

APPROXIMATE HERMITE—HADAMARD TYPE INEQUALITIES FOR APPROXIMATELY CONVEX FUNCTIONS

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(Communicated by C. P. Niculescu)

Abstract. In this paper, approximate lower and upper Hermite–Hadamard type inequalities are obtained for functions that are approximately convex with respect to a given Chebyshev system.

1. Introduction

Throughout this paper \mathbb{R} , \mathbb{R}_+ , \mathbb{N} and \mathbb{Z} denote the sets of real, nonnegative real, natural and integer numbers respectively. Given a nonempty open real interval I , denote by $\Delta(I)$ and $\Delta^\circ(I)$ the sets

$$\{(x, y) \in I \times I \mid x \leq y\} \quad \text{and} \quad \{(x, y) \in I \times I \mid x < y\},$$

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respectively. We say that a pair (ω_0, ω_1) is a *Chebyshev system* over I , if $\omega_0, \omega_1 : I \rightarrow \mathbb{R}$ are continuous functions and

$$\Omega(x, y) := \left| \frac{\omega_0(x) \omega_0(y)}{\omega_1(x) \omega_1(y)} \right| > 0 \quad ((x, y) \in \Delta^\circ(I)). \quad (1.1)$$

One can easily see, that if ω_0 is a positive function, then (1.1) holds if and only if ω_1/ω_0 is strictly increasing on I . In this latter case, (ω_0, ω_1) will be called a *positive Chebyshev system* over I . On the other hand, we can always assume that ω_0 is a positive function, because for every Chebyshev system (ω_0, ω_1) , there exists $\alpha, \beta \in \mathbb{R}$ such that $\alpha\omega_0 + \beta\omega_1 > 0$ (cf. [1], [2]). In the sequel, for fixed $x, y \in I$, the partial

Mathematics subject classification (2010): Primary 39B22, 39B12.

Keywords and phrases: convexity, approximate convexity, lower and upper Hermite–Hadamard inequalities.

This research has been supported by the Hungarian Scientific Research Fund (OTKA) Grant NK81402 and by the TÁMOP-4.2.1/B-09/1/KONV-2010-0007, TÁMOP-4.2.2/B-10/1-2010-0024 projects. These projects are co-financed by the European Union and the European Social Fund.

functions $u \mapsto \Omega(u, y)$ and $u \mapsto \Omega(x, u)$ will be denoted by $\Omega(\cdot, y)$ and $\Omega(x, \cdot)$, respectively. An important property of Chebyshev systems is that for every two pairs $(x, \xi), (y, \eta) \in I \times \mathbb{R}$ with $x \neq y$ the function ω defined as

$$\omega := \xi \frac{\Omega(\cdot, y)}{\Omega(x, y)} + \eta \frac{\Omega(x, \cdot)}{\Omega(x, y)}$$

is the unique linear combination of ω_0 and ω_1 such that $\omega(x) = \xi$ and $\omega(y) = \eta$ hold.

Given a positive Chebyshev system (ω_0, ω_1) over I and a proper subinterval J of I , a function $f : J \rightarrow \mathbb{R}$ is called (ω_0, ω_1) -convex on J if, for all $x < u < y$ from J ,

$$\begin{vmatrix} f(x) & f(u) & f(y) \\ \omega_0(x) & \omega_0(u) & \omega_0(y) \\ \omega_1(x) & \omega_1(u) & \omega_1(y) \end{vmatrix} \geq 0, \tag{1.2}$$

or equivalently,

$$f(u) \leq \frac{\Omega(u, y)}{\Omega(x, y)} f(x) + \frac{\Omega(x, u)}{\Omega(x, y)} f(y). \tag{1.3}$$

If (1.2) holds with strict inequality sign “>”, then f is said to be *strictly* (ω_0, ω_1) -convex on J .

The integral average of any standard convex function $f : I \rightarrow \mathbb{R}$ can be estimated from the midpoint and the endpoints of the domain as follows:

$$f\left(\frac{x+y}{2}\right) \leq \int_0^1 f(tx + (1-t)y) dt \leq \frac{f(x) + f(y)}{2} \quad (x, y \in I). \tag{1.4}$$

This is the well known Hermite–Hadamard type inequality. The above implication was discovered by Hadamard [5]. (See also [10], [7], and [12], [4], [11], [12], [13] for a historical account.) In [3] and [1], the authors established the following connections between (ω_0, ω_1) -convexity and Hermite–Hadamard type inequality.

THEOREM A. *Let (ω_0, ω_1) be a positive Chebyshev system on an open interval I and let $\rho : I \rightarrow \mathbb{R}$ be a positive integrable function. Define, for all elements $x < y$ of I , the functions $\xi(x, y)$, $c(x, y)$, $c_1(x, y)$ and $c_2(x, y)$ by the formulas*

$$\begin{aligned} \xi(x, y) &= \left(\frac{\omega_1}{\omega_0}\right)^{-1} \left(\frac{\int_x^y \omega_1 \rho}{\int_x^y \omega_0 \rho}\right) \quad \text{and} \quad c(x, y) = \frac{\int_x^y \omega_0 \rho}{\omega_0(\xi(x, y))}, \\ c_1(x, y) &= \frac{1}{\Omega(x, y)} \left| \begin{vmatrix} \int_x^y \omega_0 \rho & \omega_0(y) \\ \int_x^y \omega_1 \rho & \omega_1(y) \end{vmatrix} \right| \quad \text{and} \quad c_2(x, y) = \frac{1}{\Omega(x, y)} \left| \begin{vmatrix} \omega_0(x) & \int_x^y \omega_0 \rho \\ \omega_0(y) & \int_x^y \omega_1 \rho \end{vmatrix} \right|. \end{aligned} \tag{1.5}$$

If a function $f : I \rightarrow \mathbb{R}$ is (ω_0, ω_1) -convex, then for all elements $x < y$ of I , it satisfies the inequality

$$c(x, y)f(\xi(x, y)) \leq \int_x^y f \rho \leq c_1(x, y)f(x) + c_2(x, y)f(y). \tag{1.6}$$

In Theorem 2.2 and Theorem 3.12 below, these results will be generalized to the context of approximate (ω_0, ω_1) -convex, i.e., to the case when f satisfies an inequality analogous to (1.3) whose right hand side involves also an error term.

Let X be a real linear space and $D \subset X$ be a convex subset. In order to describe the old and new results about the connection of approximate Jensen convexity and the approximate Hermite–Hadamard inequality with variable error terms, we need to introduce the following terminology.

For a function $f : D \rightarrow \mathbb{R}$, we say that f is *hemi- P* , if, for all $x, y \in D$, the mapping

$$t \mapsto f((1-t)x + ty) \quad (t \in [0, 1]) \tag{1.7}$$

has property P . For example f is *hemiintegrable*, if for all $x, y \in D$ the mapping defined by (1.7) is integrable. Analogously, we say that a function $h : (D - D) \rightarrow \mathbb{R}$ is *radially- P* , if for all $u \in D - D$, the mapping

$$t \mapsto h(tu) \quad (t \in [0, 1])$$

has property P on $[0, 1]$.

In [6], Háyzy and second author of this paper established a connection between an approximate lower Hermite–Hadamard type inequality and an approximate Jensen type inequality by proving the following result.

THEOREM B. *Let $\alpha : (D - D) \rightarrow \mathbb{R}_+$ be a nonnegative radially Lebesgue integrable even function. Assume that $f : D \rightarrow \mathbb{R}$ is hemi-Lebesgue integrable and approximately Jensen convex in the sense of*

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2} + \alpha(x-y) \quad (x, y \in D). \tag{1.8}$$

Then f also satisfies the approximate lower Hermite–Hadamard inequality

$$f\left(\frac{x+y}{2}\right) \leq \int_0^1 f(tx + (1-t)y)dt + \int_0^1 \alpha(t(x-y))dt \quad (x, y \in D). \tag{1.9}$$

In [8] (cf. [14], [15]) the authors established the connections between an approximate upper Hermite–Hadamard type inequality and an approximate Jensen type inequality as stated in the following theorem.

THEOREM C. *Let $\alpha : (D - D) \rightarrow \mathbb{R}_+$ be a nonnegative radially Lebesgue integrable even function and $\rho : [0, 1] \rightarrow \mathbb{R}_+$ be a nonnegative Lebesgue integrable function with $\int_0^1 \rho = 1$. Assume that $f : D \rightarrow \mathbb{R}$ is hemiintegrable on D and satisfies the approximate Jensen inequality (1.8). Then, for $x, y \in D$, f also satisfies the approximate upper Hermite–Hadamard inequality*

$$\begin{aligned} \int_0^1 f(tx + (1-t)y)\rho(t)dt &\leq \lambda f(x) + (1-\lambda)f(y) \\ &+ \sum_{n=0}^{\infty} \frac{1}{2^n} \int_0^1 \alpha(2d_{\mathbb{Z}}(2^n t)(x-y))\rho(t)dt, \end{aligned} \tag{1.10}$$

where $\lambda := \int_0^1 t\rho(t)dt$ and, for $s \in \mathbb{R}$, $d_{\mathbb{Z}}(s) := \text{dist}(s, \mathbb{Z}) = \inf\{|s-k| : k \in \mathbb{Z}\}$.

In Theorem 2.5 and Theorem 3.14 below these results will be generalized and extended to the setting of (ω_0, ω_1) -convexity.

2. From approximate (ω_0, ω_1) -convexity to approximate lower Hermite–Hadamard inequality

In this section we will investigate the implication between an (ω_0, ω_1) -convexity type inequality and a lower Hermite–Hadamard inequality. Consider the following basic assumptions.

(A1) (T, \mathcal{A}, μ) is a measure space.

(A2) $\Lambda : T \times \Delta^\circ(I) \rightarrow \mathbb{R}_+$ is μ -integrable in its first variable.

(A3) $M : T \times \Delta^\circ(I) \rightarrow \mathbb{R}$ is \mathcal{A} -measurable in its first variable and for all $t \in T$, the map $(x, y) \mapsto M(t, x, y)$ is a two-variable mean on I . $M_0 : \Delta^\circ(I) \rightarrow I$ is a strict mean such that

$$\mu\{t \in T \mid \Lambda(t, x, y) > 0, M(t, x, y) \neq M_0(x, y)\} > 0 \quad \text{if } (x, y) \in \Delta^\circ(I). \quad (2.1)$$

(A4) There exist an (ω_0, ω_1) -Chebyshev system on I such that ω_0 is positive. Furthermore, for $i \in \{0, 1\}$,

$$\omega_i(M_0(x, y)) = \int_T \Lambda(t, x, y) \omega_i(M(t, x, y)) d\mu(t) \quad ((x, y) \in \Delta^\circ(I)). \quad (2.2)$$

For all $(x, y) \in \Delta^\circ(I)$, denote

$$\begin{aligned} T'_{x,y} &:= \{t \in T \mid \Lambda(t, x, y) > 0, M(t, x, y) < M_0(x, y)\}, \\ T''_{x,y} &:= \{t \in T \mid \Lambda(t, x, y) > 0, M(t, x, y) \geq M_0(x, y)\}. \end{aligned}$$

Observe that, for all $(x, y) \in \Delta^\circ(I)$, $T'_{x,y}$ and $T''_{x,y}$ are in \mathcal{A} , moreover, by (2.1), the μ -measure of $T'_{x,y} \cup T''_{x,y}$ is positive. Define, for all $(x, y) \in \Delta^\circ(I)$, $i \in \{0, 1\}$,

$$\begin{aligned} S'_i(x, y) &= \int_{T'_{x,y}} \Lambda(t, x, y) \omega_i(M(t, x, y)) d\mu(t) \quad \text{and} \\ S''_i(x, y) &= \int_{T''_{x,y}} \Lambda(t, x, y) \omega_i(M(t, x, y)) d\mu(t). \end{aligned} \quad (2.3)$$

The following proposition describes the properties of these sets and numbers.

PROPOSITION 2.1. *If (A1)–(A4) hold, then for all $(x, y) \in \Delta^\circ(I)$,*

$$S'_i(x, y) + S''_i(x, y) = \omega_i(M_0(x, y)) \quad (i \in \{0, 1\}). \quad (2.4)$$

Furthermore, the μ -measure of the sets $T'_{x,y}$ and $T''_{x,y}$ is positive.

Proof. Let $(x, y) \in \Delta^\circ(I)$. (2.2) implies that

$$\begin{aligned} \omega_i(M_0(x, y)) &= \int_T \Lambda(t, x, y) \omega_i(M(t, x, y)) d\mu(t) \\ &= \int_{\{t \in T | \Lambda(t, x, y) > 0\}} \Lambda(t, x, y) \omega_i(M(t, x, y)) d\mu(t) = S'_i(x, y) + S''_i(x, y), \end{aligned}$$

for $i \in \{0, 1\}$. Hence (2.4) holds. To prove the positivity of $\mu(T''_{x,y})$ and $\mu(T'_{x,y})$, assume that $\mu(T'_{x,y}) = 0$. Then $S'_i(x, y) = 0$ and, in view of (2.1), it follows that $\mu(T''_{x,y}) > 0$. Thus, by (2.4), we have that

$$\omega_i(M_0(x, y)) = S'_i(x, y) + S''_i(x, y) = S''_i(x, y) = \int_{T''_{x,y}} \Lambda(t, x, y) \omega_i(M(t, x, y)) d\mu(t)$$

for $i \in \{0, 1\}$. Dividing the above identities by each other and using also the positivity of ω_0 , we get that

$$\frac{\int_{T''_{x,y}} \Lambda(t, x, y) \omega_1(M(t, x, y)) d\mu(t)}{\int_{T''_{x,y}} \Lambda(t, x, y) \omega_0(M(t, x, y)) d\mu(t)} = \frac{\omega_1(M_0(x, y))}{\omega_0(M_0(x, y))}.$$

Rearranging this equality, we obtain that

$$\int_{T''_{x,y}} \Lambda(t, x, y) \Omega(M_0(x, y), M(t, x, y)) d\mu(t) = 0.$$

Hence,

$$\int_{\{t \in T | \Lambda(t, x, y) > 0, M(t, x, y) > M_0(x, y)\}} \Lambda(t, x, y) \Omega(M_0(x, y), M(t, x, y)) d\mu(t) = 0.$$

On the other hand, for all $t \in T$ with $M(t, x, y) > M_0(x, y)$, we have that $\Omega(M_0(x, y), M(t, x, y)) > 0$ and, by (2.1), $\mu(\{t \in T | \Lambda(t, x, y) > 0, M(t, x, y) > M_0(x, y)\}) > 0$. This yields that

$$\int_{\{t \in T | \Lambda(t, x, y) > 0, M(t, x, y) > M_0(x, y)\}} \Lambda(t, x, y) \Omega(M_0(x, y), M(t, x, y)) d\mu(t) > 0,$$

which is a contradiction.

The proof for the case when $\mu(T''_{x,y}) = 0$ is analogous. \square

One of the main result of this paper is established in the following theorem.

THEOREM 2.2. *Assume that (A1)–(A4) hold. Let $f : I \rightarrow \mathbb{R}$ be a locally upper bounded Borel measurable solution of the approximate (ω_0, ω_1) -convexity type functional inequality*

$$f(u) \leq \frac{\Omega(u, y)}{\Omega(x, y)} f(x) + \frac{\Omega(x, u)}{\Omega(x, y)} f(y) + \varepsilon_{x, y}(u) \quad (u \in [x, y]), \quad (2.5)$$

where for all $(x, y) \in \Delta^\circ(I)$ and $u \in]x, y[$, the function $(v, w) \mapsto \varepsilon_{v, w}(u)$ is bounded and Borel measurable for $(v, w) \in [x, u] \times [u, y]$. Then f also satisfies the approximate lower Hermite–Hadamard type inequality

$$f(M_0(x, y)) \leq \int_T \Lambda(t, x, y) f(M(t, x, y)) d\mu(t) + E(x, y) \quad ((x, y) \in \Delta(I)), \quad (2.6)$$

where $E : \Delta^\circ(I) \rightarrow \mathbb{R}$ is defined by the following way

$$E(x, y) = \frac{\int_{T'_{x,y}} \int_{T''_{x,y}} \Lambda(t', x, y) \Lambda(t'', x, y) \Omega(M(t', x, y), M(t'', x, y)) \varepsilon_{M(t', x, y), M(t'', x, y)}(M_0(x, y)) d\mu(t'') d\mu(t')}{\int_{T'_{x,y}} \int_{T''_{x,y}} \Lambda(t', x, y) \Lambda(t'', x, y) \Omega(M(t', x, y), M(t'', x, y)) d\mu(t'') d\mu(t')} \quad (2.7)$$

REMARK 2.3. In the above theorem, the regularity condition for f can be relaxed if the error function $\varepsilon_{x, y}$ enjoys boundedness and continuity properties. For instance, if $\varepsilon_{x, y}$ is bounded on $[x, y]$ for some $(x, y) \in \Delta^\circ(I)$, then (2.5) implies that f is bounded on $[x, y]$. Similarly, if $\limsup_{u \rightarrow x+0} \varepsilon_{x, y}(u) = 0$ for some $(x, y) \in \Delta^\circ(I)$, then (2.5) implies that f is upper semicontinuous at x from the right.

Proof. Let $(x, y) \in \Delta^\circ(I)$. Substituting in (2.5) x by $M(t', x, y)$ and y by $M(t'', x, y)$, and u by $M_0(x, y)$, where $t' \in T'_{x, y}$ and $t'' \in T''_{x, y}$, we get that

$$\begin{aligned} & \Omega(M(t', x, y), M(t'', x, y)) f(M_0(x, y)) \\ & \leq \Omega(M_0(x, y), M(t'', x, y)) f(M(t', x, y)) \\ & \quad + \Omega(M(t', x, y), M_0(x, y)) f(M(t'', x, y)) \\ & \quad + \Omega(M(t', x, y), M(t'', x, y)) \varepsilon_{M(t', x, y), M(t'', x, y)}(M_0(x, y)). \end{aligned}$$

Multiplying this inequality by $\Lambda(t', x, y) \Lambda(t'', x, y)$ and integrating with respect to $\mu \times \mu$ on $T'_{x, y} \times T''_{x, y}$, we get that

$$\begin{aligned}
 & \int_{T'_{x,y}} \int_{T''_{x,y}} \Lambda(t', x, y) \Lambda(t'', x, y) \Omega(M(t', x, y), M(t'', x, y)) d\mu(t'') d\mu(t') f(M_0(x, y)) \\
 \leq & \int_{T'_{x,y}} \int_{T''_{x,y}} \Lambda(t', x, y) \Lambda(t'', x, y) \Omega(M_0(x, y), M(t'', x, y)) f(M(t', x, y)) d\mu(t'') d\mu(t') \\
 & + \int_{T'_{x,y}} \int_{T''_{x,y}} \Lambda(t', x, y) \Lambda(t'', x, y) \Omega(M(t', x, y), M_0(x, y)) f(M(t'', x, y)) d\mu(t'') d\mu(t') \\
 & + \int_{T'_{x,y}} \int_{T''_{x,y}} \Lambda(t', x, y) \Lambda(t'', x, y) \Omega(M(t', x, y), M(t'', x, y)) \times \\
 & \quad \times \varepsilon_{M(t', x, y), M(t'', x, y)}(M_0(x, y)) d\mu(t'') d\mu(t').
 \end{aligned} \tag{2.8}$$

Applying Fubini’s theorem and the notation of (2.3), we get that

$$\begin{aligned}
 & \int_{T'_{x,y}} \int_{T''_{x,y}} \Lambda(t', x, y) \Lambda(t'', x, y) \Omega(M(t', x, y), M(t'', x, y)) d\mu(t'') d\mu(t') \\
 & = (S'_0(x, y) S''_1(x, y) - S'_1(x, y) S''_0(x, y)).
 \end{aligned} \tag{2.9}$$

Observe that $(S'_0(x, y) S''_1(x, y) - S'_1(x, y) S''_0(x, y))$ is positive. Indeed, by the definition of the Chebyshev-system, we have, for all $(t', t'') \in T'_{x,y} \times T''_{x,y}$,

$$\Omega(M(t', x, y), M(t'', x, y)) > 0.$$

By Proposition 2.1, the measure of $T'_{x,y} \times T''_{x,y}$ is positive. Hence the left hand side of (2.9) is positive. Using the identity (2.4), it follows that

$$\begin{aligned}
 & \int_{T'_{x,y}} \int_{T''_{x,y}} \Lambda(t', x, y) \Lambda(t'', x, y) \Omega(M_0(x, y), M(t'', x, y)) f(M(t', x, y)) d\mu(t'') d\mu(t') \\
 = & (\omega_0(M_0(x, y)) S''_1(x, y) - \omega_1(M_0(x, y)) S''_0(x, y)) \int_{T'_{x,y}} \Lambda(t', x, y) f(M(t', x, y)) d\mu(t') \\
 = & ((S'_0(x, y) + S''_0(x, y)) S''_1(x, y) - (S'_1(x, y) + S''_1(x, y)) S''_0(x, y)) \\
 & \times \int_{T'_{x,y}} \Lambda(t', x, y) f(M(t', x, y)) d\mu(t') \\
 = & (S'_0(x, y) S''_1(x, y) - S'_1(x, y) S''_0(x, y)) \int_{T'_{x,y}} \Lambda(t', x, y) f(M(t', x, y)) d\mu(t'),
 \end{aligned} \tag{2.10}$$

and similarly,

$$\begin{aligned} & \int_{T'_{x,y}} \int_{T''_{x,y}} \Lambda(t', x, y) \Lambda(t'', x, y) \Omega(M(t', x, y), M_0(x, y)) f(M(t'', x, y)) d\mu(t'') d\mu(t') \\ &= (S'_1(x, y) S'_0(x, y) - S''_0(x, y) S'_1(x, y)) \int_{T''_{x,y}} \Lambda(t'', x, y) f(M(t'', x, y)) d\mu(t''). \end{aligned} \quad (2.11)$$

Substituting the above formulas (2.9), (2.10), and (2.11) into (2.8) and dividing the inequality so obtained by $(S'_0(x, y) S''_1(x, y) - S'_1(x, y) S''_0(x, y))$, we get (2.6) with the error function $E : \Delta^\circ(I) \rightarrow \mathbb{R}$ defined by (2.7), which completes the proof. \square

REMARK 2.4. A direct corollary of this theorem is the lower Hermite–Hadamard type inequality established by Theorem A. Indeed, suppose that, with the notations introduced in (1.5), the assumptions of Theorem A hold. Then, the (ω_0, ω_2) -convexity of f implies that it is locally bounded and Borel measurable. We show first that the conditions of Theorem 2.2 are also valid. Let μ denote the Lebesgue measure on $[0, 1]$ and define, for all $(x, y) \in \Delta^\circ(I)$, $t \in [0, 1]$,

$$\begin{aligned} M_0(x, y) &:= \xi(x, y), \quad M(t, x, y) := (1-t)x + ty, \quad \text{and} \\ \Lambda(t, x, y) &:= \frac{(y-x)\rho((1-t)x + ty)}{c(x, y)}. \end{aligned}$$

Since $M(t, x, y) = M_0(x, y)$ can hold only for one value of t , hence (2.1) holds trivially. We also have

$$\begin{aligned} \int_0^1 \Lambda(t, x, y) \omega_1(M(t, x, y)) dt &= \frac{y-x}{c(x, y)} \int_0^1 \rho((1-t)x + ty) \omega_1((1-t)x + ty) dt \\ &= \frac{1}{c(x, y)} \int_x^y \rho \omega_1 = \omega_0(\xi(x, y)) \frac{\int_x^y \omega_1 \rho}{\int_x^y \omega_0 \rho} \\ &= \omega_0(\xi(x, y)) \frac{\omega_1}{\omega_0}(\xi(x, y)) = \omega_1(\xi(x, y)) = \omega_1(M_0(x, y)) \end{aligned}$$

and, similarly,

$$\int_0^1 \Lambda(t, x, y) \omega_0(M(t, x, y)) dt = \frac{1}{c(x, y)} \int_x^y \rho \omega_0 = \omega_0(\xi(x, y)),$$

which proves (2.2). Thus all the assumptions A(1)–A(4) are verified. Therefore, if a function $f : I \rightarrow \mathbb{R}$ is (ω_0, ω_1) -convex, i.e., satisfies (2.5) with $\varepsilon_{x,y} := 0$, then it fulfills (2.6) with $E := 0$, which, by the obvious identity $\frac{1}{c(x,y)} \int_x^y f \rho = \int_0^1 \Lambda(t, x, y) f(M(t, x, y)) dt$ is equivalent to the left hand side inequality in (1.6).

The following result could be deduced from Theorem 2.2, however a direct proof is more convenient here. Given a set D , denote $\{(x, y) \mid x, y \in D, x \neq y\}$ by D^{2*} .

THEOREM 2.5. *Let D be a convex set of a linear space X . Let \mathcal{A} be a sigma algebra containing the Borel subsets of $[0, 1]$ and μ be a probability measure on the measure space $([0, 1], \mathcal{A})$ such that the support of μ is not a singleton. Denote*

$$\mu_1 := \int_{[0,1]} t d\mu(t) \quad \text{and} \quad S(\mu) := \mu([0, \mu_1]) \int_{[\mu_1, 1]} t d\mu(t) - \mu([\mu_1, 1]) \int_{[0, \mu_1]} t d\mu(t).$$

Assume that $f : D \rightarrow \mathbb{R}$ is and hemi- μ -integrable solution of the functional inequality

$$f((1-t)x + ty) \leq (1-t)f(x) + tf(y) + \eta_{x,y}(t) \quad ((x, y) \in D^{2*}, t \in [0, 1]) \quad (2.12)$$

where, for all $(x, y) \in D^{2*}$, $\eta_{x,y} : [0, 1] \rightarrow \mathbb{R}$ is a function such that

$$I(x, y) := \int_{[\mu_1, 1]} \int_{[0, \mu_1]} (t'' - t') \eta_{(1-t')x + t'y, (1-t'')x + t''y} \left(\frac{\mu_1 - t'}{t'' - t'} \right) d\mu(t') d\mu(t'')$$

exists in $[-\infty, \infty]$ for all $(x, y) \in D^{2*}$. Then, for all $(x, y) \in D^{2*}$, the function f also satisfies the lower Hermite–Hadamard type inequality

$$f((1 - \mu_1)x + \mu_1 y) \leq \int_{[0,1]} f((1-t)x + ty) d\mu(t) + \frac{1}{S(\mu)} I(x, y) \quad ((x, y) \in D^{2*}). \quad (2.13)$$

REMARK 2.6. In the above theorem, the hemi- μ -integrability condition for f can be relaxed if the error function $\eta_{x,y}$ enjoys boundedness and continuity properties. For instance, if $\eta_{x,y}$ is upper bounded on $[x, y]$ for some $(x, y) \in D^{2*}$, then (2.5) implies that $f((1-t)x + ty)$ is upper bounded for $t \in [0, 1]$. Similarly, if $\limsup_{t \rightarrow 0+0} \eta_{x,y}(t) = 0$ for some $(x, y) \in D^{2*}$, then (2.5) implies that $f((1-t)x + ty)$ is an upper semicontinuous function of t at zero from the right.

Proof. Let $(x, y) \in D^{2*}$. Substituting in (2.12) x by $(1-t')x + t'y$, y by $(1-t'')x + t''y$ and t by $\frac{\mu_1 - t'}{t'' - t'}$, where $0 \leq t' \leq \mu_1$ and $\mu_1 < t'' \leq 1$, we get that

$$f((1 - \mu_1)x + \mu_1 y) \leq \frac{t'' - \mu_1}{t'' - t'} f((1-t')x + t'y) + \frac{\mu_1 - t'}{t'' - t'} f((1-t'')x + t''y) + \eta_{(1-t')x + t'y, (1-t'')x + t''y} \left(\frac{\mu_1 - t'}{t'' - t'} \right). \quad (2.14)$$

Multiplying (2.14) by $t'' - t'$ and integrating on $[0, \mu_1] \times [\mu_1, 1]$ with respect to the product measure $\mu \times \mu$, we obtain

$$\begin{aligned}
& \int_{] \mu_1, 1]} \int_{[0, \mu_1]} (t'' - t') d\mu(t') d\mu(t'') f((1 - \mu_1)x + \mu_1 y) \\
& \leq \int_{] \mu_1, 1]} (t'' - \mu_1) d\mu(t'') \int_{[0, \mu_1]} f((1 - t')x + t'y) d\mu(t') \\
& \quad + \int_{[0, \mu_1]} (\mu_1 - t') d\mu(t') \int_{] \mu_1, 1]} f((1 - t'')x + t''y) d\mu(t'') \\
& \quad + \int_{] \mu_1, 1]} \int_{[0, \mu_1]} (t'' - t') \eta_{(1-t')x+t'y, (1-t'')x+t''y} \left(\frac{\mu_1 - t'}{t'' - t'} \right) d\mu(t') d\mu(t'').
\end{aligned} \tag{2.15}$$

Applying Fubini's theorem, we get that

$$\begin{aligned}
\int_{] \mu_1, 1]} \int_{[0, \mu_1]} (t'' - t') d\mu(t') d\mu(t'') &= \mu([0, \mu_1]) \int_{] \mu_1, 1]} t'' d\mu(t'') - \mu(] \mu_1, 1]) \int_{[0, \mu_1]} t' d\mu(t') \\
&= S(\mu).
\end{aligned} \tag{2.16}$$

Using that the support of μ is not a singleton, we can see that the left hand side of (2.16) is positive and hence so is $S(\mu)$.

Applying also Fubini's theorem, it follows that

$$\begin{aligned}
\int_{] \mu_1, 1]} (t'' - \mu_1) d\mu(t'') &= \mu([0, 1]) \int_{] \mu_1, 1]} t'' d\mu(t'') - \mu(] \mu_1, 1]) \int_{[0, 1]} t d\mu(t) \\
&= \left(\mu([0, \mu_1]) + \mu(] \mu_1, 1]) \right) \int_{] \mu_1, 1]} t'' d\mu(t'') \\
&\quad - \mu(] \mu_1, 1]) \left(\int_{[0, \mu_1]} t' d\mu(t') + \int_{] \mu_1, 1]} t'' d\mu(t'') \right) = S(\mu)
\end{aligned} \tag{2.17}$$

and, similarly,

$$\int_{[0, \mu_1]} (\mu_1 - t') d\mu(t') = S(\mu). \tag{2.18}$$

Substituting the above formulas (2.16), (2.17), and (2.18) into (2.15) and dividing the inequality so obtained by $S(\mu)$, we arrive at (2.13). This completes the proof. \square

The following corollary is analogous to the result of [6].

COROLLARY 2.7. *Assume that $f : D \rightarrow \mathbb{R}$ a hemi-Lebesgue integrable solution of the functional inequality (2.12), where, for all $(x, y) \in D^{2*}$, $\eta_{x,y} : [0, 1] \rightarrow \mathbb{R}$ is a*

function, such that

$$I(x, y) := \int_{\frac{1}{2}}^1 \int_0^{\frac{1}{2}} (t'' - t') \eta_{(1-t'')x+t'y, (1-t'')x+t''y} \left(\frac{\frac{1}{2} - t'}{t'' - t'} \right) dt' dt'' \tag{2.19}$$

exists in $[-\infty, \infty]$ for all $(x, y) \in D^{2*}$. Then, for all $x, y \in D^{2*}$, the function f also satisfies

$$f\left(\frac{x+y}{2}\right) \leq \int_0^1 f((1-t)x + ty) dt + 8I(x, y). \tag{2.20}$$

Proof. We apply Theorem 2.5, when \mathcal{A} is the family of Lebesgue measurable subsets of $[0, 1]$, μ is the Lebesgue measure. Then $\mu_1 = \frac{1}{2}$ and $S(\mu) = \frac{1}{8}$ and the result directly follows from Theorem 2.5. \square

REMARK 2.8. In what follows, we deduce the conclusion of Theorem B from the above corollary under stronger regularity assumption on f . Let $\alpha : (D - D) \rightarrow \mathbb{R}_+$ be a nonnegative radially Lebesgue integrable function and assume that $f : D \rightarrow \mathbb{R}$ is hemi-upper bounded and approximately Jensen convex in the sense of (1.8). Then, by [9, Thm. 8], f fulfils the following approximate convexity inequality:

$$f((1-t)x + ty) \leq (1-t)f(x) + tf(y) + \sum_{n=0}^{\infty} \frac{1}{2^n} \alpha(2d_{\mathbb{Z}}(2^n t)(x-y))$$

$$((x, y) \in D^2, t \in [0, 1]),$$

i.e., (2.12) holds with $\eta_{x,y}$ defined as

$$\eta_{x,y}(t) := \sum_{n=0}^{\infty} \frac{1}{2^n} \alpha(2d_{\mathbb{Z}}(2^n t)(x-y)) \quad ((x, y) \in D^2, t \in [0, 1]).$$

Thus, by Corollary 2.7, the inequality (2.20) holds with

$$I(x, y) = \int_{\frac{1}{2}}^1 \int_0^{\frac{1}{2}} (t'' - t') \eta_{(1-t'')x+t'y, (1-t'')x+t''y} \left(\frac{\frac{1}{2} - t'}{t'' - t'} \right) dt' dt''$$

$$= \sum_{n=0}^{\infty} \frac{1}{2^n} \int_{\frac{1}{2}}^1 \int_0^{\frac{1}{2}} (t'' - t') \alpha \left(2d_{\mathbb{Z}} \left(2^n \frac{\frac{1}{2} - t'}{t'' - t'} \right) (t'' - t')(x-y) \right) dt' dt''$$

$$= \sum_{n=0}^{\infty} \frac{1}{2^n} \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} (t+s) \alpha \left(2d_{\mathbb{Z}} \left(\frac{2^n t}{t+s} \right) (t+s)(x-y) \right) dt ds$$

$$= \sum_{n=0}^{\infty} \frac{2}{2^n} \int_0^{\frac{1}{2}} \int_0^t (t+s) \alpha \left(2d_{\mathbb{Z}} \left(\frac{2^n t}{t+s} \right) (t+s)(x-y) \right) ds dt.$$

The last equality above is the consequence of the symmetry of the integrand with respect to the variables s, t . For $n = 0$,

$$\begin{aligned} \frac{2}{2^n} \int_0^{\frac{1}{2}} \int_0^t (t+s) \alpha \left(2d_{\mathbb{Z}} \left(\frac{2^n t}{t+s} \right) (t+s)(x-y) \right) ds dt &= 2 \int_0^{\frac{1}{2}} \int_0^t (t+s) \alpha (2s(x-y)) ds dt \\ &= 2 \int_0^{\frac{1}{2}} \int_s^{\frac{1}{2}} (t+s) \alpha (2s(x-y)) dt ds = \int_0^{\frac{1}{2}} (1-2s) \left(\frac{3}{2}s + \frac{1}{4} \right) \alpha (2s(x-y)) ds \\ &= \frac{1}{8} \int_0^1 (1-\sigma)(3\sigma+1) \alpha (\sigma(x-y)) d\sigma. \end{aligned}$$

To compute the the double integral for $n \geq 1$, we will split its domain according to the position of $\frac{2^n t}{t+s}$ related to integer numbers. For all $n \in \mathbb{N}$ and $0 < s < t \leq \frac{1}{2}$, there exists a unique $m \in \{2^{n-1}, \dots, 2^n - 1\}$ (namely $m := \left\lfloor \frac{2^n t}{t+s} \right\rfloor$) such that

$$\text{either } m \leq \frac{2^n t}{t+s} < m + \frac{1}{2} \quad \text{or} \quad m + \frac{1}{2} \leq \frac{2^n t}{t+s} < m + 1.$$

This, for all $m \in \{2^{n-1}, \dots, 2^n - 1\}$, in terms of t yields the following inequalities for s :

$$\frac{2^n - m - \frac{1}{2}}{m + \frac{1}{2}} t < s \leq \frac{2^n - m}{m} t \quad \text{and} \quad \frac{2^n - m - 1}{m + 1} t < s \leq \frac{2^n - m - \frac{1}{2}}{m + \frac{1}{2}} t,$$

respectively. On these intervals, we have that

$$d_{\mathbb{Z}} \left(\frac{2^n t}{t+s} \right) (t+s) = \begin{cases} \left(\frac{2^n t}{t+s} - m \right) (t+s) = (2^n - m)t - ms, \\ \text{if } \frac{2^n - m - \frac{1}{2}}{m + \frac{1}{2}} t < s \leq \frac{2^n - m}{m} t, \\ (m + 1 - \frac{2^n t}{t+s}) (t+s) = (m + 1 - 2^n)t + (m + 1)s, \\ \text{if } \frac{2^n - m - 1}{m + 1} t < s \leq \frac{2^n - m - \frac{1}{2}}{m + \frac{1}{2}} t. \end{cases}$$

Thus, we get that

$$\begin{aligned}
 & \int_0^{\frac{1}{2}} \int_0^t (t+s) \alpha \left(2d_{\mathbb{Z}} \left(2^n \frac{t}{t+s} \right) (t+s)(x-y) \right) ds dt \\
 &= \int_0^{\frac{1}{2}} \sum_{m=2^{n-1}}^{2^n-1} \left(\int_{\frac{2^n-m-\frac{1}{2}}{m+\frac{1}{2}}t}^{\frac{2^n-m}{m}} (t+s) \alpha \left(2((2^n-m)t-ms)(x-y) \right) ds \right. \\
 &\quad \left. + \int_{\frac{2^n-m-1}{m-1}t}^{\frac{2^n-m-\frac{1}{2}}{m+\frac{1}{2}}t} (t+s) \alpha \left(2((m+1-2^n)t+(m+1)s)(x-y) \right) ds \right) dt \\
 &= \sum_{m=2^{n-1}}^{2^n-1} \int_0^{\frac{1}{2}} \left(\int_0^{\frac{2^{n+1}t}{2m+1}} \alpha(\sigma(x-y)) \left(\frac{\sigma+2^{n+1}t}{(2m+2)^2} + \frac{2^{n+1}t-\sigma}{(2m)^2} \right) d\sigma \right) dt \\
 &= \sum_{m=2^{n-1}}^{2^n-1} \int_0^{\frac{2^n}{2m+1}} \left(\int_{\frac{(2m+1)\sigma}{2^{n+1}}}^{\frac{1}{2}} \alpha(\sigma(x-y)) \left(\frac{\sigma+2^{n+1}t}{(2m+2)^2} + \frac{2^{n+1}t-\sigma}{(2m)^2} \right) dt \right) d\sigma \\
 &= \frac{1}{16} \sum_{m=2^{n-1}}^{2^n-1} \int_0^{\frac{2^n}{2m+1}} \alpha(\sigma(x-y)) \left(1 - \frac{2m+1}{2^n} \sigma \right) \left(\frac{\sigma(2m+3)+2^n}{(m+1)^2} + \frac{\sigma(2m-1)+2^n}{m^2} \right) d\sigma \\
 &= \frac{1}{16} \sum_{m=2^{n-1}}^{2^n-1} \int_0^1 \alpha(\sigma(x-y)) \left(1 - \frac{2m+1}{2^n} \sigma \right)^+ \left(\frac{\sigma(2m+3)+2^n}{(m+1)^2} + \frac{\sigma(2m-1)+2^n}{m^2} \right) d\sigma.
 \end{aligned}$$

(Here x^+ stands for the positive part of x .) Summarizing our computations, for $8I(x, y)$, we get

$$8I(x, y) = \int_0^1 \alpha(\sigma(x-y)) \Phi(\sigma) d\sigma,$$

where

$$\begin{aligned}
 \Phi(\sigma) &:= (1-\sigma)(3\sigma+1) + \sum_{n=1}^{\infty} \sum_{m=2^{n-1}}^{2^n-1} \left(1 - \frac{2m+1}{2^n} \sigma \right)^+ \left(\frac{\sigma(2m+3)+2^n}{2^n(m+1)^2} + \frac{\sigma(2m-1)+2^n}{2^n m^2} \right) \\
 &= (1-\sigma)(3\sigma+1) + \sum_{m=1}^{\infty} \left(1 - \frac{2m+1}{2^{\lceil \log_2 m \rceil + 1}} \sigma \right)^+ \\
 &\quad \times \left(\frac{\sigma(2m+3)+2^{\lceil \log_2 m \rceil + 1}}{2^{\lceil \log_2 m \rceil + 1} (m+1)^2} + \frac{\sigma(2m-1)+2^{\lceil \log_2 m \rceil + 1}}{2^{\lceil \log_2 m \rceil + 1} m^2} \right).
 \end{aligned}$$

One can easily see that Φ is a continuous function over $[0, 1]$ with $\Phi(t) \geq 1$ for $0 \leq t \leq \frac{2}{3}$ and $\Phi(1) = 0$. Hence the error term $8I(x, y)$ obtained in (2.20) is not comparable with that in (1.9).

In what follows, we examine the case, when X is a normed space and $\eta_{x,y}(t)$ is a linear combination of the products of the powers of t , $1 - t$, and of $\|x - y\|$, i.e., for all $(x, y) \in D^{2*}$ $\eta_{x,y}$ is of the form

$$\eta_{x,y}(t) := \int_{[0,\infty]^2} t^p(1-t)^q \|x-y\|^{p+q-1} d\nu(p,q) \quad ((x,y) \in D^{2*}), \tag{2.21}$$

where ν is a σ -finite Borel measure on $[0, \infty]^2$. An important particular case is when ν is of the form $\sum_{i=1}^k c_i \delta_{(p_i, q_i)}$, where $c_1, \dots, c_k > 0$, $(p_1, q_1), \dots, (p_k, q_k) \in [0, \infty]^2$.

THEOREM 2.9. *Let \mathcal{A} be a sigma algebra containing the Borel subsets of $[0, 1]$ and μ be a probability measure on the measure space $([0, 1], \mathcal{A})$ such that the support of μ is not a singleton. Let ν be a σ -finite Borel measure on $[0, \infty]^2$ such that, for all $s \in \{\|x - y\| \mid (x, y) \in D^{2*}\}$,*

$$J(s) := \int_{[0,\infty]^2} \left(\int_{[0,\mu_1]} (\mu_1 - t')^p d\mu(t') \int_{[\mu_1,1]} (t'' - \mu_1)^q d\mu(t'') \right) s^{p+q-1} d\nu(p,q)$$

exists in $[-\infty, \infty]$. Assume that $f : D \rightarrow \mathbb{R}$ is a hemi- μ -integrable solution of the functional inequality

$$f((1-t)x + ty) \leq (1-t)f(x) + tf(y) + \int_{[0,\infty]^2} t^p(1-t)^q \|x-y\|^{p+q-1} d\nu(p,q) \tag{2.22}$$

for all $(x, y) \in D^{2}$ and $t \in [0, 1]$. Then f also fulfils the Hermite–Hadamard type inequality*

$$f((1 - \mu_1)x + \mu_1 y) \leq \int_{[0,1]} f((1-t)x + ty) d\mu(t) + \frac{1}{S(\mu)} J(\|x - y\|) \quad ((x, y) \in D^{2*}). \tag{2.23}$$

REMARK 2.10. In the above theorem, the hemi- μ -integrability condition for f can be relaxed if the measure ν is finite with compact support contained in $]0, \infty[^2$. Then the function $\eta_{x,y}$ defined by (2.21) is continuous on $[x, y]$ and $\eta_{x,y}(0) = \eta_{x,y}(1) = 0$, hence (2.22) implies that $t \mapsto f((1-t)x + ty)$ is upper bounded on $[0, 1]$ and upper semicontinuous at the endpoint of $[0, 1]$. Thus f is hemi-upper bounded and upper hemicontinuous on D , which yields its hemi- μ -integrability.

Proof. We want to apply Theorem 2.5. Let $(x, y) \in D^{2*}$ be arbitrary. Let $\eta_{x,y} : [0, 1] \rightarrow \mathbb{R}$ defined by (2.21). Then, (2.22) is equivalent to (2.12). To deduce (2.23), by

Theorem 2.5, we obtain that

$$\begin{aligned}
 I(x, y) &= \int_{[\mu_1, 1][0, \mu_1]} \int_{[0, \infty]^2} (t'' - t') \int_{[\mu_1 - t', 1]} \left(\frac{\mu_1 - t'}{t'' - t'} \right)^p \left(\frac{t'' - \mu_1}{t'' - t'} \right)^q \| (t'' - t')(x - y) \|^{p+q-1} d\nu(p, q) d\mu(t') d\mu(t'') \\
 &= \int_{[0, \infty]^2} \left(\int_{[0, \mu_1]} (\mu_1 - t')^p d\mu(t') \int_{[\mu_1, 1]} (t'' - \mu_1)^q d\mu(t'') \right) \| x - y \|^{p+q-1} d\nu(p, q) \\
 &= J(\|x - y\|),
 \end{aligned}$$

which proves the statement. \square

COROLLARY 2.11. Let ν be a σ -finite Borel measure on $[0, \infty]^2$, such that for all $s \in \{\|x - y\| : (x, y) \in D^{2*}\}$

$$\int_{[0, \infty]^2} \frac{s^{p+q-1}}{2^{p+q-1}(p+1)(q+1)} d\nu(p, q)$$

exists in $[-\infty, \infty]$. Assume that $f : D \rightarrow \mathbb{R}$ is a hemi-Lebesgue integrable solution of the functional inequality

$$f((1-t)x + ty) \leq (1-t)f(x) + tf(y) + \int_{[0, \infty]^2} t^p (1-t)^q \|x - y\|^{p+q-1} d\nu(p, q), \tag{2.24}$$

where $(x, y) \in D^{2*}$ and $t \in [0, 1]$. Then, f also satisfies the Hermite–Hadamard type inequality

$$f\left(\frac{x+y}{2}\right) \leq \int_{[0, 1]} f((1-t)x + ty) dt + \int_{[0, \infty]^2} \frac{\|x - y\|^{p+q-1}}{2^{p+q-1}(p+1)(q+1)} d\nu(p, q) \quad ((x, y) \in D^{2*}). \tag{2.25}$$

Proof. Observe that (2.24) is equivalent to (2.12), where for all $(x, y) \in D^{2*}$, $\eta_{x,y} : [0, 1] \rightarrow \mathbb{R}$ is defined by (2.21). We have $S(\mu) = \frac{1}{8}$ and using Theorem 2.9, we obtain that

$$\begin{aligned}
 J(s) &= \int_{[0, \infty]^2} \int_0^{\frac{1}{2}} \left(\frac{1}{2} - t\right)^p dt \int_{\frac{1}{2}}^1 \left(t - \frac{1}{2}\right)^q dt s^{p+q-1} d\nu(p, q) \\
 &= \int_{[0, \infty]^2} \frac{s^{p+q-1}}{2^{p+q+2}(p+1)(q+1)} d\nu(p, q),
 \end{aligned}$$

which yields (2.25). \square

3. From approximate (ω_0, ω_1) -convexity to approximate upper Hermite–Hadamard inequality

In the first part of this section we will investigate the implication between the (ω_0, ω_1) -convexity type inequality and upper Hermite–Hadamard inequality. Consider the following assumptions.

- (B1) (T, \mathcal{A}, μ) is a measure space.
 (B2) $\Lambda : T \times \Delta^\circ(I) \rightarrow \mathbb{R}_+$ is integrable (with respect to μ) in its first variable.
 (B3) $M : T \times \Delta^\circ(I) \rightarrow \mathbb{R}$ is measurable in its first variable and for all $t \in T$, the map $(x, y) \mapsto M(t, x, y)$ is a two-variable mean on I . $M_0 : \Delta^\circ(I) \rightarrow I$ is a strict mean.
 (B4) There exist an (ω_0, ω_1) -Chebyshev system on I such that ω_0 is positive and $i \in \{0, 1\}$ (2.2) holds.

THEOREM 3.12. *Assume that (B1)–(B4) hold. Let $f : I \rightarrow \mathbb{R}$ be a locally bounded Borel measurable solution of the approximate (ω_0, ω_1) -convexity inequality (2.5), where for all $(x, y) \in \Delta^\circ(I)$, $\eta_{x,y} : [x, y] \rightarrow \mathbb{R}$ is a bounded and Borel measurable function. Then f also satisfies the following approximate upper Hermite–Hadamard type inequality*

$$\int_T \Lambda(t, x, y) f(M(t, x, y)) d\mu(t) \leq \frac{\Omega(M_0(x, y), y)}{\Omega(x, y)} f(x) + \frac{\Omega(x, M_0(x, y))}{\Omega(x, y)} f(y) + E(x, y), \quad (3.1)$$

with $E : D^{2*} \rightarrow \mathbb{R}$ defined by

$$E(x, y) = \int_T \Lambda(t, x, y) \varepsilon_{x,y}(M(t, x, y)) d\mu(t). \quad (3.2)$$

Proof. Let $(x, y) \in \Delta^\circ(I)$ be arbitrary. Substituting in (2.5) u by $M(t, x, y)$, we get that

$$f(M(t, x, y)) \leq \frac{\Omega(M(t, x, y), y)}{\Omega(x, y)} f(x) + \frac{\Omega(x, M(t, x, y))}{\Omega(x, y)} f(y) + \varepsilon_{x,y}(M(t, x, y)) \quad (t \in T).$$

Multiplying this inequality by $\Lambda(t, x, y)$ and integrating with respect to μ on T , we get that

$$\begin{aligned} & \int_T \Lambda(t, x, y) f(M(t, x, y)) d\mu(t) \\ & \leq \frac{\int_T \Lambda(t, x, y) \Omega(M(t, x, y), y) d\mu(t)}{\Omega(x, y)} f(x) + \frac{\int_T \Lambda(t, x, y) \Omega(x, M(t, x, y)) d\mu(t)}{\Omega(x, y)} f(y) \\ & \quad + \int_T \Lambda(t, x, y) \varepsilon_{x,y}(M(t, x, y)) d\mu(t). \end{aligned} \quad (3.3)$$

Applying (2.2), it follows that

$$\int_T \Lambda(t, x, y) \Omega(M(t, x, y), y) d\mu(t) = \Omega(M_0(x, y), y) \tag{3.4}$$

and

$$\int_T \Lambda(t, x, y) \Omega(x, M(t, x, y)) d\mu(t) = \Omega(x, M_0(x, y)). \tag{3.5}$$

Substituting (3.4) and (3.5) to (3.3) we have (3.1). \square

REMARK 3.13. An immediate corollary of this theorem is the second inequality of Theorem A. Assume that the assumptions of Theorem A hold. It is easy to see that the conditions of Theorem 3.12 are also valid. For all $(x, y) \in \Delta^\circ(I)$ let $\eta_{x,y}, \mu, M(t, x, y), M_0(x, y)$ and $\Lambda(t, x, y)$ be defined as in Remark 2.4. Then (2.2) holds. Therefore, by (1.5) and using also Remark 2.4,

$$c_1(x, y) = \frac{1}{\Omega(x, y)} \left| \int_x^y \omega_0 \rho \omega_0(y) \right| = \frac{1}{\Omega(x, y)} \left| \omega_0(M_0(x, y)) \omega_0(y) \right| = \frac{\Omega(M_0(x, y), y)}{\Omega(x, y)}.$$

Similarly, it can be seen, that $c_2(x, y) = \frac{\Omega(x, M_0(x, y))}{\Omega(x, y)}$. Thus, by Theorem 3.12, we get the second inequality in Theorem A.

THEOREM 3.14. Let D be a convex set of a linear space X . Let \mathcal{A} be a sigma algebra containing the Borel subsets of $[0, 1]$ and μ be a probability measure on the measure space $([0, 1], \mathcal{A})$. Denote $\mu_1 := \int_{[0,1]} t d\mu(t)$. Assume that $f : D \rightarrow \mathbb{R}$ is a hemi- μ -integrable solution of the approximate convexity inequality (2.12), where, for all $(x, y) \in D^{2*}$, $\eta_{x,y} : [0, 1] \rightarrow \mathbb{R}$ is a bounded and Borel measurable function. Then, for all $(x, y) \in D^{2*}$, the function f also satisfies the approximate upper Hermite–Hadamard inequality

$$\int_{[0,1]} f((1-t)x + ty) d\mu(t) \leq (1 - \mu_1)f(x) + \mu_1 f(y) + \int_{[0,1]} \eta_{x,y}(t) d\mu(t). \tag{3.6}$$

REMARK 3.15. In the above theorem, the regularity condition for f can be relaxed if the error function $\eta_{x,y}$ enjoys boundedness and continuity properties. For instance, if $\eta_{x,y}$ is upper bounded on $[x, y]$ for some $(x, y) \in D^{2*}$, then (2.5) implies that $f((1-t)x + ty)$ is upper bounded for $t \in [0, 1]$. Similarly, if $\limsup_{t \rightarrow 0+0} \eta_{x,y}(t) = 0$ for some $(x, y) \in D^{2*}$, then (2.5) implies that $f((1-t)x + ty)$ is an upper semicontinuous function of t at zero from the right.

Proof. Let $(x, y) \in D^{2*}$ be fixed. Integrating (2.12) with respect to the variable t and the measure μ on $[0, 1]$ we get (3.6). \square

REMARK 3.16. Assume that the conditions of Theorem C hold. To prove a similar result as in Theorem C, we have to assume that also $\alpha : (D - D) \rightarrow \mathbb{R}$ is radially

bounded and radially continuous at 0. By [14] and [9], we can get that f is approximately convex in the following sense

$$f((1-t)x+ty) \leq (1-t)f(x)+tf(y)+ \sum_{n=0}^{\infty} \frac{1}{2^n} \alpha(2d_{\mathbb{Z}}(2^n t)(x-y)) \quad (x,y \in D, t \in [0,1]).$$

Let $\eta_{x,y}(t) := \sum_{n=0}^{\infty} \frac{1}{2^n} \alpha(2d_{\mathbb{Z}}(2^n t)(x-y))$ for $t \in [0,1]$ and $x,y \in D$. Let \mathcal{A} be the class of Lebesgue measurable subsets of $[0,1]$ and let the measure μ be defined by $d\mu(t) = \rho(t)dt$. Then $\mu_1 = \int_0^1 t\rho(t)dt = \lambda$. Thus applying Theorem 3.14 and the Fubini's theorem, we get (1.10), which completes the proof of Theorem C.

In what follows, we examine the case, when X is a normed space and $\eta_{x,y}(t)$ is a linear combination of the products of the powers of t , $1-t$, and of $\|x-y\|$, i.e., for all $(x,y) \in D^{2^*}$ $\eta_{x,y}$ is of the form

$$\eta_{x,y}(t) := \int_{[0,\infty]^3} t^p(1-t)^q \|x-y\|^r d\nu(p,q,r) \quad ((x,y) \in D^{2^*}), \tag{3.7}$$

where ν is a σ -finite Borel measure on $[0,\infty]^3$. An important particular case is when ν is of the form $\sum_{i=1}^k c_i \delta_{(p_i,q_i,r_i)}$, where $c_1, \dots, c_k > 0$ and $(p_1, q_1, r_1), \dots, (p_k, q_k, r_k) \in [0,\infty]^3$.

COROLLARY 3.17. *Let μ be a Borel probability measure on $[0,1]$, denote $\mu_1 := \int_{[0,1]} t d\mu(t)$. Let ν be a σ -finite Borel measure on $[0,\infty]^3$ such that, for all $s \in \{\|x-y\| \mid (x,y) \in D^{2^*}\}$,*

$$\int_{[0,\infty]^3} \int_{[0,1]} t^p(1-t)^q s^r d\mu(t) d\nu(p,q,r)$$

exists in $[-\infty,\infty]$. Assume that $f : D \rightarrow \mathbb{R}$ is and hemi- μ -integrable solution of the functional inequality

$$f((1-t)x+ty) \leq (1-t)f(x)+tf(y)+ \int_{[0,\infty]^3} t^p(1-t)^q \|x-y\|^r d\nu(p,q,r) \tag{3.8}$$

for all $(x,y) \in D^{2^}$ and $t \in [0,1]$. Then f also fulfills the following approximate upper Hermite–Hadamard inequality,*

$$\begin{aligned} & \int_{[0,1]} f((1-t)x+ty) d\mu(t) \\ & \leq (1-\mu_1)f(x)+\mu_1f(y)+ \int_{[0,\infty]^3} \int_{[0,1]} t^p(1-t)^q \|x-y\|^r d\mu(t) d\nu(p,q,r). \end{aligned} \tag{3.9}$$

Proof. We apply Theorem 3.14. For all $x,y \in D^{2^*}$, let $\eta_{x,y} : [0,1] \rightarrow \mathbb{R}$ defined by (3.7). Then it is easy to see that (3.8) is equivalent to (2.12). Hence, by Theorem 3.14 we get (3.9). \square

Denote by B the so-called beta-function, defined by

$$B(p_1, p_2) = \int_0^1 t^{p_1-1} (1-t)^{p_2-1} dt \quad (p_1, p_2 > 0).$$

COROLLARY 3.18. *Let ν be a σ -finite Borel measure on $[0, \infty[^3$ such that, for all $s \in \{\|x - y\| \mid (x, y) \in D^{2*}\}$,*

$$\int_{[0, \infty[^3} B(p + 1, q + 1) s^r d\nu(p, q, r)$$

exists in $[-\infty, \infty]$. Assume that $f : D \rightarrow \mathbb{R}$ is a hemi-Lebesgue integrable solution of the functional inequality

$$f((1-t)x + ty) \leq (1-t)f(x) + tf(y) + \int_{[0, \infty[^3} t^p (1-t)^q \|x - y\|^r d\nu(p, q, r)$$

for all $(x, y) \in D^{2}$ and $t \in [0, 1]$. Then f also fulfills the approximate upper Hermite–Hadamard inequality*

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x) + f(y)}{2} + \int_{[0, \infty[^3} B(p + 1, q + 1) \|x - y\|^r d\nu(p, q, r) \quad ((x, y) \in D^{2*}). \tag{3.10}$$

Proof. We apply Corollary 3.17 when μ is the Lebesgue measure. Then, for all $(x, y) \in D^{2*}$,

$$\begin{aligned} E(x, y) &= \int_{[0, \infty[^3} \int_{[0, 1]} t^p (1-t)^q \|x - y\|^r d\mu(t) d\nu(p, q, r) \\ &= \int_{[0, \infty[^3} B(p + 1, q + 1) \|x - y\|^r d\nu(p, q, r). \end{aligned}$$

Thus, the result directly follows from Corollary 3.17. \square

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(Received November 8, 2011)

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