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ON THE RATIO BETWEEN SUCCESSIVE RADII OF A SYMMETRIC CONVEX BODY

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Abstract. In this note we study the upper bound for the ratio between the so called successive inner and outer radii of a 0-symmetric convex body K. This problem was studied by Perel'man and Pukhov and it is a natural generalization of the classical results of Jung and Steinhagen.

1. Introduction

Let $\mathcal{K}^n$ be the set of all convex bodies, i.e., compact convex sets, in the n-dimensional Euclidean space $\mathbb{R}^n$. The subset of $\mathcal{K}^n$ consisting of all 0-symmetric convex bodies, i.e., such that if $x \in K$ then $-x \in K$, is denoted by $\mathcal{K}_0^n$. Let $\langle \cdot,\cdot \rangle$ and $|\cdot|_2$ be the standard inner product and Euclidean norm in $\mathbb{R}^n$, respectively, and B_n the n-dimensional Euclidean unit ball.

The set of all i-dimensional linear subspaces of $\mathbb{R}^n$ is denoted by $\mathcal{L}_i^n$. For $L \in \mathcal{L}_i^n$, $L^\perp$ denotes its orthogonal complement and for $K \in \mathcal{K}^n$ and $L \in \mathcal{L}_i^n$ the orthogonal projection of K onto L is denoted by $K|L$. We use e_i for i-th canonical unit vector in $\mathbb{R}^n$, and with $\mathrm{lin}\{u_1,\dots,u_m\}$ we represent the linear hull of the vectors $\{u_1,\dots,u_m\}$. Finally, for $S \subset \mathbb{R}^n$, we denote by $\mathrm{conv}\, S$ its convex hull, and we write $\mathrm{relbd}\, S$ to denote the relative boundary of S, i.e., the boundary of S relative to its affine hull $\mathrm{aff}\, S$.

The width in the (unit) direction u, the diameter, the minimal width, the circumradius and the inradius of a convex body K are denoted by $\omega(K,u)$, $\mathrm{D}(K)$, $\omega(K)$, $\mathrm{R}(K)$ and $\mathrm{r}(K)$, respectively. For more information on these functionals and their properties we refer to [3, pp. 56–59]. If f is a functional on $\mathcal{K}^n$ depending on the dimension in which a convex body K is embedded, and if K is contained in an affine subspace A, then we write $f(K;A)$ to denote that f has to be evaluated with respect to the subspace A. With this notation we define the following successive outer and inner radii.

DEFINITION 1.1. For $K \in \mathcal{K}^n$ and all $i = 1,\dots,n$ let

$$\mathrm{R}_i(K) = \min_{L \in \mathcal{L}_i^n} \mathrm{R}(K|L) \quad \text{and} \quad \widetilde{\mathrm{r}}_i(K) = \max_{L \in \mathcal{L}_i^n} \mathrm{r}(K|L;L).$$

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If we replace in the definition of $\widetilde{\mathrm{r}}_i$ projections by sections, we obtain another serie of successive inner radii:

DEFINITION 1.2. For $K \in \mathscr{K}^n$ and all $i = 1, \dots, n$ let

$$\mathrm{r}_i(K) = \max_{L \in \mathscr{L}_i^n} \max_{x \in L^\perp} \mathrm{r}\big(K \cap (x+L); x+L\big).$$

Observe that $\mathrm{R}_i(K)$ is the smallest radius of solid cylinder containing K with i-dimensional spherical cross section, whereas $\mathrm{r}_i(K)$ is the radius of the greatest i-dimensional ball contained in K. We obviously have

$$\mathrm{R}_n(K) = \mathrm{R}(K), \quad \mathrm{R}_1(K) = \frac{\omega(K)}{2},$$
$$\mathrm{r}_n(K) = \widetilde{\mathrm{r}}_n(K) = \mathrm{r}(K) \text{ and } \mathrm{r}_1(K) = \widetilde{\mathrm{r}}_1(K) = \frac{\mathrm{D}(K)}{2}.$$

It is clear that the outer radii are increasing in i, whereas the inner radii are decreasing in i. Moreover, $\mathrm{r}_i(K) \leqslant \widetilde{\mathrm{r}}_i(K)$ for all $i = 1, \dots, n$ and any convex body $K \in \mathscr{K}^n$. The first systematic study of the successive radii was developed in [1]. For more information on these radii we refer to [1, 2, 4, 5, 6, 7, 8, 9] and the references inside. Here our interest is focussed in an open problem concerning the ratio $\mathrm{R}_{n-i+1}(K)/\mathrm{r}_i(K)$.

The well known relations between diameter and circumradius, and minimal width and inradius, were obtained respectively by Jung and Steinhagen (see e.g. [3, pp. 84, 86]); we express them in terms of the successive radii:

$$\frac{\mathrm{R}_n(K)}{\mathrm{r}_1(K)} \leqslant \sqrt{\frac{2n}{n+1}} \quad \text{and} \quad \frac{\mathrm{R}_1(K)}{\mathrm{r}_n(K)} \leqslant \begin{cases} \sqrt{n} & \text{for } n \text{ odd,} \\ \frac{n+1}{\sqrt{n+2}} & \text{for } n \text{ even.} \end{cases} \tag{1.1}$$

The regular n-simplex gives equality in both inequalities. These relations would appear as particular cases of a more general formula by determining the best upper bound for the ratio $\mathrm{R}_{n-i+1}/\mathrm{r}_i$. Pukhov [11] and Perel'man [10] showed the following result:

THEOREM 1.1. *Let $K \in \mathscr{K}^n$ and $1 \leqslant i \leqslant n$. Then it holds*

$$\frac{\mathrm{R}_{n-i+1}(K)}{\mathrm{r}_i(K)} < i+1. \tag{1.2}$$

But the optimal bound is still not known. It is conjectured that the regular n-simplex provides the optimal upper bound (we write S_n to denote the regular simplex with inradius 1): if $i = 1, n$, then $\mathrm{R}_{n-i+1}(S_n)/\mathrm{r}_i(S_n)$ takes the values of (1.1); for $i = 2$ and n even,

$$\frac{\mathrm{R}_{n-1}(S_n)}{\mathrm{r}_2(S_n)} = \frac{(2n-1)\sqrt{3}}{\sqrt{2n(n+1)}};$$

in the remaining cases

$$\frac{\mathrm{R}_{n-i+1}(S_n)}{\mathrm{r}_i(S_n)} = \sqrt{1 - \frac{i}{n+1}}\sqrt{i(i+1)} \sim (i+1)\sqrt{1 - \frac{i}{n+1}}.$$

The values of the successive radii of the simplex S_n can be found in [4].

In [1] the best possible lower bound for the above ratio was obtained: $\mathrm{R}_{n-i+1}(K)/\mathrm{r}_i(K) \geqslant 1$, with equality for the ball.

Moreover, in the particular case of $n=3$ (and $i=2$), Perel'man [10] improved the result reducing the bound in (1.2) from 3 to 2.151...:

$$\frac{\mathrm{R}_2(K)}{\mathrm{r}_2(K)} < 2.151\ldots. \tag{1.3}$$

On the other hand, in the case of a 0-symmetric convex body, $K \in \mathscr{K}_0^n$, Pukhov proved in [11] that

$$\frac{\mathrm{R}_{n-i+1}(K)}{\mathrm{r}_i(K)} < \sqrt{e}\min\left\{\sqrt{i}, \sqrt{n-i+1}\right\}. \tag{1.4}$$

The optimal bound is also not known. It is conjectured that both, the regular cube and the regular cross-polytope provide the optimal upper bound: $\mathrm{R}_{n-i+1}(K)/\mathrm{r}_i(K) \leqslant \sqrt{n-i+1}\sqrt{i/n}$. The values of the successive radii of the regular cube and cross-polytope can be also found in [4].

Notice that if $K \in \mathscr{K}_0^3$, Pukhov's result gives $\mathrm{R}_2(K)/\mathrm{r}_2(K) < \sqrt{2e} \approx 2.33\ldots$ Here we get a slight better bound when $n=3$, for a 0-symmetric convex body; it improves both (1.3) and (1.4).

THEOREM 1.2. *Let $K \in \mathscr{K}_0^3$. Then it holds*

$$\frac{\mathrm{R}_2(K)}{\mathrm{r}_2(K)} < 2.$$

We can also state the same problem but considering the inner radii defined via projections. In this respect we prove the following result.

THEOREM 1.3. *Let $K \in \mathscr{K}_0^n$ and $1 \leqslant i \leqslant n$. Then*

$$\frac{\mathrm{R}_{n-i+1}(K)}{\widetilde{\mathrm{r}}_i(K)} \leqslant \sqrt{n-i+1}. \tag{1.5}$$

Observe that if $i=n$, equality holds for all $K \in \mathscr{K}_0^n$.

The paper is organized as follows. In Section 2 we present the proofs of the above theorems. Section 3 is devoted to some additional remarks and properties.

2. Proofs of the main results

First we obtain the announced upper bound for the ratio $\mathrm{R}_2(K)/\mathrm{r}_2(K)$ when $K \in \mathscr{K}_0^3$, by proving Theorem 1.2.

Proof of Theorem 1.2. We assume, without loss of generality, that e_1 is the direction giving the diameter of K, i.e., $\mathrm{D}(K) = \omega(K, \mathrm{e}_1)$. For the sake of brevity we write $\mathrm{R} := \mathrm{R}(K) = \mathrm{D}(K)/2$.

First we show that the diameter of the convex body $K|e_1^\perp$ is not greater than $4r_2(K)$. In order to do it, we suppose that $R' := D(K|e_1^\perp)/2 > 2r_2(K)$. Since K is 0-symmetric, then the segment $[-Re_1, Re_1] \subset K$, and from $R' = D(K|e_1^\perp)/2$, we can assure the existence of $u \in K|e_1^\perp$ (and its 0-symmetral) such that $|u|_2 = R'$. Doing a rotation with axis $\mathrm{lin}\{e_1\}$ we can assume that $u \in \mathrm{lin}\{e_1, e_2\}$, i.e., that $u = (0, R', 0)$. Since $u = (0, R', 0) \in K|e_1^\perp$, there exists a point $(a, R', 0) \in K$ (and its 0-symmetral), and therefore it holds that $(a, R', 0)|e_1^\perp = u$.

The length of $[-Re_1, Re_1]$ is $2R = D(K)$ and since $(a, R', 0) \in K$ we have that $d\big((a, R', 0), 0\big)^2 = a^2 + (R')^2 \leqslant R^2$. We can suppose without loss of generality that $a > 0$; notice also that $a \leqslant R$. We have constructed in this way a (planar) parallelogram $P = \mathrm{conv}\{\pm(a, R', 0), \pm(R, 0, 0)\}$ which, because of the convexity, is contained in K. Its inradius, considered as a planar set, is the smallest distance from the origin 0 to the facets, more precisely, the distance from 0 to the longest facet. It is an easy computation to check that this distance, and thus the inradius of P, is given by

$$r\big(P; \mathrm{lin}\{e_1, e_2\}\big) = \frac{R'R}{\sqrt{(R+a)^2 + (R')^2}}.$$

From $a^2 + (R')^2 \leqslant R^2$ we get that

$$\sqrt{(R+a)^2 + (R')^2} \leqslant \sqrt{2R^2 + 2aR} \leqslant \sqrt{2R^2 + 2R^2} = \sqrt{4R^2} = 2R,$$

and thus

$$r\big(P; \mathrm{lin}\{e_1, e_2\}\big) = \frac{R'R}{\sqrt{(R+a)^2 + (R')^2}} \geqslant \frac{R'}{2}.$$

Since we assumed that $R' > 2r_2(K)$, then $r\big(P; \mathrm{lin}\{e_1, e_2\}\big) > r_2(K)$, which is a contradiction because $r_2(K)$ is the radius of the greatest circumference contained in K. Therefore, $D(K|e_1^\perp) \leqslant 4r_2(K)$.

Finally, since $K|e_1^\perp$ is a 0-symmetric set and it verifies moreover that $D(K|e_1^\perp) \leqslant 4r_2(K)$, we get $R(K|e_1^\perp) = D(K|e_1^\perp)/2 \leqslant 2r_2(K)$. Now, taking into account that $e_1^\perp$ is a 2-dimensional linear subspace, we can conclude that $R_2(K) \leqslant R(K|e_1^\perp) \leqslant 2r_2(K)$. It finishes the proof. □

As mentioned before, the optimal bound for $n = 3$ is still far away, since we think it should be $2/\sqrt{3}$.

Proof of Theorem 1.3. Let $L_1 \in \mathscr{L}_i^n$ be an arbitrary linear subspace and we consider $K|L_1$, which is also a 0-symmetric convex body. For the sake of brevity we write $\widetilde{r}_i = \widetilde{r}_i(K)$. Then it holds $\rho_1 = r(K|L_1; L_1) \leqslant \widetilde{r}_i$. Let $u_1 \in L_1$ be the unit vector such that $\omega(K|L_1; L_1) = \omega(K|L_1, u_1; L_1)$. Then,

$$K|L_1 \subset \left\{ y \in L_1 : |\langle y, u_1 \rangle| \leqslant \frac{\omega(K|L_1; L_1)}{2} \right\} = \{ y \in L_1 : |\langle y, u_1 \rangle| \leqslant \rho_1 \}$$

because $K|L_1$ is 0-symmetric, which implies that $\omega(K|L_1; L_1) = 2\rho_1$. Moreover, since we are working with the orthogonal projection onto L_1, it holds

$$K \subset \{ x \in \mathbb{R}^n : |\langle x, u_1 \rangle| \leqslant \rho_1 \}.$$

Notice that we can assume $i \leqslant n-1$, because for $i=n$ the result is just Steinhagen theorem (see e.g. [3, p. 86]). Now we consider $u_1^{\perp} \in \mathscr{L}_{n-1}^n$ and let L_2 be an i-dimensional linear subspace of $u_1^{\perp}$. With an analogous argument to the above one we know that there exists a suitable $u_2 \in L_2$ such that

$$K|L_2 \subset \{y \in L_2 : |\langle y, u_2\rangle| \leqslant \rho_2\},$$

with $\rho_2 = \mathrm{r}(K|L_2;L_2) \leqslant \widetilde{\mathrm{r}}_i$. Again we can conclude that

$$K \subset \{x \in \mathbb{R}^n : |\langle x, u_2\rangle| \leqslant \rho_2\}.$$

Next, if $i \leqslant n-2$, we consider $\mathrm{lin}\{u_1,u_2\}^{\perp} \in \mathscr{L}_{n-2}^n$, and we take L_3 an i-dimensional subspace of $\mathrm{lin}\{u_1,u_2\}^{\perp}$.

Using an iterative argument, in the $(n-i+1)$-step we obtain $n-i+1$ pairwise orthogonal unit vectors $u_1,\dots,u_{n-i+1}$ (by the construction) and positive real numbers $\rho_j \leqslant \widetilde{\mathrm{r}}_i$, for $j=1,\dots,n-i+1$, such that

$$\begin{aligned} K &\subset \bigcap_{j=1}^{n-i+1} \{x \in \mathbb{R}^n : |\langle x, u_j\rangle| \leqslant \rho_j\} \\ &= \Big\{x \in \mathbb{R}^n : |\langle x, u_j\rangle| \leqslant \rho_j \text{ for } j=1,\dots,n-i+1\Big\}. \end{aligned} \tag{2.1}$$

Thus writing $H_{n-i+1} = \mathrm{lin}\{u_1,\dots,u_{n-i+1}\}$ and denoting by $C_{l_1,\dots,l_{n-i+1}}$ the 0-symmetric orthogonal box contained in H_{n-i+1} with edge lengths $l_1,\dots,l_{n-i+1}$, we get as a consequence of (2.1) that

$$K|H_{n-i+1} \subseteq C_{2\rho_1,\dots,2\rho_{n-i+1}} \subseteq C_{2\widetilde{\mathrm{r}}_i,\dots,2\widetilde{\mathrm{r}}_i}, \tag{2.2}$$

i.e., it is contained in the $(n-i+1)$-cube of H_{n-i+1} with edge length $2\widetilde{\mathrm{r}}_i$. Hence

$$\mathrm{R}(K|H_{n-i+1}) \leqslant \mathrm{R}(C_{2\widetilde{\mathrm{r}}_i,\dots,2\widetilde{\mathrm{r}}_i}) = \sqrt{n-i+1}\,\widetilde{\mathrm{r}}_i,$$

and therefore

$$\mathrm{R}_{n-i+1}(K) \leqslant \mathrm{R}(K|H_{n-i+1}) \leqslant \sqrt{n-i+1}\,\widetilde{\mathrm{r}}_i = \sqrt{n-i+1}\,\widetilde{\mathrm{r}}_i(K). \quad \square$$

If $K \in \mathscr{K}^n$ is an arbitrary convex body (not necessarily 0-symmetric) then, a similar argument to the above one allows to show that a suitable projection of K onto an $(n-i+1)$-dimensional linear subspace H is contained in an orthogonal box $C_{\omega_1,\dots,\omega_{n-i+1}}$ (see (2.2)) with edge-lengths $\omega_j := \omega(K|L_j;L_j)$, where $L_j \in \mathscr{L}_i^n$, $j = 1,\dots,n-i+1$, are suitably chosen. Using Steinhagen's theorem (see (1.1)) in the subspace L_j, namely,

$$\omega(K|L_j;L_j) \leqslant \begin{cases} 2\sqrt{i}\,\mathrm{r}(K|L_j;L_j) & \text{for } i \text{ odd}, \\ 2\frac{i+1}{\sqrt{i+2}}\,\mathrm{r}(K|L_j;L_j) & \text{for } i \text{ even}, \end{cases}$$

and since $\mathrm{r}(K|L_j;L_j) \leqslant \widetilde{\mathrm{r}}_i(K)$ for all $1 \leqslant j \leqslant n-i+1$, we finally obtain the following result.

PROPOSITION 2.1. *Let $K \in \mathscr{K}^n$ and $1 \leqslant i \leqslant n$. Then*

$$\frac{\mathrm{R}_{n-i+1}(K)}{\widetilde{\mathrm{r}}_i(K)} \leqslant \begin{cases} \sqrt{i}\sqrt{n-i+1} & \text{for } i \text{ odd,} \\ \frac{i+1}{\sqrt{i+2}}\sqrt{n-i+1} & \text{for } i \text{ even.} \end{cases}$$

Observe that in order to avoid the parity distinction for i, both bounds above should be replaced by $\sqrt{i+1/3}\sqrt{n-i+1}$. We also notice that these bounds for the ratio $\mathrm{R}_{n-i+1}(K)/\widetilde{\mathrm{r}}_i(K)$, depending on the values of n and i, can improve Pukhov's bound $i+1$.

3. Some final remarks

We notice that the problem on bounding the ratio $\mathrm{R}_j/\mathrm{r}_i$, $1 \leqslant i,j \leqslant n$, has only interest when $j = n-i+1$:

PROPOSITION 3.1. *If $j > n-i+1$ there is no upper bound for $\mathrm{R}_j(K)/\mathrm{r}_i(K)$.*

Proof. Notice that since we assume $j > n-i+1$ then $i > 1$. We are going to find a convex body such that for $j > n-i+1$, the above ratio is arbitrarily large. It suffices to consider the $(i-1)$-dimensional ball $B_{i-1} = B_n \cap L$, with $L \in \mathscr{L}^n_{i-1}$. On one hand, since $\dim B_{i-1} = i-1$, then $\mathrm{r}_i(B_{i-1}) = 0$; on the other hand, we can assume that $B_{i-1} \subset (RB_j) \times (L')^{\perp}$, for suitable $R > 0$, where B_j is the unit ball of $L' \in \mathscr{L}^n_j$. Since

$$\dim L + \dim L' = i-1+j > i-1+n-i+1 = n,$$

then L and L' have, at least, a common straight line ℓ. Hence

$$B_{i-1} \cap \ell = [-u,u] \subset RB_j,$$

with $|u|_2 = 1$, and thus $R \geqslant 1$. Therefore $\mathrm{R}_j(B_{i-1}) \geqslant R \geqslant 1$, and then the quotient $\mathrm{R}_j/\mathrm{r}_i$ is not bounded by above. It suffices to consider the convex hull of B_{i-1} and suitable sufficiently close points in order to obtain a convex body in $\mathbb{R}^n$ with non-empty interior and verifying the same property. □

As it was already noticed in [10], observe that if $j < n-i+1$, since the successive outer radii form an increasing sequence, knowing the optimal bound for the ratio $\mathrm{R}_{n-i+1}/\mathrm{r}_i$ would give immediately the required upper bound for $\mathrm{R}_j/\mathrm{r}_i$. Therefore, $\mathrm{R}_{n-i+1}/\mathrm{r}_i$ is the only ratio needed to be considered.

In order to conclude this note, we briefly comment a relation between the inner radii defined via sections and projections for 0-symmetric convex bodies. From the definition of inner radii we trivially have $\mathrm{r}_i(K) \leqslant \widetilde{\mathrm{r}}_i(K)$ for all $i = 1,\dots,n$ and any $K \in \mathscr{K}^n$. We would like to point out the existence of a reverse relation: the following lemma provides a (not sharp) lower bound for $\mathrm{r}_i(K)$ in terms of $\widetilde{\mathrm{r}}_i(K)$ when K is 0-symmetric.

LEMMA 3.1. *Let $K \in \mathscr{K}^n_0$ and $1 \leqslant i \leqslant n$. Then $\widetilde{\mathrm{r}}_i(K) \leqslant \sqrt{i}\,\mathrm{r}_i(K)$.*

Proof. Without loss of generality we assume that $L = \mathrm{lin}\{\mathrm{e}_1,\dots,\mathrm{e}_i\} \in \mathscr{L}_i^n$ is the i-dimensional linear subspace such that $\widetilde{\mathrm{r}}_i(K) = \mathrm{r}(K|L;L)$. The central symmetry of K ensures that $\widetilde{\mathrm{r}}_i(K)B_i \subseteq K|L$, where $B_i = B_n \cap L$ denotes the i-dimensional unit ball of L. Now let

$$u_j = \widetilde{\mathrm{r}}_i(K)\mathrm{e}_j \in \big(\widetilde{\mathrm{r}}_i(K)\,\mathrm{relbd}\,B_i\big) \subseteq K|L, \quad j = 1,\dots,i.$$

These points u_j are projections of points of the original body K, i.e., there exist numbers $a_k^l \in \mathbb{R}$ for $k = i+1,\dots,n$, $l = 1,\dots,i$, such that

$$v_j := u_j + \big(0,\dots,0,a_{i+1}^j,\dots,a_n^j\big) \in K, \quad j = 1,\dots,i,$$

and since K is a 0-symmetric convex body, $C = \mathrm{conv}\{\pm v_1,\dots,\pm v_i\} \subseteq K$. Next we show that $\mathrm{r}(C;\mathrm{lin}\,C) \geqslant \mathrm{r}\big(\mathrm{conv}\{\pm u_1,\dots,\pm u_i\};L\big)$.

Since C is 0-symmetric, then $\mathrm{r}(C;\mathrm{lin}\,C) = \min_{x\in\mathrm{relbd}\,C}|x|_2$ and so we may choose $x \in \mathrm{relbd}\,C$ such that $\mathrm{r}(C;\mathrm{lin}\,C) = |x|_2$. Let $x = \sum_{j=1}^i(\lambda_j - \mu_j)v_j$, with $\lambda_j,\mu_j \geqslant 0$ for $j = i,\dots,i$ and $\sum_{j=1}^i(\lambda_j+\mu_j) = 1$. Then

$$\begin{aligned} |x|_2^2 &= \left|\sum_{j=1}^i(\lambda_j-\mu_j)u_j\right|_2^2 + \left|\sum_{j=1}^i(\lambda_j-\mu_j)\big(0,\dots,0,a_{i+1}^j,\dots,a_n^j\big)\right|_2^2, \\ &\geqslant \left|\sum_{j=1}^i(\lambda_j-\mu_j)u_j\right|_2^2 = \big|x|L\big|_2^2. \end{aligned} \tag{3.1}$$

Since $x|L \in \mathrm{relbd\,conv}\{\pm u_1,\dots,\pm u_i\}$, we get that

$$\mathrm{r}(C;\mathrm{lin}\,C) = |x|_2 \geqslant \big|x|L\big|_2 \geqslant \mathrm{r}\big(\mathrm{conv}\{\pm u_1,\dots,\pm u_i\};L\big).$$

Thus we can conclude that

$$\begin{aligned} \mathrm{r}_i(K) &\geqslant \mathrm{r}(C;\mathrm{lin}\,C) \geqslant \mathrm{r}\big(\mathrm{conv}\{\pm u_1,\dots,\pm u_i\};L\big) \\ &= \widetilde{\mathrm{r}}_i(K)\mathrm{r}\big(\mathrm{conv}\{\pm\mathrm{e}_1,\dots,\pm\mathrm{e}_i\};L\big) = \widetilde{\mathrm{r}}_i(K)\frac{1}{\sqrt{i}}, \end{aligned} \tag{3.2}$$

and we get the required inequality, $\widetilde{\mathrm{r}}_i(K) \leqslant \sqrt{i}\,\mathrm{r}_i(K)$. □

For $n = 3$ and $i = 2$, Theorem 1.3 and Lemma 3.1 together also show Theorem 1.2:

$$\frac{\mathrm{R}_2(K)}{\mathrm{r}_2(K)} \leqslant \sqrt{2}\frac{\mathrm{R}_2(K)}{\widetilde{\mathrm{r}}_2(K)} \leqslant \sqrt{2}\sqrt{2} = 2.$$

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