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INEQUALITIES INVOLVING LOGARITHMIC MEAN OF ARBITRARY ORDER

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Abstract. Inequalities involving logarithmic mean of arbitrary order are obtained. These results are derived from the Lazarević and Causa-Huygens inequalities for hyperbolic functions. Some inequalities for the weighted sums of powers are also utilized. In particular cases obtained results simplify to known inequalities for the logarithmic mean of a low order.

1. Introduction

The history of mean values is long and laden with detail. Among means of two variables the logarithmic mean has attracted attention of several researchers. A two-parameter generalizations of the logarithmic mean have been introduced by K. B. Stolarsky (see [15]). A particular case of Stolarsky mean is called the logarithmic mean of arbitrary order (see (2.1)). The goal of this note is to establish new inequalities satisfied by the latter mean. Some known inequalities involving logarithmic mean of order one are special cases of the main results established in this paper. In Section 2 we give definitions of bivariate means used in the sequel. Also, some known inequalities involving hyperbolic functions are included in this section. The main results of this note are established in Section 3.

2. Definitions and preliminaries

Throughout the sequel we will assume that x and y are positive and unequal numbers. We begin this section with definitions of certain bivariate means used in the sequel. The logarithmic mean of order $t \in \mathbb{R}$ of x and y, denoted by $L_t(x,y) \equiv L_t$, is defined as follows [11]:

$$L_t(x,y) = \begin{cases} L(x^t,y^t)^{\frac{1}{t}} & \text{if } t \neq 0, \\ G(x,y) & \text{if } t = 0, \end{cases} \tag{2.1}$$

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where $L(x,y) \equiv L = \dfrac{x-y}{\ln x - \ln y}$ is the logarithmic mean of order one and $G(x,y) \equiv G = \sqrt{xy}$ is the geometric mean of x and y. Another mean used in this paper is the power mean $A_t(x,y) \equiv A_t$ of order $t \in \mathbb{R}$:

$$A_t(x,y) = \begin{cases} \left(\dfrac{x^t+y^t}{2}\right)^{\frac{1}{t}} & \text{if } t \neq 0, \\ G(x,y) & \text{if } t = 0. \end{cases} \tag{2.2}$$

It is worth mentioning that all means defined above belong to a two-parameter family of mens introduced by K.B. Stolarsky in [15]. These means have been studied by several researchers. See, e.g., [10], [6] and the references therein.

The key inequality used in this paper is the following one

$$(\cosh x)^{1/3} < \frac{\sinh x}{x} < \frac{2+\cosh x}{3} \tag{2.3}$$

$(x \neq 0)$. First inequality in (2.3) is due to Lazarević [2] while the second one is commonly referred to as the Cusa-Huygens inequality for hyperbolic functions. Inequalities (2.3) are special cases of inequalities established in [9].

For later use let us recall a result which has been established in [5] (see Theorem 3.2).

THEOREM 2.1. *Let u, v, γ and δ be positive numbers which satisfy the following conditions*

(i) $min(u,v) < 1 < max(u,v)$,

(ii) $1 < u^\gamma v^\delta$,

(iii) $\gamma + \delta < \gamma\frac{1}{u} + \delta\frac{1}{v}$.

Then the following inequality

$$2 < \left(\frac{1}{u}\right)^{\gamma p} + \left(\frac{1}{v}\right)^{\delta p} < u^{\gamma p} + v^{\delta p}. \tag{2.4}$$

holds true provided $\gamma \geqslant 1$, $\delta \geqslant 1$, and $p \geqslant 1$. Second inequality in (2.4) is valid if $p > 0$.

We will also utilize the following result (see [5], Theorem 3.1).

THEOREM 2.2. *Assume that the numbers u, v, γ and δ satisfy assumptions of Theorem 2.1. Further, let α and β be positive numbers and assume that $v < 1 < u$. Then*

$$\alpha + \beta < \alpha u^p + \beta v^q \tag{2.5}$$

if either

$$p > 0 \qquad \text{and} \qquad q \leqslant p\frac{\delta\alpha}{\gamma\beta}, \tag{2.6}$$

or if

$$q \leqslant p \leqslant -1 \qquad \text{and} \qquad \delta\alpha \leqslant \gamma\beta. \tag{2.7}$$

Conditions of validity of (2.5) when $u < 1 < v$ are also obtained in [5]. We omit further details.

3. Main results

In this section we shall establish inequalities involving logarithmic mean L_t. For the later use let us introduce a variable $\lambda = (t/2)\ln(x/y)$ $(t \in \mathbb{R})$. One can easily verify, using (2.1) - (2.2), that

$$A\left(e^{\lambda}, e^{-\lambda}\right) = \cosh\lambda = \left(\frac{A_t}{G}\right)^t \tag{3.1}$$

and

$$L\left(e^{\lambda}, e^{-\lambda}\right) = \frac{\sinh\lambda}{\lambda} = \left(\frac{L_t}{G}\right)^t. \tag{3.2}$$

This implies that

$$\frac{\tanh\lambda}{\lambda} = \left(\frac{L_t}{A_t}\right)^t \tag{3.3}$$

Our first result reads as follows.

THEOREM 3.1. *Let x and y be positive and unequal numbers, let $t \neq 0$, and let $p \geqslant 1$. Then*

$$2 < \left(\frac{G}{L_t}\right)^{2pt} + \left(\frac{A_t}{L_t}\right)^{pt} < \left(\frac{L_t}{G}\right)^{2pt} + \left(\frac{L_t}{A_t}\right)^{pt}. \tag{3.4}$$

Second inequality in (3.4) holds true for $p > 0$.

Proof. We shall prove the assertion using Theorem 2.1 with

$$u = \frac{\sinh z}{z}, v = \frac{\tanh z}{z}, \gamma = 2, \delta = 1.$$

It is well known that $v < 1 < u$ holds for all $z \neq 0$. Moreover, the first inequality in (2.3) can be written as $1 < u^2 v$ while the second one is the same as $3 < 2\frac{1}{u} + \frac{1}{v}$. Letting $z = \lambda$, where λ is the same as above, we obtain

$$2 < \left(\frac{\lambda}{\sinh\lambda}\right)^{2pt} + \left(\frac{\lambda}{\tanh\lambda}\right)^{pt} < \left(\frac{\sinh\lambda}{\lambda}\right)^{2pt} + \left(\frac{\tanh\lambda}{\lambda}\right)^{pt}.$$

Application of (3.2) and (3.3) completes the proof. $\square$

Particular cases of inequality (3.4) have been obtained in [4].

COROLLARY 3.2. *The following inequalities*

$$\frac{2L}{L+G} < \frac{A_{1/2}}{L} < \frac{L^2}{GA_{1/2}} < \frac{L+G}{2G} \tag{3.5}$$

hold true.

Proof. We utilize the first two members of (3.4) with $p = 1$ and $t = \frac{1}{2}$ and next apply $L_{1/2} = L^2/A_{1/2}$, to obtain

$$2 < \frac{A_{1/2}G}{L^2} + \frac{A_{1/2}}{L}. \tag{3.6}$$

Multiplying both sides of (3.6) by $L/(L+G)$ we obtain the first inequality in (3.5). The second inequality in (3.5) is equivalent to $A_{1/2}^2 G < L^3$ (see [13] and [7]), while the third one is equivalent to the first inequality in (3.5). The proof is complete. □

The first inequality in (3.5) has been established in [12].

A generalization of the inequality which connects first and third members of (3.4) reads as follows.

THEOREM 3.3. *Let $x > 0$, $y > 0$ ($x \neq y$), and let $t \neq 0$. Further, let $\alpha > 0$ and $\beta > 0$. Then*

$$\alpha + \beta < \alpha \left(\frac{L_t}{G}\right)^{pt} + \beta \left(\frac{L_t}{A_t}\right)^{qt} \tag{3.7}$$

if either

$$p > 0 \qquad and \qquad q \leqslant p\frac{\alpha}{2\beta}, \tag{3.8}$$

or if

$$q \leqslant p \leqslant -1 \qquad and \qquad \alpha \leqslant 2\beta. \tag{3.9}$$

Proof. We shall prove this result using Theorem 2.2 with

$$u = \frac{\sinh z}{z}, v = \frac{\tanh z}{z}, \gamma = 2, \delta = 1.$$

As pointed out in the proof of Theorem 3.1 that they satisfy conditions (i) - (iii). Letting $z = \lambda$, where λ is the same as in the proof of Theorem 3.1, we conclude, using inequality (2.5), that

$$\alpha + \beta < \alpha \left(\frac{\sinh \lambda}{\lambda}\right)^p + \beta \left(\frac{\tanh \lambda}{\lambda}\right)^q.$$

Making use of (3.2) and (3.3) we obtain the desired result. This completes the proof. □

To this end we will assume that $\alpha > 0$ and $\beta > 0$. Several inequalities can be derived from (3.7). For the sake of presentation we define the weights

$$w_1 = \alpha/(\alpha+\beta) \qquad \text{and} \qquad w_2 = \beta/(\alpha+\beta).$$

Clearly $w_1 + w_2 = 1$. We shall now prove the following.

COROLLARY 3.4. *Let $t \neq 0$. If $\alpha \leqslant 2\beta$, then*

$$L_t^t < w_1 G^t + w_2 A_t^t. \tag{3.10}$$

Also, if $\alpha \geqslant 2\beta$, then

$$L_t^{-t} < w_1 G^{-t} + w_2 A_t^{-t}. \tag{3.11}$$

Proof. In order to establish (3.10) it suffices to use Theorem 3.3 with $p = q = -1$. Similarly, (3.11) can be obtained using Theorem 3.3 with $p = q = 1$. This completes the proof. □

Letting in (3.10) $t = 1$ and $t = 1/2$ we obtain, respectively,

$$L < w_1 G + w_2 A$$

and

$$L < w_1 (A_{1/2} G)^{1/2} + w_2 A_{1/2}$$

provided $\alpha \leqslant 2\beta$. The last two inequalities are known in mathematical literature in the case when $\alpha = 2$ and $\beta = 1$ (see [1], [7], and [14]). Similarly, letting in (3.11) $t = -1$ and $t = -1/2$ we obtain, respectively,

$$L^{-1} < w_1 G^{-1} + w_2 A^{-1}$$

and

$$L^{-1} < w_1 (A_{1/2} G)^{-1/2} + w_2 A_{1/2}^{-1}$$

provided $\alpha \geqslant 2\beta$. For more inequalities involving L^{-1}, the interested reader is referred to [8].

Inequalities for the extended logarithmic mean E_t, where $E_t^{t-1} = L_t^t / L$ have been derived in [3]. They can be used to obtain more inequalities for the mean discussed in this paper.

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