function, let Ψ be its complementary function, and let w be a weight function. Suppose that Φ satisfies the Δ_2-condition. Then the following are equivalent*:

(i) *There is a constant $C > 0$ such that for all $x \in L_{\Phi,w}$ and all sub-σ-algebras $\mathscr{A}$ of Σ,*

$$\big\|\mathbb{E}[x|\mathscr{A}]\big\|_{\text{w-}L_{\Phi,w}} \leqslant C\|x\|_{L_{\Phi,w}}. \tag{15}$$

(ii) *There are constants a and b such that $0 < a \leqslant w \leqslant b$ a.e.*

(iii) *There is a constant* $C > 0$ *such that for all* $u \in (0,\infty)$ *and all sub-*σ*-algebras* $\mathscr{A}$ *of* Σ,

$$\Psi\left(\frac{\Phi(u)\mathbb{E}[w|\mathscr{A}]}{Cwu}\right)w \in L_1$$

and

$$\mathbb{E}\left[\Psi\left(\frac{\Phi(u)\mathbb{E}[w|\mathscr{A}]}{Cwu}\right)w \,\middle|\, \mathscr{A}\right] \leqslant \Phi(u)\mathbb{E}[w|\mathscr{A}] \ \ a.e.$$

(iv) $L_{\Phi,w}$ *is a Banach function space over* (Ω,Σ,μ) *and can be equivalently renormed so as to be rearrangement-invariant.*

Moreover, if $L_{\Phi,w}$ *satisfies these conditions, then* $\text{w-}L_{\Phi,w} = \text{w-}L_\Phi = M^*(\varphi_{L_\Phi})$ *and for all* $x \in \text{w-}L_{\Phi,w}$,

$$\min\{a,1\}\,\|x\|_{M^*(\varphi_{L_\Phi})} \leqslant \|x\|_{\text{w-}L_{\Phi,w}} \leqslant \max\{b,1\}\,\|x\|_{M^*(\varphi_{L_\Phi})}, \tag{16}$$

where a *and* b *are constants in* (ii).

For the proof of Theorem 2, we will need a lemma. Before stating it, we note that if $\mu(A) > 0$, then $\mu_w(A) > 0$ and

$$\|1_A\|_{L_{\Phi,w}}^{-1} = \Phi^{-1}\big(\mu_w(A)^{-1}\big),$$

where Φ^{-1} denotes the inverse of Φ (see [11, p. 58]).

LEMMA 5. *Let* Φ *and* w *be as in Theorem 2. Suppose that* Φ *satisfies the* Δ_2*-condition and that* (i) *of Theorem 2 holds. Then there are constants* $\delta > 0$ *and* $K > 0$ *such that if* $0 < \mu(A) = \mu(B) < \delta$, *then*

$$\mu_w(B) \leqslant K\mu_w(A). \tag{17}$$

Proof. Let $k > 0$ and $u_0 \geqslant 0$ be constants which satisfy (14). We choose $\delta > 0$ so that if $0 < \mu(B) < \delta$, then $\Phi(u_0) \leqslant \mu_w(B)^{-1}$. Suppose $0 < \mu(A) = \mu(B) < \delta$. Then, by arguing as in the proof of Lemma 3, we have

$$\|1_B\|_{L_{\Phi,w}} \leqslant (1+2C)\,\|1_A\|_{L_{\Phi,w}},$$

which implies

$$\Phi^{-1}\big(\mu_w(A)^{-1}\big) \leqslant (1+2C)\Phi^{-1}\big(\mu_w(B)^{-1}\big).$$

Choose $m \in \mathbb{N}$ so that $(1+2C) \leqslant 2^m$. Then

$$\mu_w(A)^{-1} \leqslant \Phi\big(2^m\Phi^{-1}\big(\mu_w(B)^{-1}\big)\big) \leqslant k^m\mu_w(B)^{-1},$$

where the second inequality follows from (14) and the fact that

$$u_0 \leqslant \Phi^{-1}\big(\mu_w(B)^{-1}\big).$$

Thus (17) holds with $K = k^m$. $\square$

Proof of Theorem 2. (i)$\Rightarrow$(ii) Suppose that (i) holds. Let $\delta > 0$ and $K > 0$ be as in Lemma 5, and let $0 < t < \delta$. By [3, (5.8)] there exist A and B in $\Sigma(t)$ such that

$$\mu_w(A) = \int_{1-t}^{1} w^*(s)\,ds \quad \text{and} \quad \mu_w(B) = \int_0^t w^*(s)\,ds.$$

From Lemma 5 it follows that

$$\frac{1}{t}\int_0^t w^*(s)\,ds \leqslant \frac{K}{t}\int_{1-t}^{1} w^*(s)\,ds.$$

Letting $t \downarrow 0$, we obtain

$$\operatorname*{ess\,sup}_{\Omega} w \leqslant K \operatorname*{ess\,inf}_{\Omega} w$$

(see [3, (9.6)]). Since $\mathbb{E}[w] = 1$, this implies (ii).

(ii)$\Rightarrow$(iv) Suppose that (ii) holds. Then it is clear that $L_{\Phi,w} = L_{\Phi}$. It suffices to show that the norms of these spaces are equivalent. To this end, suppose $x \in L_{\Phi}$. If we let $\beta = \max\{b,1\}$ and $c = \beta\,\|x\|_{L_\Phi}$, then

$$\begin{aligned}\mathbb{E}\big[\Phi\big(c^{-1}|x|\big)w\big] &\leqslant \mathbb{E}\big[\Phi\big(c^{-1}|x|\big)\beta\big]\\ &\leqslant \mathbb{E}\big[\Phi\big(c^{-1}\beta|x|\big)\big] = \mathbb{E}\big[\Phi\big(\|x\|_{L_\Phi}^{-1}|x|\big)\big] \leqslant 1,\end{aligned}$$

and hence $\|x\|_{L_{\Phi,w}} \leqslant c = \beta\,\|x\|_{L_\Phi}$.

Now suppose $x \in L_{\Phi,w}$. If we let $\alpha = \min\{a,1\}$ and $c = \alpha^{-1}\,\|x\|_{L_{\Phi,w}}$, then

$$\begin{aligned}\mathbb{E}\big[\Phi\big(c^{-1}|x|\big)\big] &\leqslant \mathbb{E}\big[\Phi\big(c^{-1}|x|\big)\alpha^{-1}w\big]\\ &\leqslant \mathbb{E}\big[\Phi\big(c^{-1}\alpha^{-1}|x|\big)w\big] = \mathbb{E}\big[\Phi\big(\|x\|_{L_{\Phi,w}}^{-1}|x|\big)w\big] \leqslant 1,\end{aligned}$$

and hence $\|x\|_{L_\Phi} \leqslant c = \alpha^{-1}\,\|x\|_{L_{\Phi,w}}$. Thus

$$\min\{a,1\}\,\|x\|_{L_\Phi} \leqslant \|x\|_{L_{\Phi,w}} \leqslant \max\{b,1\}\,\|x\|_{L_\Phi} \tag{18}$$

for all $x \in L_\Phi = L_{\Phi,w}$, and (iv) holds.

(iv)$\Rightarrow$(i) Suppose that (iv) holds. Then by [6, Lemma 2], the inequality

$$\big\|\mathbb{E}[x|\mathscr{A}]\big\|_{L_{\Phi,w}} \leqslant C\,\|x\|_{L_{\Phi,w}}$$

holds for all $x \in L_{\Phi,w}$ and all sub-σ-algebras $\mathscr{A}$. In particular, (15) holds for all x and all $\mathscr{A}$, as required.

(ii)$\Rightarrow$(iii) Suppose that (ii) holds and let $C = a^{-1}b$. Then by [2, Lemma 2.1][1],

$$\Psi\left(\frac{\Phi(u)\mathbb{E}[w|\mathscr{A}]}{Cwu}\right) \leqslant \Psi\left(\frac{\Phi(u)}{u}\right) \leqslant \Phi(u)$$

[1]In [2, Lemma 2.1], both Φ and Ψ are assumed to be N-functions; but this assumption is unnecessary.

for all $u \in (0,\infty)$. This implies (iii).

(iii)$\Rightarrow$(i) Suppose that (iii) holds, and let $x \in L_\infty$. Then by [7, Lemma 2][2],

$$\Phi\big(c^{-1}\mathbb{E}\big[|x|\big|\mathscr{A}\big]\big)\mathbb{E}[w|\mathscr{A}] \leqslant \mathbb{E}\big[\Phi(2Cc^{-1}|x|)w\big|\mathscr{A}\big] \text{ a.e.}$$

for any $c > 0$. Setting $c = 2C\|x\|_{L_{\Phi,w}}$ and taking the expectation of the both sides, we have

$$\mathbb{E}\big[\Phi\big(c^{-1}\mathbb{E}\big[|x|\big|\mathscr{A}\big]\big)w\big] \leqslant \mathbb{E}\big[\Phi\big(\|x\|_{L_{\Phi,w}}^{-1}|x|\big)w\big] \leqslant 1,$$

and therefore

$$\big\|\mathbb{E}[x|\mathscr{A}]\big\|_{L_{\Phi,w}} \leqslant \big\|\mathbb{E}\big[|x|\big|\mathscr{A}\big]\big\|_{L_{\Phi,w}} \leqslant c = 2C\|x\|_{L_{\Phi,w}}.$$

In particular, we have

$$\big\|\mathbb{E}[x|\mathscr{A}]\big\|_{\text{w-}L_{\Phi,w}} \leqslant 2C\|x\|_{L_{\Phi,w}}.$$

It is easy to check that this inequality also holds for $x \in L_{\Phi,w}\setminus L_\infty$. Thus (i) holds.

Finally we prove the last statement. Suppose again that (ii) holds. Then by (18) we have that

$$\min\{a,1\}\|x\|_{\text{w-}L_\Phi} \leqslant \|x\|_{\text{w-}L_{\Phi,w}} \leqslant \max\{b,1\}\|x\|_{\text{w-}L_\Phi} \tag{19}$$

for all $x \in \text{w-}L_\Phi = \text{w-}L_{\Phi,w}$. Furthermore, since L_Φ is rearrangement-invariant, (6) holds with X replaced by L_Φ. This together with (19) implies (16) and completes the proof. □

5. Examples

Let Φ, Ψ, and w be as in Theorem 2, and let $X = L_{\Phi,w}$. If $\mathbb{E}\big[\Psi(w^{-1})w\big] < \infty$ and if w is unbounded, then X is a Banach function space over (Ω,Σ,μ) for which the equivalent conditions of Theorem 1 do not hold. We first give an example of a Banach function space X which is not a weighted Orlicz space and for which the equivalent conditions of Theorem 1 do not hold.

EXAMPLE 1. Let X_1 and X_2 be a pair of rearrangement-invariant Banach function spaces such that

$$\lim_{t\to 0+}\frac{\varphi_{X_2}(t)}{\varphi_{X_1}(t)} = \infty. \tag{20}$$

For instance, if $X_1 = L_p$, $X_2 = L_q$, and $1 \leqslant p < q \leqslant \infty$, then this condition is satisfied.

Let $\{\Omega_1,\Omega_2\} \subset \Sigma$ be a partition of Ω such that $\mu(\Omega_1) = \mu(\Omega_2) = 2^{-1}$. Define X to be the set of all $x \in L_0$ such that

$$\|x\|_X := \big\|x1_{\Omega_1}\big\|_{X_1} + \big\|x1_{\Omega_2}\big\|_{X_2} < \infty.$$

[2] In [7, Lemma 2], Φ is assumed to be an N-function; but this assumption is unnecessary.

It is easily checked that X forms a Banach function space. We claim that (iv) of Theorem 1 does not hold. To see this, suppose for contradiction that $\|x\|_{M(\overline{\varphi}_X)} \leqslant C\|x\|_X$ for all $x \in X$, where C is a constant which is independent of x. Given $t \in (0, 2^{-1})$, choose $A, B \in \Sigma(t)$ so that $A \subset \Omega_1$ and $B \subset \Omega_2$. Then

$$\varphi_{X_2}(t) = \|1_B\|_X \leqslant \overline{\varphi}_X(t) = \|1_A\|_{M(\overline{\varphi}_X)} \leqslant C\|1_A\|_X = C\varphi_{X_1}(t).$$

Thus $\varphi_{X_2}(t)\varphi_{X_1}(t)^{-1} \leqslant C$ for all $t \in (0, 2^{-1})$. This contradicts (20), and thus (iv) of Theorem 1 does not hold, as claimed.

The next example shows that a Banach function space X may not be equivalently renormed so as to be rearrangement-invariant even if the equivalent conditions of Theorem 1 hold.

EXAMPLE 2. Let X_1 and X_2 be a pair of rearrangement-invariant Banach function spaces such that:

(i) X_2 is a proper subset of X_1.

(ii) There is a constant $k \geqslant 1$ such that

$$k^{-1}\varphi_{X_1}(t) \leqslant \varphi_{X_2}(t) \leqslant k\varphi_{X_1}(t)$$

for all $t \in [0, 1]$.

For instance, if $X_1 = L_{p,\infty}$, $X_2 = L_{p,1}$, and $1 < p < \infty$, then these conditions are satisfied.

Let $\{\Omega_1, \Omega_2\}$ and X be as in Example 1. We claim that the equivalent conditions of Theorem 1 hold, but that X cannot be equivalently renormed so as to be rearrangement-invariant. To show that $X \hookrightarrow M(\overline{\varphi}_X)$, it suffices to show that $X \subset M(\overline{\varphi}_X)$ (see [1, Theorem 1.8, p. 7]). Observe that if $t \in [0, 1]$ and $A \in \Sigma(t)$, then

$$k^{-1}\varphi_{X_1}(2^{-1}t) \leqslant \|1_A\|_X \leqslant (1+k)\varphi_{X_1}(t).$$

Since φ_{X_1} is quasi-concave, we have $2^{-1}\varphi_{X_1}(t) \leqslant \varphi_{X_1}(2^{-1}t)$ and hence

$$(2k)^{-1}\varphi_{X_1}(t) \leqslant \overline{\varphi}_X(t) \leqslant (1+k)\varphi_{X_1}(t)$$

for all $t \in [0, 1]$. Therefore $M(\overline{\varphi}_X) = M(\varphi_{X_1}) \supset X_1 \supset X$, as required.

To show that X cannot be equivalently renormed so as to be rearrangement-invariant, it suffices to show that there exist $x, y \in L_0$ such that $x \in X$ and $y \notin X$ though they have the same distribution. Choose $z \in X_1 \setminus X_2$ so that $z^* = 0$ on $[2^{-1}, 1]$. Since (Ω, Σ, μ) is nonatomic, there exist $x, y \in L_0$ such that

$$\{\omega \in \Omega : x(\omega) \neq 0\} \subset \Omega_1, \quad \{\omega \in \Omega : y(\omega) \neq 0\} \subset \Omega_2,$$

and

$$x^* = y^* = z^* \text{ on } [0, 1]$$

(see [3, (5.6)]). Since $z \in X_1 \setminus X_2$, it follows that $x \in X_1$ and $y \notin X_2$. Hence $x \in X$ and $y \notin X$ though they have the same distribution. This completes the proof of the claim.

The next example shows that (i) and (ii) of Theorem 2 are not necessarily equivalent when Φ does not satisfy the Δ_2-condition.

EXAMPLE 3. Let w be a weight function such that $w^*(t) = (3/4)\cdot t^{-1/4}$ for all $t \in (0,1]$. Of course, (ii) of Theorem 2 does not hold. Let Φ be the Young function defined by

$$\Phi(u) = e^u - u - 1, \quad u \in [0,\infty).$$

We claim that (i) and (iv) of Theorem 2 hold. To see this, it suffices to show that $L_{\Phi,w}$ and L_Φ coincide and the norms of these spaces are equivalent. To this end, suppose first that $x \in L_\Phi$. Let $a = \mathbb{E}\big[w^2\big] = 9/8$ and let $b = 2a\,\|x\|_{L_\Phi}$. Then, since $\Phi(u)^2 \leqslant \Phi(2u)$ for all $u \in [0,\infty)$,

$$\begin{aligned}
\mathbb{E}\big[\Phi\big(b^{-1}|x|\big)w\big] &\leqslant \mathbb{E}\big[w^2\big]^{1/2}\,\mathbb{E}\big[\Phi\big(b^{-1}|x|\big)^2\big]^{1/2} \\
&\leqslant \mathbb{E}\big[w^2\big]^{1/2}\,\mathbb{E}\big[\Phi\big(2b^{-1}|x|\big)\big]^{1/2} \\
&= \mathbb{E}\big[w^2\big]^{1/2}\,\mathbb{E}\big[\Phi\big(a^{-1}\,\|x\|_{L_\Phi}^{-1}\,|x|\big)\big]^{1/2} \\
&\leqslant \mathbb{E}\big[w^2\big]^{1/2}\,a^{-1/2}\,\mathbb{E}\big[\Phi\big(\|x\|_{L_\Phi}^{-1}\,|x|\big)\big]^{1/2} \\
&= \mathbb{E}\big[\Phi\big(\|x\|_{L_\Phi}^{-1}\,|x|\big)\big]^{1/2} \leqslant 1.
\end{aligned}$$

It follows that $x \in L_{\Phi,w}$ and $\|x\|_{L_{\Phi,w}} \leqslant b = 2a\,\|x\|_{L_\Phi}$. Suppose now that $x \in L_{\Phi,w}$. Let $a = \mathbb{E}\big[w^{-1}\big] = 16/15$ and let $b = 2a\,\|x\|_{L_{\Phi,w}}$. Then

$$\begin{aligned}
\mathbb{E}\big[\Phi\big(b^{-1}|x|\big)\big] &\leqslant \mathbb{E}\big[w^{-1}\big]^{1/2}\,\mathbb{E}\big[\Phi\big(b^{-1}|x|\big)^2 w\big]^{1/2} \\
&\leqslant \mathbb{E}\big[w^{-1}\big]^{1/2}\,\mathbb{E}\big[\Phi\big(2b^{-1}|x|\big)w\big]^{1/2} \\
&= \mathbb{E}\big[w^{-1}\big]^{1/2}\,\mathbb{E}\big[\Phi\big(a^{-1}\,\|x\|_{L_{\Phi,w}}^{-1}\,|x|\big)w\big]^{1/2} \\
&\leqslant \mathbb{E}\big[w^{-1}\big]^{1/2}\,a^{-1/2}\,\mathbb{E}\big[\Phi\big(\|x\|_{L_{\Phi,w}}^{-1}\,|x|\big)w\big]^{1/2} \\
&= \mathbb{E}\big[\Phi\big(\|x\|_{L_{\Phi,w}}^{-1}\,|x|\big)w\big]^{1/2} \leqslant 1.
\end{aligned}$$

It follows that $x \in L_\Phi$ and $\|x\|_{L_\Phi} \leqslant b = 2a\,\|x\|_{L_{\Phi,w}}$. Thus the proof of the claim is complete.

6. Application to martingale inequalities

In this section we discuss some maximal inequalities for martingales. For basic results and notions concerning martingales, we refer the reader to [4] or [13].

We denote by $\mathbb{M}$ the set of all martingales on Ω and by $\mathbb{M}_{\mathrm{u}}$ the set of all uniformly integrable martingales on Ω. For each $f = (f_n)_{n\in\mathbb{Z}_+} \in \mathbb{M}$, we let

$$\mathrm{M}f = \sup_{n\in\mathbb{Z}_+} |f_n|.$$

Recall that every martingale in $\mathbb{M}_u$ converges a.e. For each $f = (f_n)_{n\in\mathbb{Z}_+} \in \mathbb{M}_u$, we let

$$f_\infty = \lim_{n\to\infty} f_n \quad \text{a.e.}$$

It is well known (see [13, p. 150] or [4, p. 17]) that if $f = (f_n) \in \mathbb{M}_u$, then

$$\|\mathrm{M}f\|_{\text{w-}L_1} \leqslant \sup_{n\in\mathbb{Z}_+} \|f_n\|_{L_1} = \lim_{n\to\infty} \|f_n\|_{L_1} = \|f_\infty\|_{L_1}.$$

The following theorem is a generalization of this result.

THEOREM 3. *Let X be a Banach function space. Then the following are equivalent*:

(i) *There is a constant $C > 0$ such that for all $f = (f_n)_{n\in\mathbb{Z}_+} \in \mathbb{M}_u$,*

$$\|\mathrm{M}f\|_{\text{w-}X} \leqslant C\,\|f_\infty\|_X. \tag{21}$$

(ii) *There is a constant $C > 0$ such that for all $f = (f_n)_{n\in\mathbb{Z}_+} \in \mathbb{M}$,*

$$\|\mathrm{M}f\|_{\text{w-}X} \leqslant C \varliminf_{n\to\infty} \|f_n\|_X. \tag{22}$$

(iii) *There is a constant $C > 0$ such that for all $f = (f_n)_{n\in\mathbb{Z}_+} \in \mathbb{M}$,*

$$\|\mathrm{M}f\|_{\text{w-}X} \leqslant C \varlimsup_{n\to\infty} \|f_n\|_X.$$

(iv) *The equivalent conditions of Theorem 1 hold.*

Proof. (i)$\Rightarrow$(ii) Suppose that (i) holds. Let $f = (f_n) \in \mathbb{M}$ and $n \in \mathbb{Z}_+$. Since the stopped martingale $f^{\langle n\rangle} := (f_{n\wedge k})_{k\in\mathbb{Z}_+}$ is uniformly integrable, it follows that

$$\|\mathrm{M}_n f\|_{\text{w-}X} = \|\mathrm{M}f^{\langle n\rangle}\|_{\text{w-}X} \leqslant C\,\|f_n\|_X,$$

where $\mathrm{M}_n f = \max_{k\leqslant n} |f_k|$. This implies (22).

(ii)$\Rightarrow$(iii) Obvious.

(iii)$\Rightarrow$(iv) Suppose that (iii) holds. Let $x \in X$ and let $\mathscr{A}$ be a sub-σ-algebra of Σ. Define $f = (f_n) \in \mathbb{M}$ by

$$f_n = \begin{cases} \mathbb{E}[x|\mathscr{A}] & \text{if } n = 0, \\ x & \text{if } n \geqslant 1. \end{cases}$$

Then by (iii),

$$\big\|\mathbb{E}[x|\mathscr{A}]\big\|_{\text{w-}X} \leqslant \|\mathrm{M}f\|_{\text{w-}X} \leqslant C \varlimsup_{n\to\infty} \|f_n\|_X = C\,\|x\|_X.$$

Thus (ii) of Theorem 1 holds.

(iv)$\Rightarrow$(i) Suppose that (ii) of Theorem 1 holds. Let $f = (f_n) \in \mathbb{M}_{\mathrm{u}}$ and define a filtration $\mathscr{F} = (\mathscr{F}_n)$ by $\mathscr{F}_n = \sigma\{f_0, f_1, \ldots, f_n\}$, $n \in \mathbb{Z}_+$. Given $\lambda > 0$, let τ be the $\mathscr{F}$-stopping time defined by

$$\tau(\omega) = \min\{n \in \mathbb{Z}_+ : |f_n(\omega)| > \lambda\}$$

with the convention that $\min \emptyset = \infty$. Then

$$\{\omega \in \Omega : (\mathrm{M}f)(\omega) > \lambda\} = \{\omega \in \Omega : \tau(\omega) < \infty\}$$

and

$$\lambda 1_{\{\omega \in \Omega : (\mathrm{M}f)(\omega) > \lambda\}} \leqslant |f_\tau| 1_{\{\omega \in \Omega : \tau(\omega) < \infty\}} \leqslant \mathbb{E}\big[|f_\infty| \big| \mathscr{F}_\tau\big] \text{ a.e.}$$

Therefore

$$\begin{aligned} \lambda \left\| 1_{\{\omega \in \Omega : (\mathrm{M}f)(\omega) > \lambda\}} \right\|_X &= \lambda \left\| 1_{\{\omega \in \Omega : (\mathrm{M}f)(\omega) > \lambda\}} \right\|_{\text{w-}X} \\ &\leqslant \left\| \mathbb{E}\big[|f_\infty| \big| \mathscr{F}_\tau\big] \right\|_{\text{w-}X} \leqslant C \|f_\infty\|_X . \end{aligned}$$

Thus (21) holds, as required. □

REMARK 1. In Theorem 3, $\mathbb{M}$ and $\mathbb{M}_{\mathrm{u}}$ can be replaced by the set of all nonnegative submartingales and the set of all uniformly integrable nonnegative submartingales, respectively.

We conclude with the following theorem.

THEOREM 4. *Let Φ be a strictly increasing Young function and let w be a weight function. Suppose that Φ satisfies the Δ_2-condition. Then the following are equivalent*:

(i) *There is a constant $C > 0$ such that for all $f = (f_n)_{n \in \mathbb{Z}_+} \in \mathbb{M}_{\mathrm{u}}$,*

$$\|\mathrm{M}f\|_{\text{w-}L_{\Phi,w}} \leqslant C \|f_\infty\|_{L_{\Phi,w}} .$$

(ii) *There is a constant $C > 0$ such that for all $f = (f_n)_{n \in \mathbb{Z}_+} \in \mathbb{M}$,*

$$\|\mathrm{M}f\|_{\text{w-}L_{\Phi,w}} \leqslant C \varliminf_{n \to \infty} \|f_n\|_{L_{\Phi,w}} .$$

(iii) *There is a constant $C > 0$ such that for all $f = (f_n)_{n \in \mathbb{Z}_+} \in \mathbb{M}$,*

$$\|\mathrm{M}f\|_{\text{w-}L_{\Phi,w}} \leqslant C \varlimsup_{n \to \infty} \|f_n\|_{L_{\Phi,w}} .$$

(iv) *The equivalent conditions of Theorem 2 hold.*

Proof. The argument of Theorem 3 applies with X replaced by $L_{\Phi,w}$. □

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