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ON A HALF–DISCRETE MULHOLLAND–TYPE INEQUALITY

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Abstract. By means of weight functions and Hadamard's inequality, a half-discrete Mulholland-type inequality with a best constant factor is given. A best extension with multi-parameters, some equivalent forms as well as the operator expressions are also considered.

1. Introduction

Assuming that $f,g \in L^2(R_+)$, $||f|| = \{\int_0^\infty f^2(x)dx\}^{\frac{1}{2}} > 0$, $||g|| > 0$, we have the following Hilbert's integral inequality (cf. [1]):

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy < \pi ||f|| ||g||, \tag{1}$$

where the constant factor π is best possible. If $a = \{a_n\}_{n=1}^\infty$, $b = \{b_n\}_{n=1}^\infty \in l^2$, $||a|| = \{\sum_{n=1}^\infty a_n^2\}^{\frac{1}{2}} > 0$, $||b|| > 0$, then we have the following analogous discrete Hilbert's inequality

$$\sum_{m=1}^\infty \sum_{n=1}^\infty \frac{a_m b_n}{m+n} < \pi ||a|| ||b||, \tag{2}$$

with the same best constant factor π. Inequalities (1) and (2) are important in analysis and its applications (cf. [2], [3], [4]). On the other-hand, we have the following Mulholland's inequality with the same best constant factor (cf. [1], [5]):

$$\sum_{m=2}^\infty \sum_{n=2}^\infty \frac{a_m b_n}{\ln mn} < \pi \left\{ \sum_{m=2}^\infty m a_m^2 \sum_{n=2}^\infty n b_n^2 \right\}^{\frac{1}{2}}. \tag{3}$$

In 1998, by introducing an independent parameter $\lambda \in (0,1]$, Yang [6] gave an extension of (1). By generalizing the results from [6], Yang [7] gave some best extensions of (1) and (2): If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\lambda_1 + \lambda_2 = \lambda$, $k_\lambda(x,y)$ is a non-negative homogeneous function of degree $-\lambda$ satisfying $k(\lambda_1) = \int_0^\infty k_\lambda(t,1)t^{\lambda_1 - 1}dt \in R_+$, $\phi(x) =$

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$x^{p(1-\lambda_1)-1}$, $\psi(x) = x^{q(1-\lambda_2)-1}$, $f(\geqslant 0) \in L_{p,\phi}(R_+) = \{f|||f||_{p,\phi} := \{\int_0^\infty \phi(x)|f(x)|^p dx\}^{\frac{1}{p}} < \infty\}$, $g(\geqslant 0) \in L_{q,\psi}(R_+)$, and $||f||_{p,\phi}$, $||g||_{q,\psi} > 0$, then

$$\int_0^\infty \int_0^\infty k_\lambda(x,y) f(x) g(y) dx dy < k(\lambda_1) ||f||_{p,\phi} ||g||_{q,\psi}, \tag{4}$$

where the constant factor $k(\lambda_1)$ is best possible. Moreover if $k_\lambda(x,y)$ is finite and $k_\lambda(x,y)x^{\lambda_1-1}(k_\lambda(x,y)y^{\lambda_2-1})$ is decreasing for $x > 0$ $(y > 0)$, then for $a_m, b_n \geqslant 0$, $a = \{a_m\}_{m=1}^\infty \in l_{p,\phi} = \{a|||a||_{p,\phi} := \{\sum_{n=1}^\infty \phi(n)|a_n|^p\}^{\frac{1}{p}} < \infty\}$, and $b = \{b_n\}_{n=1}^\infty \in l_{q,\psi}$, $||a||_{p,\phi}$, $||b||_{q,\psi} > 0$, we have

$$\sum_{m=1}^\infty \sum_{n=1}^\infty k_\lambda(m,n) a_m b_n < k(\lambda_1) ||a||_{p,\phi} ||b||_{q,\psi}, \tag{5}$$

where the constant $k(\lambda_1)$ is still best value. Clearly, for $p = q = 2$, $\lambda = 1$, $k_1(x,y) = \frac{1}{x+y}$, $\lambda_1 = \lambda_2 = \frac{1}{2}$, (4) reduces to (1), while (5) reduces to (2).

Some other results about Hilbert-type inequalities can be found in [8]–[16]. On half-discrete Hilbert-type inequalities with the general non-homogeneous kernels, Hardy et al. provided a few results in Theorem 351 of [1]. But they did not prove that the constant factors are best possible. In 2005, Yang [17] gave a result with the kernel $\frac{1}{(1+nx)^\lambda}$ by introducing a variable and proved that the constant factor is best possible. Very recently, Yang [18] gave the following half-discrete Hilbert's inequality with best constant factor:

$$\int_0^\infty f(x) \sum_{n=1}^\infty \frac{a_n}{(x+n)^\lambda} dx < \pi ||f|| ||a||. \tag{6}$$

In this paper, by means of weight functions and Hadamard's inequality, a half-discrete Mulholland-type inequality similar to (3) and (6) with a best constant factor is given as follows:

$$\int_0^\infty f(x) \sum_{n=1}^\infty \frac{a_n}{\ln e(2n)^x} dx < \pi ||f|| \left\{ \sum_{n=1}^\infty n a_n^2 \right\}^{\frac{1}{2}}. \tag{7}$$

Moreover, a best extension of (7) with multi-parameters, some equivalent forms as well as the operator expressions are considered.

2. Some lemmas

LEMMA 1. *If* $0 < \lambda \leqslant 2$, $\alpha \geqslant 2$, *and* $\omega(n)$ *and* $\varpi(x)$ *are weight functions given by*

$$\omega(n) := (\ln \alpha n)^{\frac{\lambda}{2}} \int_0^\infty \frac{1}{\ln^\lambda e(\alpha n)^x} x^{\frac{\lambda}{2}-1} dx, n \in \mathbf{N}, \tag{8}$$

$$\varpi(x) := x^{\frac{\lambda}{2}} \sum_{n=1}^\infty \frac{1}{n \ln^\lambda e(\alpha n)^x} (\ln \alpha n)^{\frac{\lambda}{2}-1}, x \in (0,\infty), \tag{9}$$

then we have

$$\varpi(x) < \omega(n) = B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right). \tag{10}$$

Proof. Substituting $t = x\ln\alpha n$ in (8), and by simple calculation, we have

$$\omega(n) = \int_0^\infty \frac{1}{(1+t)^\lambda} t^{\frac{\lambda}{2}-1} dt = B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right).$$

For fixed $x > 0$, in view of the conditions, it is easy to see that

$$h(x,y) := \frac{(\ln\alpha y)^{\frac{\lambda}{2}-1}}{y\ln^\lambda e(\alpha y)^x} = \frac{1}{y(1+x\ln\alpha y)^\lambda(\ln\alpha y)^{1-\frac{\lambda}{2}}}$$

is decreasing and strictly convex for $y \in (\frac{1}{2}, \infty)$. Hence by Hadamard's inequality, we find

$$\varpi(x) < x^{\frac{\lambda}{2}} \int_{\frac{1}{2}}^\infty \frac{1}{y(1+x\ln\alpha y)^\lambda(\ln\alpha y)^{1-\frac{\lambda}{2}}} dy$$

$$\overset{t=x\ln\alpha y}{=} \int_{x\ln(\alpha/2)}^\infty \frac{t^{\frac{\lambda}{2}-1}}{(1+t)^\lambda} dt \leqslant B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right),$$

and (10) follows. □

LEMMA 2. *Let the assumptions of Lemma 1 be fulfilled and additionally, let* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $a_n \geqslant 0$, $n \in \mathbf{N}, f(x)$ *is a non-negative measurable function in* $(0,\infty)$. *Then we have the following inequalities:*

$$J := \left\{\sum_{n=1}^\infty \frac{1}{n}(\ln\alpha n)^{\frac{p\lambda}{2}-1}\left[\int_0^\infty \frac{f(x)}{\ln^\lambda e(\alpha n)^x} dx\right]^p\right\}^{\frac{1}{p}}$$

$$\leqslant \left[B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right)\right]^{\frac{1}{q}} \left\{\int_0^\infty \varpi(x) x^{p(1-\frac{\lambda}{2})-1} f^p(x) dx\right\}^{\frac{1}{p}}, \tag{11}$$

$$L_1 := \left\{\int_0^\infty \frac{x^{\frac{q\lambda}{2}-1}}{[\varpi(x)]^{q-1}}\left[\sum_{n=1}^\infty \frac{a_n}{\ln^\lambda e(\alpha n)^x}\right]^q dx\right\}^{\frac{1}{q}}$$

$$\leqslant \left\{B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) \sum_{n=1}^\infty n^{q-1}(\ln\alpha n)^{q(1-\frac{\lambda}{2})-1} a_n^q\right\}^{\frac{1}{q}}. \tag{12}$$

Proof. By Hölder's inequality (cf. [19]) and (10), it follows

$$\left[\int_0^\infty \frac{f(x)dx}{\ln^\lambda e(\alpha n)^x}\right]^p$$

$$= \left\{\int_0^\infty \frac{1}{\ln^\lambda e(\alpha n)^x}\left[\frac{x^{(1-\frac{\lambda}{2})/q}}{(\ln\alpha n)^{(1-\frac{\lambda}{2})/p}} \frac{f(x)}{n^{\frac{1}{p}}}\right]\left[\frac{(\ln\alpha n)^{(1-\frac{\lambda}{2})/p}}{x^{(1-\frac{\lambda}{2})/q}} n^{\frac{1}{p}}\right] dx\right\}^p$$

$$\leqslant \int_0^\infty \frac{1}{\ln^\lambda e(\alpha n)^x} \frac{x^{(1-\frac{\lambda}{2})(p-1)} f^p(x)}{n(\ln \alpha n)^{1-\frac{\lambda}{2}}} dx \left\{ \int_0^\infty \frac{n^{q-1}}{\ln^\lambda e(\alpha n)^x} \frac{(\ln \alpha n)^{(1-\frac{\lambda}{2})(q-1)}}{x^{1-\frac{\lambda}{2}}} dx \right\}^{p-1}$$

$$= \left\{ \omega(n) n^{q-1} (\ln \alpha n)^{q(1-\frac{\lambda}{2})-1} \right\}^{p-1} \int_0^\infty \frac{1}{\ln^\lambda e(\alpha n)^x} \frac{x^{(1-\frac{\lambda}{2})(p-1)} f^p(x)}{n(\ln \alpha n)^{1-\frac{\lambda}{2}}} dx$$

$$= \left[B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) \right]^{p-1} n(\ln \alpha n)^{1-\frac{p\lambda}{2}} \int_0^\infty \frac{1}{\ln^\lambda e(\alpha n)^x} \frac{x^{(1-\frac{\lambda}{2})(p-1)} f^p(x)}{n(\ln \alpha n)^{1-\frac{\lambda}{2}}} dx.$$

Then by Lebesgue term by term integration theorem (cf. [20]), we have

$$J \leqslant \left[B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) \right]^{\frac{1}{q}} \left\{ \sum_{n=1}^\infty \int_0^\infty \frac{1}{\ln^\lambda e(\alpha n)^x} \frac{x^{(1-\frac{\lambda}{2})(p-1)}}{n(\ln \alpha n)^{1-\frac{\lambda}{2}}} f^p(x) dx \right\}^{\frac{1}{p}}$$

$$= \left[B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) \right]^{\frac{1}{q}} \left\{ \int_0^\infty \sum_{n=1}^\infty \frac{1}{\ln^\lambda e(\alpha n)^x} \frac{x^{(1-\frac{\lambda}{2})(p-1)}}{n(\ln \alpha n)^{1-\frac{\lambda}{2}}} f^p(x) dx \right\}^{\frac{1}{p}}$$

$$= \left[B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) \right]^{\frac{1}{q}} \left\{ \int_0^\infty \varpi(x) x^{p(1-\frac{\lambda}{2})-1} f^p(x) dx \right\}^{\frac{1}{p}},$$

hence, (11) follows. By Hölder's inequality again, we have

$$\left[\sum_{n=1}^\infty \frac{a_n}{\ln^\lambda e(\alpha n)^x} \right]^q$$

$$= \left\{ \sum_{n=1}^\infty \frac{1}{\ln^\lambda e(\alpha n)^x} \left[\frac{x^{(1-\frac{\lambda}{2})/q}}{(\ln \alpha n)^{(1-\frac{\lambda}{2})/p}} \frac{1}{n^{\frac{1}{p}}} \right] \left[\frac{(\ln \alpha n)^{(1-\frac{\lambda}{2})/p}}{x^{(1-\frac{\lambda}{2})/q}} n^{\frac{1}{p}} a_n \right] \right\}^q$$

$$\leqslant \left\{ \sum_{n=1}^\infty \frac{1}{\ln^\lambda e(\alpha n)^x} \frac{x^{(1-\frac{\lambda}{2})(p-1)}}{n(\ln \alpha n)^{1-\frac{\lambda}{2}}} \right\}^{q-1} \sum_{n=1}^\infty \frac{n^{q-1}}{\ln^\lambda e(\alpha n)^x} \frac{(\ln \alpha n)^{(1-\frac{\lambda}{2})(q-1)}}{x^{1-\frac{\lambda}{2}}} a_n^q$$

$$= \frac{[\varpi(x)]^{q-1}}{x^{\frac{q\lambda}{2}-1}} \sum_{n=1}^\infty \frac{n^{q-1}}{\ln^\lambda e(\alpha n)^x} x^{\frac{\lambda}{2}-1} (\ln \alpha n)^{(1-\frac{\lambda}{2})(q-1)} a_n^q.$$

By Lebesgue term by term integration theorem, we have

$$L_1 \leqslant \left\{ \int_0^\infty \sum_{n=1}^\infty \frac{n^{q-1}}{\ln^\lambda e(\alpha n)^x} x^{\frac{\lambda}{2}-1} (\ln \alpha n)^{(1-\frac{\lambda}{2})(q-1)} a_n^q dx \right\}^{\frac{1}{q}}$$

$$= \left\{ \sum_{n=1}^\infty \left[(\ln \alpha n)^{\frac{\lambda}{2}} \int_0^\infty \frac{x^{\frac{\lambda}{2}-1}}{\ln^\lambda e(\alpha n)^x} dx \right] n^{q-1} (\ln \alpha n)^{q(1-\frac{\lambda}{2})-1} a_n^q \right\}^{\frac{1}{q}}$$

$$= \left\{ \sum_{n=1}^\infty \omega(n) n^{q-1} (\ln \alpha n)^{q(1-\frac{\lambda}{2})-1} a_n^q \right\}^{\frac{1}{q}},$$

and in view of (10), inequality (12) follows. □

3. Main results

We introduce the functions

$$\Phi(x) := x^{p(1-\frac{\lambda}{2})-1}(x>0), \text{ and } \Psi(n) := n^{q-1}(\ln\alpha n)^{q(1-\frac{\lambda}{2})-1}(n\in\mathbf{N}).$$

Observe that $[\Phi(x)]^{1-q} = x^{\frac{q\lambda}{2}-1}$, and $[\Psi(n)]^{1-p} = \frac{1}{n}(\ln\alpha n)^{\frac{p\lambda}{2}-1}$.

THEOREM 1. *If* $0<\lambda\leqslant 2,\ \alpha\geqslant 2,\ p>1,\ \frac{1}{p}+\frac{1}{q}=1,\ f(x),\ a_n\geqslant 0,\ f\in L_{p,\Phi}(R_+), a=\{a_n\}_{n=1}^{\infty}\in l_{q,\Psi},\ ||f||_{p,\Phi}>0$ *and* $||a||_{q,\Psi}>0$, *then we have the following equivalent inequalities:*

$$I := \sum_{n=1}^{\infty}\int_0^{\infty}\frac{a_n f(x)dx}{\ln^{\lambda} e(\alpha n)^x} = \int_0^{\infty}\sum_{n=1}^{\infty}\frac{a_n f(x)dx}{\ln^{\lambda} e(\alpha n)^x} < B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)||f||_{p,\Phi}||a||_{q,\Psi}, \quad (13)$$

$$J = \left\{\sum_{n=1}^{\infty}[\Psi(n)]^{1-p}\left[\int_0^{\infty}\frac{f(x)}{\ln^{\lambda} e(\alpha n)^x}dx\right]^p\right\}^{\frac{1}{p}} < B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)||f||_{p,\Phi}, \quad (14)$$

$$L := \left\{\int_0^{\infty}[\Phi(x)]^{1-q}\left[\sum_{n=1}^{\infty}\frac{a_n}{\ln^{\lambda} e(\alpha n)^x}\right]^q dx\right\}^{\frac{1}{q}} < B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)||a||_{q,\Psi}, \quad (15)$$

where the constant $B(\frac{\lambda}{2},\frac{\lambda}{2})$ *is the best possible in the above inequalities.*

Proof. The two expressions for I in (13) follow from Lebesgue's term by term integration theorem. By (11) and (10), we have (14). By Hölder's inequality, we have

$$I = \sum_{n=1}^{\infty}\left[\Psi^{\frac{-1}{q}}(n)\int_0^{\infty}\frac{1}{\ln^{\lambda} e(\alpha n)^x}f(x)dx\right][\Psi^{\frac{1}{q}}(n)a_n] \leqslant J||a||_{q,\Psi}. \quad (16)$$

Then by (14), we have (13). On the other-hand, assume that (13) is valid. Setting

$$a_n := [\Psi(n)]^{1-p}\left[\int_0^{\infty}\frac{1}{\ln^{\lambda} e(\alpha n)^x}f(x)dx\right]^{p-1}, n\in\mathbf{N},$$

where $J^{p-1} = ||a||_{q,\Psi}$. By (11), we find $J<\infty$. If $J=0$, then (14) is trivially valid; if $J>0$, then by (13), we have

$$||a||_{q,\Psi}^q = J^{q(p-1)} = J^p = I < B(\frac{\lambda}{2},\frac{\lambda}{2})||f||_{p,\Phi}||a||_{q,\Psi},$$

therefore $||a||_{q,\Psi}^{q-1} = J < B(\frac{\lambda}{2},\frac{\lambda}{2})||f||_{p,\Phi}$, that is, (14) is equivalent to (13). On the other-hand, by (10) we have $[\varpi(x)]^{1-q} > [B(\frac{\lambda}{2},\frac{\lambda}{2})]^{1-q}$. Then in view of (12), we have (15). By Hölder's inequality, we find

$$I = \int_0^{\infty}[\Phi^{\frac{1}{p}}(x)f(x)]\left[\Phi^{\frac{-1}{p}}(x)\sum_{n=1}^{\infty}\frac{1}{\ln^{\lambda} e(\alpha n)^x}a_n\right]dx \leqslant ||f||_{p,\Phi}L. \quad (17)$$

Then by (15), we have (13). On the other-hand, assume that (13) is valid. Setting

$$f(x) := [\Phi(x)]^{1-q} \left[\sum_{n=1}^{\infty} \frac{1}{\ln^{\lambda} e(\alpha n)^{x}} a_n \right]^{q-1}, x \in (0, \infty),$$

then $L^{q-1} = ||f||_{p,\Phi}$. By (12), we find $L < \infty$. If $L = 0$, then (15) is trivially valid; if $L > 0$, then by (13), we have

$$||f||_{p,\Phi}^{p} = L^{q} = I < B(\frac{\lambda}{2}, \frac{\lambda}{2}) ||f||_{p,\Phi} ||a||_{q,\Psi},$$

therefore $||f||_{p,\Phi}^{p-1} = L < B(\frac{\lambda}{2}, \frac{\lambda}{2}) ||a||_{q,\Psi}$, that is, (15) is equivalent to (13). Hence, (13), (14) and (15) are equivalent.

For $0 < \varepsilon < \frac{p\lambda}{2}$, setting $\widetilde{f}(x) = x^{\frac{\lambda}{2} + \frac{\varepsilon}{p} - 1}$, $x \in (0,1)$; $\widetilde{f}(x) = 0$, $x \in [1, \infty)$, and $\widetilde{a}_n = \frac{1}{n} (\ln \alpha n)^{\frac{\lambda}{2} - \frac{\varepsilon}{q} - 1}$, $n \in \mathbf{N}$, if there exists a positive number $k (\leqslant B(\frac{\lambda}{2}, \frac{\lambda}{2}))$, such that (13) is valid as we replace $B(\frac{\lambda}{2}, \frac{\lambda}{2})$ with k, then in particular, it follows that

$$\begin{aligned}
\widetilde{I} &:= \sum_{n=1}^{\infty} \int_0^{\infty} \frac{1}{\ln^{\lambda} e(\alpha n)^{x}} \widetilde{a}_n \widetilde{f}(x) dx < k ||\widetilde{f}||_{p,\Phi} ||\widetilde{a}||_{q,\Psi} \\
&= k \left\{ \int_0^1 \frac{dx}{x^{-\varepsilon+1}} \right\}^{\frac{1}{p}} \left\{ \frac{1}{(\ln \alpha)^{\varepsilon+1}} + \sum_{n=2}^{\infty} \frac{1}{n (\ln \alpha n)^{\varepsilon+1}} \right\}^{\frac{1}{q}} \\
&< k \left(\frac{1}{\varepsilon} \right)^{\frac{1}{p}} \left\{ \frac{1}{(\ln \alpha)^{\varepsilon+1}} + \int_1^{\infty} \frac{1}{x (\ln \alpha x)^{\varepsilon+1}} dx \right\}^{\frac{1}{q}} \\
&= \frac{k}{\varepsilon} \left\{ \frac{\varepsilon}{(\ln \alpha)^{\varepsilon+1}} + \frac{1}{(\ln \alpha)^{\varepsilon}} \right\}^{\frac{1}{q}},
\end{aligned} \tag{18}$$

$$\begin{aligned}
\widetilde{I} &= \sum_{n=1}^{\infty} \frac{1}{n} (\ln \alpha n)^{\frac{\lambda}{2} - \frac{\varepsilon}{q} - 1} \int_0^1 \frac{1}{\ln^{\lambda} e(\alpha n)^{x}} x^{\frac{\lambda}{2} + \frac{\varepsilon}{p} - 1} dx \\
&\overset{t = x \ln \alpha n}{=} \sum_{n=1}^{\infty} \frac{1}{n (\ln \alpha n)^{\varepsilon+1}} \int_0^{\ln \alpha n} \frac{1}{(t+1)^{\lambda}} t^{\frac{\lambda}{2} + \frac{\varepsilon}{p} - 1} dt \\
&= B\left(\frac{\lambda}{2} + \frac{\varepsilon}{p}, \frac{\lambda}{2} - \frac{\varepsilon}{p} \right) \sum_{n=1}^{\infty} \frac{1}{n (\ln \alpha n)^{\varepsilon+1}} - A(\varepsilon) \\
&> B\left(\frac{\lambda}{2} + \frac{\varepsilon}{p}, \frac{\lambda}{2} - \frac{\varepsilon}{p} \right) \int_1^{\infty} \frac{1}{y (\ln \alpha y)^{\varepsilon+1}} dy - A(\varepsilon) \\
&= \frac{1}{\varepsilon (\ln \alpha)^{\varepsilon}} B\left(\frac{\lambda}{2} + \frac{\varepsilon}{p}, \frac{\lambda}{2} - \frac{\varepsilon}{p} \right) - A(\varepsilon), \\
A(\varepsilon) &:= \sum_{n=1}^{\infty} \frac{1}{n (\ln \alpha n)^{\varepsilon+1}} \int_{\ln \alpha n}^{\infty} \frac{1}{(t+1)^{\lambda}} t^{\frac{\lambda}{2} + \frac{\varepsilon}{p} - 1} dt.
\end{aligned} \tag{19}$$

We find

$$0 < A(\varepsilon) \leqslant \sum_{n=1}^{\infty} \frac{1}{n(\ln \alpha n)^{\varepsilon+1}} \int_{\ln \alpha n}^{\infty} \frac{1}{t^{\lambda}} t^{\frac{\lambda}{2}+\frac{\varepsilon}{p}-1} dt$$

$$= \frac{1}{\frac{\lambda}{2}-\frac{\varepsilon}{p}} \sum_{n=1}^{\infty} \frac{1}{n(\ln \alpha n)^{\frac{\lambda}{2}+\frac{\varepsilon}{q}+1}} < \infty,$$

and so $A(\varepsilon) = O(1)(\varepsilon \to 0^+)$. Hence by (18) and (19), it follows that

$$\frac{1}{(\ln \alpha)^{\varepsilon}} B\left(\frac{\lambda}{2}+\frac{\varepsilon}{p}, \frac{\lambda}{2}-\frac{\varepsilon}{p}\right) - \varepsilon O(1) < k \left\{ \frac{\varepsilon}{(\ln \alpha)^{\varepsilon+1}} + \frac{1}{(\ln \alpha)^{\varepsilon}} \right\}^{\frac{1}{q}},$$

and $B(\frac{\lambda}{2},\frac{\lambda}{2}) \leqslant k(\varepsilon \to 0^+)$. Hence $k = B(\frac{\lambda}{2},\frac{\lambda}{2})$ is the best value of (13).

By the equivalence of the inequalities, the constant factor $B(\frac{\lambda}{2},\frac{\lambda}{2})$ in (14) and (15) is the best possible. □

REMARK 1. (i) Define the first type half-discrete Hilbert-type operator $T_1 : L_{p,\Phi}(R_+) \to l_{p,\Psi^{1-p}}$ as follows: For $f \in L_{p,\Phi}(R_+)$, we define $T_1 f \in l_{p,\Psi^{1-p}}$ by

$$T_1 f(n) = \int_0^{\infty} \frac{1}{\ln^{\lambda} e(\alpha n)^x} f(x) dx, n \in \mathbf{N}.$$

Then by (14), $||T_1 f||_{p,\Psi^{1-p}} \leqslant B(\frac{\lambda}{2},\frac{\lambda}{2})||f||_{p,\Phi}$ and so T_1 is a bounded operator with $||T_1|| \leqslant B(\frac{\lambda}{2},\frac{\lambda}{2})$. Since by Theorem 1, the constant factor in (14) is best possible, we have $||T_1|| = B(\frac{\lambda}{2},\frac{\lambda}{2})$.

(ii) Define the second type half-discrete Hilbert-type operator $T_2 : l_{q,\Psi} \to L_{q,\Phi^{1-q}}(R_+)$ as follows: For $a \in l_{q,\Psi}$, we define $T_2 a \in L_{q,\Phi^{1-q}}(R_+)$ by

$$T_2 a(x) = \sum_{n=1}^{\infty} \frac{1}{\ln^{\lambda} e(\alpha n)^x} a_n, x \in (0,\infty).$$

Then by (15), $||T_2 a||_{q,\Phi^{1-q}} \leqslant B(\frac{\lambda}{2},\frac{\lambda}{2})||a||_{q,\Psi}$ and so T_2 is a bounded operator with $||T_2|| \leqslant B(\frac{\lambda}{2},\frac{\lambda}{2})$. Since by Theorem 1, the constant factor in (15) is best possible, we have $||T_2|| = B(\frac{\lambda}{2},\frac{\lambda}{2})$.

REMARK 2. For $p = q = 2$, $\lambda = 1$, $\alpha = 2$ in (13), (14) and (15), we have (7) and the following equivalent inequalities:

$$\left\{ \sum_{n=1}^{\infty} \frac{1}{n} \left[\int_0^{\infty} \frac{f(x)}{\ln e(2n)^x} dx \right]^2 \right\}^{\frac{1}{2}} < \pi ||f||, \tag{20}$$

$$\int_0^{\infty} \left[\sum_{n=1}^{\infty} \frac{a_n}{\ln e(2n)^x} \right]^2 dx < \pi^2 \sum_{n=1}^{\infty} n a_n^2. \tag{21}$$

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