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POLAROID AND $p-*-$PARANORMAL OPERATORS

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Abstract. In this paper we define the $p-*-$ paranormal operators and we show some properties of this class of operators. We also prove that a $p-*-$paranormal operator is polaroid and we show a necessary and sufficient condition for the Riesz idempotent associated to a non-zero isolated point of the spectrum of a $p-*-$paranormal operator to be self-adjoint. Finally, we show that generalized a-Weyl's theorem holds for $p-*-$ paranormal operators and we present some finite operators.

1. Introduction and preliminaries

Let us denote by H the complex Hilbert space and with $B(H)$ the space of all bounded linear operators defined in Hilbert space H. In the following we will mention some known classes of operators defined in Hilbert space H. Let T be an operator in $B(H)$. An operator is said to be positive (denoted $T \geqslant 0$) if $(Tx,x) \geqslant 0$ for all $x \in H$. The operator T is said to be a p-hyponormal operator if and only if $(T^*T)^p \geqslant (TT^*)^p$ for a positive number p. In [31] is defined the class of log-hyponormal operators as follows: T is a log-hyponormal operator if it is invertible and satisfies the following relation $\log T^*T \geqslant \log TT^*$. Class of p-hyponormal operators and class of log-hyponormal operators were defined as extension class of hyponormal operators, i.e, $T^*T \geqslant TT^*$. An operator T is said to be M-paranormal if $M||T^2x|| \geqslant ||Tx||^2$, for every unit vector $x \in H$ (see [7]). An operator T is called M-hyponormal if it satisfies the following relation: $||T^*(x)|| \leqslant M||Tx||$ (see [8]). An operator T belong to the class (M,k) (see [16]) if it satisfies the following relation: $T^{*k}T^k \geqslant (T^*T)^k$, for $k \geqslant 2$ and $T \in (M,k)^*$ if $T^{*k}T^k \geqslant (TT^*)^k$, for $k \geqslant 1$. It is well known that every p-hyponormal operator is a q- hyponormal operator for $p \geqslant q > 0$, by the Löwner-Heinz theorem "$A \geqslant B \geqslant 0$ ensures $A^\alpha \geqslant B^\alpha$ for any $\alpha \in [0,1]$", and every invertible p-hyponormal operator is a log-hyponormal operator since log is an operator monotone function. An operator T is paranormal if $||T^2x|| \geqslant ||Tx||^2$. It is also well known that there exists a hyponormal operator T such that T^2 is not a hyponormal operator (see [22]). In [18] authors, Furuta, Ito and Yamazaki introduced the A class of operators, respectively $A(k)$ class of operators defined as follows: for each $k > 0$, an operator T is $A(k)$ class operator if

$$\left(T^*|T|^{2k}T\right)^{\frac{1}{k+1}} \geqslant |T|^2, \tag{1}$$

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(for $k = 1$ it defines the A class operators) which includes the class of log-hyponormal operators (see Theorem 2, in [18]) and is included in the class of paranormal operators, in case where $k = 1$ (see Theorem 1 in [18]). In the same paper were introduced the absolute-k-paranormal operators as follows: for each $k > 0$, an operator T is absolute-k-paranormal operator if

$$\left\| |T|^k Tx \right\| \geqslant ||Tx||^{k+1}, \tag{2}$$

for every unit vector $x \in H$.

The $A(k)$ class of operators is included in the absolute-k-paranormal operators for any $k > 0$ (see Theorem 2 in [18]).

$$\left(T^*|T|^{2k}T\right)^{\frac{1}{k+1}} \geqslant |T^*|^2, \tag{3}$$

In case when $k = 1$ it defines the A^* class operators. In paper [30] were introduced the absolute-k^*-paranormal class of operators as follows:

$$A_{(k^*)}P = \{T \in H : |||T|^k Tx|| \geqslant ||T^*x||^{k+1}, x \in H, ||x|| = 1\}$$

for any $k > 0$. We will also show the behavior of the $M - A(k^*)$-class of operators which is defined as follows: for each $k > 0, M > 0$ an operator T is $M - A(k^*)$ class operator if

$$\left(T^*|T|^{2k}T\right)^{\frac{1}{k+1}} \geqslant M|T^*|^2,$$

and absolute-k^*-M-paranormal operators, if for each $k > 0$, $M > 0$

$$\left\| |T|^k Tx \right\| \geqslant M||T^*x||^{k+1},$$

for every unit vector $x \in H$.

an operator T is said to be $p - * -$ paranormal if,

$$|||T|^p U|T|^p x|| \cdot ||x|| \geqslant |||T|^p U^* x||^2, \tag{4}$$

for every vector $x \in H$, where $T = U|T|$ is polar decomposition of the operator T. It easy to see that by relation (4) is defined a class of operators which is extension of the $*-$ paranormal operators (for $p = 1$) (see [6], [5]). If $T \in B(\mathscr{H})$, we shall write $N(T)$ and $R(T)$ for the null space and the range of T, respectively. Also, let $\sigma(T)$ and $\sigma_a(T)$ denote the spectrum and the approximate point spectrum of T, respectively. An operator T is called Fredholm if $R(T)$ is closed, $\alpha(T) = \dim N(T) < \infty$ and $\beta(T) = \dim \mathscr{H}/R(T) < \infty$. Moreover if $i(T) = \alpha(T) - \beta(T) = 0$, then T is called Weyl. The essential spectrum $\sigma_e(T)$ and the Weyl spectrum $\sigma_W(T)$ are defined by

$$\sigma_e(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Fredholm}\}$$

and

$$\sigma_W(A) = \{\lambda \in \mathbb{C} : A - \lambda \text{ is not Weyl}\},$$

respectively. It is known that $\sigma_e(T) \subset \sigma_W(T) \subset \sigma_e(T) \cup \text{acc}\ \sigma(T)$ where we write acc K for the set of all accumulation points of $K \subset \mathbb{C}$. If we write iso $K = K \setminus \text{acc}\ K$, then we let

$$\pi_{00}(T) = \{\lambda \in \text{iso}\ \sigma(T) : 0 < \alpha(T-\lambda) < \infty\}.$$

We say that Weyl's theorem holds for T if

$$\sigma(T) \setminus \sigma_W(T) = \pi_{00}(T).$$

The operator T is called Browder if is Fredholm of finite ascent and descent. The Browder spectrum of T is given by $\sigma_b(T) = \{\lambda \in \mathbb{C} : T-\lambda \text{ is not Browder}\}$. We say that Browder's theorem holds for T if

$$\sigma(T) \setminus w(T) = p_{00}(T).$$

In [35], H. Weyl proved that Weyl's theorem holds for hermitian operators. Weyl's theorem has been extended from hermitian operators to hyponormal operators [15], algebraically hyponormal operators [23], p-hyponormal operators [14]and quasi-$*$-class A [38].

More generally, M. Berkani investigated generalized Weyl's theorem which extends Weyl's theorem, and proved that generalized Weyl's theorem holds for hyponormal operators ([10, 11, 12]). In a recent paper [25] the author showed that generalized Weyl's theorem holds for (p,k)-quasi-hyponormal operators. Recently, X. Cao, M. Guo and B. Meng [13] proved Weyl type theorems for p-hyponormal operators. M. Berkani investigated B-Fredholm theory as follows (see [1, 10, 11, 12]). An operator T is called B-Fredholm if there exists $n \in \mathbb{N}$ such that $R(T^n)$ is closed and the induced operator

$$T_{[n]} : R(T^n) \ni x \to Tx \in R(T^n)$$

is Fredholm, i.e., $R(T_{[n]}) = R(T^{n+1})$ is closed, $\alpha(T_{[n]}) = \dim N(T_{[n]}) < \infty$ and $\beta(T_{[n]}) = \dim R(T^n)/R(T_{[n]}) < \infty$. Similarly, a B-Fredholm operator T is called B-Weyl if $i(T_{[n]}) = 0$. The following results is due to M. Berkani and M. Sarih [12].

PROPOSITION 1.1. *Let $T \in B(\mathscr{H})$.*

(1) *If $R(T^n)$ is closed and $T_{[n]}$ is Fredholm, then $R(T^m)$ is closed and $T_{[m]}$ is Fredholm for every $m \geqslant n$. Moreover,* ind $T_{[m]} =$ ind $T_{[n]} (=$ ind $T)$.

(2) *An operator T is B-Fredholm (B-Weyl) if and only if there exist T-invariant subspaces $\mathscr{M}$ and $\mathscr{N}$ such that $T = T|\mathscr{M} \oplus T|\mathscr{N}$ where $T|\mathscr{M}$ is Fredholm (Weyl) and $T|\mathscr{N}$ is nilpotent.*

The B-Weyl spectrum $\sigma_{BW}(T)$ are defined by

$$\sigma_{BW}(T) = \{\lambda \in \mathbb{C} : T-\lambda \text{ is not } B\text{-Weyl}\} \subset \sigma_W(T).$$

We say that generalized Weyl's theorem holds for T if

$$\sigma(T) \setminus \sigma_{BW}(T) = E(T)$$

where $E(T)$ denotes the set of all isolated points of the spectrum which are eigenvalues (no restriction on multiplicity). Note that, if the generalized Weyl's theorem holds for T, then so does Weyl's theorem [11]. Recently in [10] M. Berkani and A. Arroud showed that if T is hyponormal, then generalized Weyl's theorem holds for T.

We define $T \in SF_{+}^{-}$ if $R(T)$ is closed, $\dim N(T) < \infty$ and $\operatorname{ind} T \leqslant 0$. Let $\pi_{00}^{a}(T)$ denote the set of all isolated points λ of $\sigma_a(T)$ with $0 < \dim\ker(T-\lambda) < \infty$. Let $\sigma_{SF_{+}^{-}}(T) = \{\lambda \mid T-\lambda \notin SF_{+}^{-}\} \subset \sigma_W(T)$. We say that a-Weyl's theorem holds for T if

$$\sigma_a(T) \setminus \sigma_{SF_{+}^{-}}(T) = \pi_{00}^{a}(T).$$

V. Rakočević [32, Corollary 2.5] proved that if a-Weyl's theorem holds for T, then Weyl's theorem holds for T.

It is easily seen that quasi-nilpotent operators do not satisfy a-Weyl's theorem, in general. For instance, if

$$T(x_1, x_2, \cdots) = (0, \frac{x_2}{2}, \frac{x_3}{3}, \cdots),\ (x_n) \in l^2(\mathscr{N}).$$

then T is quasi-nilpotent but a-Weyl's theorem fails for T , since $\sigma(T) = \sigma_a(T) = \sigma_{SF_{-}^{+}}(T) = \{0\} = \pi_{00}^{a}(T)$.

We define $T \in SBF_{+}^{-}$ if there exists a positive integer n such that $R(T^n)$ is closed, $T_{[n]} : R(T^n) \ni x \to Tx \in R(T^n)$ is upper semi-Fredholm (i.e., $R(T_{[n]}) = R(T^{n+1})$ is closed, $\dim N(T_{[n]}) = \dim N(T) \cap R(T^n) < \infty$) and $0 \geqslant \operatorname{ind} T_{[n]} (= \operatorname{ind} T)$ ([12]). We define $\sigma_{SBF_{+}^{-}}(T) = \{\lambda \mid T-\lambda \notin SBF_{+}^{-}\} \subset \sigma_{SF_{+}^{-}}(T)$. Let $E^a(T)$ denote the set of all isolated points λ of $\sigma_a(T)$ with $0 < \dim\ker(T-\lambda)$. We say that generalized a-Weyl's theorem holds for T if

$$\sigma_a(T) \setminus \sigma_{SBF_{+}^{-}}(T) = E^a(T).$$

M. Berkani and J.J. Koliha [11] proved that if generalized a-Weyl's theorem holds for T, then a-Weyl's theorem holds for T.

Let μ be an isolated point of the spectrum of T. Then the Riesz idempotent E of T with respect to μ is defined by

$$E := \frac{1}{2\pi i}\int_{\partial D} (\mu I - T)^{-1} d\mu,$$

where D is a closed disk centered at μ which contains no other points of the spectrum of T. In [33], Stampfli showed that if T satisfies the growth condition G_1, then E is self-adjoint and $E(H) = N(T-\mu)$. Recently, Jeon and Kim [24] and A. Uchiyama [34] obtained Stampfli's result for quasi-class A operators and paranormal operators. In general even though T is a paranormal operator, the Riesz idempotent E of T with respect to $\mu \in \operatorname{iso}\sigma(T)$ is not necessary self-adjoint.

In this paper we show some properties of $p-*-$paranormal operators. We also prove that a $p-*-$paranormal operator is polaroid and we show a necessary and sufficient condition for the Riesz idempotent associated to a non-zero isolated point of the spectrum of a $p-*-$paranormal operator to be self-adjoint. Finally, We show that generalized a-Weyl's theorem holds for $p-*-$ paranormal operators and we present some finite operators.

2. $p-*-$ paranormal operators

In this section we will show the behavior of the class $p-*-$ paranormal operators.

LEMMA 2.1. *Every* $1-*-$*paranormal operator is* $*-$*paranormal operator.*

Proof. Let us consider operator T which is $p-*-$paranormal operator, for every $p \geqslant 1$. Let $p=1$, then it follows that the following relation is valid:

$$|||T|U|T|x|| \geqslant |||T|U^*x||^2, \tag{5}$$

for every unit vector $x \in H$. The left hand side of the relation (5) is exactly $||T^2x||$, really

$$\begin{aligned} |||T|U|T|x||^2 &= (|T|U|T|x, |T|U|T|x) = (|T|^2U|T|x, Tx) \\ &= (T^*TTx, Tx) = (T^2x, T^2x) = ||T^2x||^2. \end{aligned}$$

And from the right side of the relation (5) we get the following relation:

$$|||T|U^*x||^2 \geqslant ||U|T|U^*x||^2 = ||T^*x||^2,$$

from which follows that T is $*-$paranormal operator. □

In what follows we will give a necessary and sufficient condition under which an operator $T \in B(H)$ is $p-*-$paranormal operator.

LEMMA 2.2. [30] *Let* T *be an operator with polar decomposition* $T = U|T|$. *Then* T *is* $p-*-$ *paranormal operator if and only if*

$$|T|^pU^*|T|^{2p}U|T|^p + 2\lambda|T|^{2p} + \lambda^2 \geqslant 0,$$

for all real $\lambda \in \mathbb{R}$.

THEOREM 2.3. *If* $*-$ *paranormal operator* T *double commutes with operator* S *from class* $(M,2)^*$, *then the product* TS *is* $*-$*paranormal operator.*

Proof. Let us denote by $\{E(t)\}$ the resolution of the identity for the self-adjoint operator SS^* (see [21]). By hypothesis T^*T and $T^{*2}T^2$ both commutes with every element of the $\{E(t)\}$. Since S is from class $(M,2)^*$, it follows that the following relation

$$S^{*2}S^2 \geqslant (SS^*)^2,$$

holds. On the other hand for every $\lambda > 0$ we get this estimation:

$$(TS)^{*2}(TS)^2 - 2\lambda(TS)(TS)^* + \lambda^2 = (T^{*2}T^2)(S^{*2}S^2) - 2\lambda(TT^*)(SS^*) + \lambda^2 \geqslant$$

$$(T^{*2}T^2)(SS^*)^2 - 2\lambda(TT^*)(SS^*) + \lambda^2 = \int_0^\infty (t^2T^{*2}T^2 - 2\lambda tTT^* + \lambda^2)dE(t) \geqslant 0,$$

from fact that T is $*-$paranormal operator. Hence TS is $*-$paranormal operator. □

LEMMA 2.4. [17] *If $T \in B(H)$ satisfies relation $|T^2| \geqslant |T^*|^2$, (or if T belongs to class $A(1^*)$) then it is $*-$paranormal operator.*

COROLLARY 2.5. *If T is operator from $(M,2)^*$ which double commutes with operator S from class $(M,2)^*$, then the product TS is $*-$paranormal operator.*

Proof. From Proposition 4.5 in [8], it follows that T is $*-$ paranormal operator. Then proof of the corollary follows directly from Theorem 2.3. □

3. Riesz idempotent for $p-*-$paranormal operators

In [27] the author introduced the class of quasi-$*$-paranormal operator.

DEFINITION 3.1. An operator T is called quasi $*$- paranormal if it satisfies the following inequality:

$$||T^*Tx||^2 \leqslant ||T^3x|| ||Tx||$$

for all unit vector $x \in H$.

It is well known that for any operators A,B and C,

$$A^*A - 2\lambda B^*B + \lambda^2 C^*C \geqslant 0 \text{ for all } \lambda > 0 \Leftrightarrow ||Bx||^2 \leqslant ||Ax|| ||Cx|| \text{ for all } x \in H.$$

Thus we have. An operator $T \in B(H)$ is quasi $*$-paranormal if and only if

$$T^*(T^{*2}T^2 - 2\lambda TT^* + \lambda^2)T \geqslant 0, \text{ for all } \lambda > 0.$$

It is well known that T is $*$-paranormal if and only if

$$T^{*2}T^2 - 2\lambda TT^* + \lambda^2 \geqslant 0, \text{ for all } \lambda > 0.$$

Thus every $*$-paranormal operator is quasi-$*$-paranormal and we have the following implications:

$$\text{Hyponormal} \Rightarrow *\text{-paranormal} \Rightarrow \text{quasi} *\text{-paranormal}.$$

The following proposition is proved in [27]. For the convenience of the reader I will give the proof.

PROPOSITION 3.2. *A quasi $*$-paranormal operator T is normaloid.*

Proof. It suffices to show

$$||T^{2m}|| = ||T||^{2m} \tag{$*$}$$

for $m = 1,2,3,\ldots$. We use the induction argument in the proof. First we prove (*) for $m = 1$. Since T is quasi $*$-paranormal,

$$||T||^4 = ||T^*T||^2 \leqslant ||T^3|| ||T|| \leqslant ||T^2|| ||T||^2 \leqslant ||T||^4.$$

Hence $||T||^2 = ||T^2||$. Now assume that (*) is true for $m=k$. Since

$$||T^3x||^2 + \lambda^2||Tx||^2 \geqslant 2\lambda||T^*Tx||^2,$$

we have

$$||T^{2(k+1)}x|| + \lambda^2||T^{2k}x|| \geqslant 2\lambda||T^*T^{2k}x||^2$$
$$\Rightarrow ||T^{2(k+1)}||^2 + \lambda^2||T^{2k}||^2 \geqslant 2\lambda||T^*T^{2k}||^2$$
$$\Rightarrow ||T||^{2(2k-1)}[||T^{2(k+1)}||^2 + \lambda^2||T^{2k}||^2] \geqslant 2\lambda||T||^{2(2k-1)}||T^*T^{2k}||^2 \geqslant 2\lambda||T^{*2k}T^{2k}||^2$$
$$\Rightarrow ||T||^{2(2k-1)}[||T^{2(k+1)}||^2 + \lambda^2||T^{2k}||^2] \geqslant 2\lambda||T^{2k}||^4.$$

Since (*) is true for $m=k$, we find

$$[||T^{2(k+1)}||^2 + \lambda^2||T||^{4k}] \geqslant 2\lambda||T||^{4k+2}.$$

Let $\lambda = ||T||^2$. Then the last inequality gives

$$||T^{2(k+1)}||^2 + ||T||^4||T||^{4k} \geqslant 2||T||^{4k+4}$$

Hence

$$2||T||^{4k+4} \geqslant ||T^{2(k+1)}||^2 + ||T||^{4k+4} \geqslant 2||T||^{4k+4}.$$

Clearly $||T||^{2(k+1)} = ||T^{2(k+1)}||$.This proves the result. □

PROPOSITION 3.3. *Every $p-*-$paranormal operator T is normaloid.*

Proof. Let T be $p-*-$paranormal with $T=U|T|$ the polar decomposition of the operator T. Let $T_p = U|T|^p$. Then T_p is $*$-paranormal [5], and hence is normaloid. It is shown [19, Lemma 4.3] that if T_p is normaloid for $p>1$, then $T=U|T|$ is normaloid . Now assume that $p=1$, then T is a $*$-paranormal operator. Hence Proposition 3.2 implies that T is normaloid. □

PROPOSITION 3.4. [30] *If T is an invertible $p-*-$paranormal operator, then T^{-1} is a $p-*-$paranormal operator.*

PROPOSITION 3.5. *if T is a $p-*-$paranormal operator on H and $\sigma(T)$ lies on the unit circle, then T is an invertible isometry.*

Proof. T being invertible, both T and T^{-1} are normaloid, being $p-*-$ paranormal. Hence $||T|| = ||T^{-1}|| = 1$ and $||x|| = ||T^{-1}Tx|| \leqslant ||Tx|| \leqslant ||x||$, for all $x \in H$. This shows that T is an invertible isometry. □

Recall that an operator $T \in B(H)$ is called isoloid if every isolated point of the spectrum of T is an eigenvalue of T. An operator T is said to be polaroid if points in $iso\sigma(T)$ are poles of the resolvent of T. It is clear that if T is polaroid, then T is isoloid. Let $P_T\{\lambda\}$ denote the algebraic eigenprojection associated with $\{\lambda\}$ whenever $\lambda \in \mathbb{C}$ is an isolated point of $\sigma(T)$.

THEOREM 3.6. *Let $T \in B(H)$ be $p-*-$paranormal. Then every isolated point of the spectrum of T is a simple pole of the resolvent, $R_\lambda(T)$ of T, i.e., T is polaroid.*

Proof. Assume that μ is an isolated point of $\sigma(T)$. We consider two case:

Case 1. If $\mu = 0$, consider the $p-*-$paranormal operator $T/R(P_T(0))$. Since $\sigma(T/R(P_T(0))) = 0$, $T/R(P_T(0)) = 0$. Thus 0 is a simple pole of the resolvent of T [4, p. 306].

Case 2. If $\mu \neq 0$, consider $T_1 = \frac{1}{\mu}(T/R(P_T(\mu)))$. Then T_1 is $*$-paranormal with $\sigma(T_1) = \{1\}$. Hence T_1 and T_1^{-1} both are isometries and $||T_1^n|| = 1$ for $n = \pm1, \pm2, \pm3, \ldots$. Moreover,we have $T = I + Q$, where Q is some quasinilpotent operator. It follows from [20, Theorem 3] that $T_1 = I$. Therefore $(T_1 - \mu)R(P_T(\mu)) = 0$ and μ is a simple pole of the resolvent of T. Hence T is polaroid. □

COROLLARY 3.7. *Let $T \in B(H)$ be $p-*-$paranormal. Then T is isoloid.*

THEOREM 3.8. *Let $T \in B(H)$ be $p-*-$paranormal. Assume $0 \neq \mu \in \operatorname{iso}\sigma(T)$ and E is the Riesz idempotent of T with respect to μ. Then E is self-adjoint.*

Proof. Since E is the Riesz idempotent of T with respect to μ and T is $p-*-$paranormal, it results from Proposition 3.6 that

$$R(E) = N(T-\mu) \,\text{and}\, N(E) = R(T-\mu).$$

Since

$$N(T-\mu) \subseteq N(T^* - \overline{\mu}),$$

we have $N(T-\mu)$ and $R(T-\mu)$ are orthogonal. Hence $R(E)^\perp = N(E)$, and so E is self-adjoint. □

4. Finite operators

Let $A, B \in B(H)$. We define the generalized derivation $\delta_{A,B} : B(H) \mapsto B(H)$ by $\delta_{A,B}(X) = AX - XB$, we note $\delta_{A,A} = \delta_A$. If the inequality $||T - (AX - XA)|| \geqslant ||T||$ holds for all $X \in B(H)$ and for all $T \in ker\delta_A$, then we say that the range of δ_A is orthogonal to the kernel of δ_A in the sense of Birkhoff. The operator $A \in B(H)$ is said to be finite [36] if

$$||I - (AX - XA)|| \geqslant 1 \tag{6}$$

for all $X \in B(H)$, where I is the identity operator. J.P.Williams [36] has shown that the class of finite operators contains every normal, hyponormal operators. In [25], J.P.Williams results are generalized to a more classes of operators containing the classes of normal and hyponormal operators. The well-known inequality (6), due to J.P. Williams [36] is the starting point of the topic of commutator approximation (a Topic which has its roots in quantum theory [37]).

Let $A \in B(H)$, the approximate reduced spectrum of A, $\sigma_{ra}(A)$, is the set of scalars λ for which there exists a normed sequence $\{x_n\}$ in H satisfying

$$(A-\lambda I)x_n \to 0, (A-\lambda I)^* x_n \to 0.$$

In this section we present some new classes of finite operators containing the classes of normal and hyponormal operators. Recall that an operator $A \in B(H)$ is said to be spectraloid if $\omega(A) = r(A)$, where $r(A)$ (resp. $\omega(A)$) is the spectral radius (resp. numerical radius) of A. We have

hyponormal $\subset$ $p-$hyponormal $\subset$ paranormal $\subset$ normaloid $\subset$ spectraloid.

LEMMA 4.1. [25] *Let $A \in \mathscr{L}(H)$. Then $\partial W(A) \cap \sigma(A) \subset \sigma_{ar}(A)$, where $W(A)$ is the numerical range of the operator A.*

LEMMA 4.2. [25] *If $\sigma_{ar}(A) \neq \phi$, then A is finite.*

THEOREM 4.3. *Let $A \in \mathscr{L}(H)$ be spectraloid. Then A is finite.*

Proof. Since A is spectraloid, we have $\omega(A) = r(A)$. Then there exists $\lambda \in \sigma(A) \subset \overline{W(A)}$ such that $|\lambda| = \omega(A)$, where $W(A)$ is the numerical range of A. Thus $\lambda \in \partial W(A)$. This implies that $\partial W(A) \cap \sigma(A) \neq \emptyset$. Now by applying Lemma 4.2, we get the result. □

COROLLARY 4.4. *Let $A \in B(H)$. If A is $p-*-$ paranormal then A is finite.*

Proof. Since A is $p-*-$ paranormal, it is normaloid and a normaloid operator is spectraloid, it suffices to apply Theorem 4.3. □

COROLLARY 4.5. *The following operators are finite:*
1. Paranormal operators
2. Quasi-$*$-paranormal operators
3. $p-*-$ paranormal operators

Theorem 4.3 extends the results showed by S.Mecheri [26] and P. J. Maher [28] for finite paranormal operators and prove the finiteness of $p-*-$paranormal operators.

5. Weyl type Theorems

An operator $T \in B(H)$ satisfies a-Browder's theorem if $\sigma_{ea}(T) = \sigma_{ab}(T)$ (where $\sigma_{ab}(T) = \{\sigma_a(T+K) : TK = KT \text{ and } K \text{ is a compact operator}\}$) and T satisfies generalized a-Browder's theorem if $\sigma_{SBF_+^-}(T) = \sigma_{ap}(T) \setminus \pi^a(T)$. Let $T \in B(X)$, where X is an infinite dimensional complex Banach space. The operator T is said to have the single-valued extension property (or SVEP) if for every open subset G of $\mathbb{C}$ and any analytic function $f : G \to X$ such that $(T-z)f(z) \equiv 0$ on G, we have $f(z) \equiv 0$ on

G. If a Banach space operator T has SVEP (everywhere), the single-valued extension property, then T and T^* satisfy Browder's (equivalently, generalized Browder's) theorem and a-Browder's (equivalently, generalized a- Browder's) theorem. A sufficient condition for an operator T satisfying Browder's (generalized Browder's) theorem to satisfy Weyl's (resp., generalized Weyl's) theorem is that T is polaroid. Observe that if $T \in B(H)$ has SVEP, then $\sigma(T) = \sigma_a(T^*)$. Hence, if T has SVEP and is polaroid, then T^* satisfies generalized a-Weyl's (so also, a-Weyl's) theorem [2, Theorem 2.14, Theorem 2.6].

THEOREM 5.1. *Let* $T \in B(H)$.

i) If T^* *is a* $p-*-$*paranormal operator, then generalized a-Weyl's theorem holds for* T.

ii) If T *is a* $p-*-$*paranormal operator, then generalized a-Weyl's theorem holds for* T^*.

Proof. (i) It is well known that T is polaroid if and only if T^* is polaroid [2, Theorem 2.11]. Now since a $p-*-$paranormal operator is polaroid by Theorem 3.6 and has SVEP [9], [2, Theorem 3.10] gives us the result of the theorem. For (ii) we can also apply [2, Theorem 3.10]. □

Since the polaroid condition entails $E(T) = \pi(T)$ and the SVEP for T entails that generalized Browder's theorem holds for T [3, Theorem 3.2], i.e. $\sigma_{BW}(T) = \sigma_D(T)$, where $\sigma_D(T)$ denotes the Drazin spectrum. Therefore, $E(T) = \pi(T) = \sigma(T) \setminus \sigma_D(T) = \sigma(T) \setminus \sigma_{BW}(T)$. Thus we have the following corollary.

COROLLARY 5.2. *If* T *is* $p-*-$*paranormal, then also* T *satisfies generalized Weyl's theorem.*

REMARK 5.3. 1. Recall [2] that if T is polaroid, then T satisfies generalized Weyl's theorem (resp. generalized a-Weyl's) theorem if and only if T satisfies Weyl's theorem (resp. a-Weyl's theorem). Hence if T is a $p-*-$paranormal operator, the above equivalences hold.

2. Let $f(z)$ be an analytic function on $\sigma(T)$. If T is polaroid, then $f(T)$ is polaroid too [2].

i) If T^* is $p-*-$paranormal, then $f(T)$ satisfies generalized a-Weyl's theorem. Indeed, since T^* is polaroid, the result holds by [2, Theorem 3.12]

ii) If T is $p-*-$paranormal, then $f(T^*)$ satisfies generalized a-Weyl's theorem. Indeed, since T is polaroid, the result holds by [2, Theorem 3.12].

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