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AN APPROXIMATION PROPERTY OF POWER FUNCTIONS

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Abstract. We will solve the inhomogeneous linear first order differential equation of the form, $xy'(x)+\lambda y(x)=\sum_{m=0}^{\infty}a_m(x-c)^m$, and prove an approximation property of power functions. More precisely, we prove the local Hyers-Ulam stability of linear first order differential equation, $xy'(x)+\lambda y(x)=0$, in a special class of analytic functions.

1. Introduction

Let Y and I be a normed space and an open subinterval of $\mathbb{R}$, respectively. If for any function $f:I\to Y$ satisfying the differential inequality

$$\left\|a_n(x)y^{(n)}(x)+a_{n-1}(x)y^{(n-1)}(x)+\cdots+a_1(x)y'(x)+a_0(x)y(x)+h(x)\right\|\leqslant\varepsilon$$

for all $x\in I$ and for some $\varepsilon\geqslant 0$, there exists a solution $f_0:I\to Y$ of the differential equation

$$a_n(x)y^{(n)}(x)+a_{n-1}(x)y^{(n-1)}(x)+\cdots+a_1(x)y'(x)+a_0(x)y(x)+h(x)=0$$

such that $\|f(x)-f_0(x)\|\leqslant K(\varepsilon)$ for any $x\in I$, where $K(\varepsilon)$ depends on ε only, then we say that the above differential equation satisfies the Hyers-Ulam stability (or the local Hyers-Ulam stability if the domain I is not the whole space $\mathbb{R}$). We may apply these terminologies for other differential equations. For more detailed definition of the Hyers-Ulam stability, refer to [2, 4, 8].

Obłoza seems to be the first author who investigated the Hyers-Ulam stability of linear differential equations (see [11, 12]). Here, we introduce a result of Alsina and Ger (see [1]): If a differentiable function $f:I\to\mathbb{R}$ is a solution of the differential inequality $|y'(x)-y(x)|\leqslant\varepsilon$, where I is an open subinterval of $\mathbb{R}$, then there exists a solution $f_0:I\to\mathbb{R}$ of the differential equation $y'(x)=y(x)$ such that $|f(x)-f_0(x)|\leqslant 3\varepsilon$ for any $x\in I$.

This result of Alsina and Ger was generalized by Takahasi, Miura and Miyajima: They proved in [14] that the Hyers-Ulam stability holds for the Banach space valued differential equation $y'(x)=\lambda y(x)$ (see also [10]). For a recent result on the Hyers-Ulam stability for second-order linear differential equations, we refer to [3].

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Using the conventional power series method, the author investigated the general solution of the inhomogeneous Hermite differential equation of the form

$$y''(x) - 2xy'(x) + 2\lambda y(x) = \sum_{m=0}^{\infty} a_m x^m$$

under some specific condition, where λ is a real number and the convergence radius of the power series is positive. This result was applied for proving that every analytic function can be approximated in a neighborhood of 0 by a Hermite function with an error bound expressed by $Cx^2e^{x^2}$ (see [5, 6]).

In §2 of this paper, using an idea from [7], we will investigate the general solution of the inhomogeneous linear differential equation of the first order,

$$xy'(x) + \lambda y(x) = \sum_{m=0}^{\infty} a_m (x-c)^m, \tag{1.1}$$

under the conditions that λ is a complex number with $n-1 < |\lambda| \leqslant n$, the coefficients a_m of the power series are given such that the radius of convergence is at least $\rho > 0$, and c is a real number satisfying $|c| \geqslant \max\left\{\rho, \frac{1}{2}(n+1)\right\}$. Moreover, we prove the local Hyers-Ulam stability of linear first order differential equation in a class of special analytic functions (see the class $\mathscr{C}_K$ in §3).

2. General solution of Eq. (1.1)

The linear first order differential equation

$$xy'(x) + \lambda y(x) = 0 \tag{2.1}$$

is one of the most famous differential equations and frequently appears in applications. As we know, every solution of Eq. (2.1) is called a power function and it has the form $y(x) = \alpha x^{-\lambda}$ (for $x \neq 0$). We note that $x = 0$ is a regular singular point of the differential equation (2.1).

THEOREM 2.1. *Let c, n, and λ be a real number, a positive integer, and a complex number, respectively. Assume that $|c| \geqslant \frac{1}{2}(n+1)$, $n-1 < |\lambda| \leqslant n$, and that the radius of convergence of power series $\sum_{m=0}^{\infty} a_m (x-c)^m$ is at least $\rho > 0$. Let us define*

$$I = \begin{cases} (c-\rho, c+\rho) & (\textit{for } c+\rho \leqslant 0 \textit{ or } c-\rho \geqslant 0), \\ (2c, 0) & (\textit{for } c < 0 \textit{ and } c+\rho > 0), \\ (0, 2c) & (\textit{for } c > 0 \textit{ and } c-\rho < 0). \end{cases}$$

Every solution $y : I \to \mathbb{C}$ of the inhomogeneous differential equation (1.1) can be expressed by

$$y(x) = y_h(x) + \sum_{m=1}^{\infty} b_m (x-c)^m, \tag{2.2}$$

where the coefficients b_m's are defined by

$$b_m = \sum_{i=0}^{m-1} \frac{(-1)^{m-i-1} i!}{m! c^{m-i}} a_i \prod_{j=1}^{m-i-1} (m-j+\lambda) \tag{2.3}$$

for every $m \in \mathbb{N}$ and $y_h(x)$ is a solution of the corresponding homogeneous differential equation (2.1).

Proof. Since each solution of Eq. (1.1) can be expressed as a power series in $x-c$, we may define $y_p(x) = y(x) - y_h(x) = \sum_{m=1}^{\infty} b_m (x-c)^m$ and prove that the function $y_p(x)$ satisfies the inhomogeneous equation (1.1). Indeed, it follows from (2.3) that

$$\begin{aligned} xy_p'(x) + \lambda y_p(x) &= (x-c)y_p'(x) + cy_p'(x) + \lambda y_p(x) \\ &= \sum_{m=1}^{\infty} \big[(m+\lambda)b_m + c(m+1)b_{m+1}\big](x-c)^m + cb_1 \\ &= \sum_{m=0}^{\infty} a_m (x-c)^m \end{aligned}$$

for all $x \in I$, since the relations

$$cb_1 = a_0 \quad \text{and} \quad (m+\lambda)b_m + c(m+1)b_{m+1} = a_m$$

hold for all $m \in \mathbb{N}$. (It is not difficult to prove the last relations by using (2.3).) This implies that $y_p(x)$ is a particular solution of the inhomogeneous equation (1.1). Since every solution of Eq. (1.1) is a sum of a solution $y_h(x)$ of the corresponding homogeneous equation and a particular solution $y_p(x)$ of the inhomogeneous equation, it can be obtained by (2.2).

Since $|c| \geqslant \frac{1}{2}(n+1)$, it holds that

$$\frac{m+n-k-1}{|c|(m-k)} \leqslant \frac{2}{n+1}\left(1+\frac{n-1}{m-k}\right) \leqslant \frac{2}{n+1}\left(1+\frac{n-1}{2}\right) = 1$$

for each $k \in \{0,1,2,\ldots,m-2\}$. Furthermore, since $|\lambda| \leqslant n$, we have

$$\begin{aligned} \frac{i!}{m!|c|^{m-i}} \prod_{j=1}^{m-i-1} |m-j+\lambda| &= \frac{|m-1+\lambda|}{|c|m} \cdot \frac{|m-2+\lambda|}{|c|(m-1)} \cdots \frac{|i+1+\lambda|}{|c|(i+2)} \cdot \frac{1}{|c|(i+1)} \\ &\leqslant \frac{1}{|c|(i+1)} \prod_{k=0}^{m-i-2} \frac{m+n-k-1}{|c|(m-k)} \\ &\leqslant 1 \end{aligned} \tag{2.4}$$

for any $i \in \{0,1,2,\ldots,m-1\}$. Thus, we get

$$|b_m| \leqslant \sum_{i=0}^{m-1} |a_i| \frac{i!}{m!|c|^{m-i}} \prod_{j=1}^{m-i-1} |m-j+\lambda| \leqslant \sum_{i=0}^{m-1} |a_i| \tag{2.5}$$

for all $m \in \mathbb{N}$.

Finally, it follows from (2.5) and [9, Problem 8.8.1 (p)] that

$$\begin{aligned}\limsup_{m\to\infty} \sqrt[m]{|b_m|} &= \limsup_{m\to\infty} \sqrt[m]{\frac{1}{m}|b_m|} \\ &\leqslant \limsup_{m\to\infty} \sqrt[m]{\frac{1}{m}\sum_{i=0}^{m-1}|a_i|} \\ &\leqslant \limsup_{m\to\infty} \sqrt[m]{|a_m|}.\end{aligned}$$

By use of the Cauchy-Hadamard theorem (see [9, Theorem 8.8.2]), the radius of convergence of the power series for $y_p(x)$ is at least ρ. Therefore, $y(x)$ in Eq. (2.2) is well defined on I. □

3. Local Hyers-Ulam stability of Eq. (2.1)

In this section, let c and λ be a fixed nonzero real number and a nonzero complex number, respectively. Assume that ρ, ρ_1, and ρ_2 are given constants satisfying $0 < \rho_1 < \rho$ and $\rho_2 = \min\{\rho_1, 1\}$. For a given $K \geqslant 0$, we denote by $\mathscr{C}_K$ the set of all functions $f : I \to \mathbb{C}$ with the properties (a) and (b):

(a) $f(x)$ is expressible by a power series $\sum_{m=0}^{\infty} b_m(x-c)^m$ whose radius of convergence is at least ρ;

(b) $\sum_{m=0}^{\infty} |a_m\rho_1^m| \leqslant K\left|\sum_{m=0}^{\infty} a_m\rho_1^m\right|$, where $a_m = (m+\lambda)b_m + c(m+1)b_{m+1}$ for any $m \in \{0,1,2,\ldots\}$.

Now we investigate an approximation property of power functions. More precisely, we prove the local Hyers-Ulam stability of the linear first order differential equation (2.1) for the functions in $\mathscr{C}_K$.

THEOREM 3.1. *Given a positive integer n, let c and λ be a real number and a complex number with $|c| \geqslant \max\left\{\rho, \frac{1}{2}(n+1)\right\}$ and $n-1 < |\lambda| \leqslant n$, respectively. Let us define $I = (c-\rho, c+\rho)$ and $I_2 = (c-\rho_2, c+\rho_2)$. If a function $y \in \mathscr{C}_K$ satisfies the differential inequality*

$$\left|xy'(x) + \lambda y(x)\right| \leqslant \varepsilon \tag{3.1}$$

for all $x \in I$ and for some $\varepsilon \geqslant 0$, then there exists a unique solution $y_h : I \to \mathbb{C}$ of the differential equation (2.1) such that

$$\left|y(x) - y_h(x)\right| \leqslant K\varepsilon\frac{\rho_2|x-c|}{\rho_2 - |x-c|}$$

for any $x \in I_2$. In particular, it holds that $y_h \in \mathscr{C}_K$.

Proof. Since y belongs to $\mathscr{C}_K$, it follows from (a) and (b) that

$$\begin{aligned} xy'(x)+\lambda y(x) &= (x-c)y'(x)+cy'(x)+\lambda y(x) \\ &= \sum_{m=0}^{\infty}\big[(m+\lambda)b_m+c(m+1)b_{m+1}\big](x-c)^m \\ &= \sum_{m=0}^{\infty} a_m(x-c)^m \end{aligned} \tag{3.2}$$

for all $x\in I$. By considering (3.1) and (3.2), we have

$$\left|\sum_{m=0}^{\infty} a_m(x-c)^m\right| \leqslant \varepsilon$$

for any $x\in I$. If we substitute $c+\rho_1$ for x in the last inequality, then we obtain

$$\left|\sum_{m=0}^{\infty} a_m\rho_1^m\right| \leqslant \varepsilon.$$

This inequality, together with (b), yields that

$$\sum_{m=0}^{\infty}\left|a_m\rho_1^m\right| \leqslant K\left|\sum_{m=0}^{\infty} a_m\rho_1^m\right| \leqslant K\varepsilon. \tag{3.3}$$

Obviously, it follows from (3.3) that

$$\sum_{i=0}^{m-1}|a_i| = \sum_{i=0}^{m-1}\left|a_i\rho_1^i\right|\frac{1}{\rho_1^i} \leqslant \begin{cases} K\varepsilon\dfrac{1}{\rho_1^{m-1}} & (\text{for } \rho_1\leqslant 1), \\ K\varepsilon & (\text{for } \rho_1>1). \end{cases} \tag{3.4}$$

Now, it follows from Theorem 2.1, (2.4), (3.2), and (3.4) that there exists a solution $y_h : I\to\mathbb{C}$ of the differential equation (2.1) such that

$$\begin{aligned} \left|y(x)-y_h(x)\right| &= \left|\sum_{m=1}^{\infty}(-1)^m(x-c)^m\sum_{i=0}^{m-1}\frac{(-1)^{i+1}i!}{m!c^{m-i}}a_i\prod_{j=1}^{m-i-1}(m-j+\lambda)\right| \\ &\leqslant \sum_{m=1}^{\infty}|x-c|^m\sum_{i=0}^{m-1}|a_i| \\ &\leqslant \begin{cases} K\varepsilon\dfrac{\rho_1|x-c|}{\rho_1-|x-c|} & (\text{for } \rho_1\leqslant 1), \\ K\varepsilon\dfrac{|x-c|}{1-|x-c|} & (\text{for } \rho_1>1) \end{cases} \\ &= K\varepsilon\frac{\rho_2|x-c|}{\rho_2-|x-c|} \end{aligned}$$

for all $x\in I_2$.

Since there exists a complex number α with $y_h(x) = \alpha x^{-\lambda}$, we can find the Taylor series of $y_h(x)$ with center at c:

$$y_h(x) = \sum_{m=0}^{\infty} b_m (x-c)^m$$

with

$$b_m = \frac{(-1)^m \alpha\lambda(\lambda+1)(\lambda+2)\cdots(\lambda+m-1)}{m!c^{\lambda+m}}$$

for each $m \in \{0,1,2,\ldots\}$. Then, we can estimate

$$\lim_{m\to\infty} \left| \frac{b_m}{b_{m+1}} \right| = \lim_{m\to\infty} \frac{|c|(m+1)}{|m+\lambda|} = |c| \geqslant \rho,$$

which implies that the radius of convergence of the Taylor series of $y_h(x)$ is at least ρ. Moreover, we have

$$a_m = (m+\lambda)b_m + c(m+1)b_{m+1} = 0$$

for any $m \in \{0,1,2,\ldots\}$, i.e., (b) is true for the sequence $\{a_m\}$. Consequently, we conclude that $y_h \in \mathscr{C}_K$.

It only remains to prove the uniqueness of y_h. Assume that $y_1, y_2 : I \to \mathbb{C}$ are solutions of the homogeneous differential equation (2.1) satisfying

$$\big|y(x) - y_i(x)\big| \leqslant K\varepsilon \frac{\rho_2|x-c|}{\rho_2 - |x-c|} \quad (i \in \{1,2\})$$

for all $x \in I_2$. Then, for each $i \in \{1,2\}$, there exists a complex number α_i with $y_i(x) = \alpha_i x^{-\lambda}$. Hence, we obtain

$$\big|\alpha_1 x^{-\lambda} - \alpha_2 x^{-\lambda}\big| = \big|y_1(x) - y_2(x)\big| \leqslant 2K\varepsilon \frac{\rho_2|x-c|}{\rho_2 - |x-c|}$$

for any $x \in I_2$. If we put $x = c$ in the last inequality, then we get

$$\frac{|\alpha_1 - \alpha_2|}{|c|^\lambda} = 0,$$

i.e., $\alpha_1 = \alpha_2$, and hence we conclude that $y_1 \equiv y_2$. □

In Theorem 3.1, $|c|$ should be large when $|\lambda|$ is large . It is an open problem whether $|c|$ can be chosen as small as we wish, even if $|\lambda|$ is large.

COROLLARY 3.2. *Given an $n \in \mathbb{N}$, let c, λ, and ρ be a real number, a complex number, and a positive constant with $|c| \geqslant \frac{1}{2}(n+1)$, $n-1 < |\lambda| \leqslant n$, and $\rho \leqslant \frac{1}{2}(n+1)$, respectively. If a function $y \in \mathscr{C}_K$ satisfies the differential inequality (3.1) for all $x \in I$ and for some $\varepsilon \geqslant 0$, where $I = (c-\rho, c+\rho)$, then there exists a unique complex number α such that*

$$\big|y(x) - \alpha x^{-\lambda}\big| = O\big(|x-c|\big)$$

as $x \to c$.

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