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SET VALUED F–VARIATIONAL INEQUALITIES AND SET VALUED VECTOR F–COMPLEMENTARITY PROBLEMS

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Abstract. In this work, we study a new type of set-valued vector F-variational inequalities and a new kind of set valued vector F-complementarity problem in Hausdorff topological vector spaces. We establish the equivalence between the set valued vector F-variational inequalities and set valued vector F-complementarity problems under certain conditions. By considering the existence of solutions for the vector F-variational inequalities and using the continuous selection theorem, we obtain some new existence theorems of solutions for the set valued vector F-variational inequalities and set valued vector F-complementarity problems, respectively.

1. Introduction

Let X,Y be two real Hausdorff topological vector spaces. A nonempty subset P of X is called convex cone if $\lambda P \subseteq P$ for all $\lambda \geqslant 0$ and $P+P=P$. A cone P is called pointed cone if P is a cone and $P \cap \{-P\} = \{0\}$ where 0 denotes the zero vector, also a cone P is called pointed if it is properly contained in X. Let $L(X,Y)$ be the space of all continuous linear mappings from X to Y, denoted by $\langle t,x\rangle$ the values of a linear operator $t \in L(X,Y)$ at $x \in X$ and K be a nonempty closed and convex cone of X. Let $T : X \to 2^Y$ be a multifunction, the graph of T denoted by $\mathscr{G}(T)$ is the set $\{(x,z) \in X \times Y |\ x \in X,\ z \in T(x)\}$. Let $C : K \to 2^Y$ be a set valued mapping such that for each $x \in K$, $C(x)$ is a pointed closed convex cone in Y with apex at the origin and $\operatorname{int} C(x) \neq \emptyset$. Assume that $T : K \to 2^{L(X,Y)}$, $F : K \to Y$ and $A : K \times L(X,Y) \to L(X,Y)$ are the mappings. A function f is called a selection of T on K if $f(x) \in T(x)$ for all $x \in K$ and it is also continuous on K. Furthermore a function f is called a continuous selection of T on K if f is a selection of T on K.

In this paper, we consider the set valued vector F-variational inequality (SVVF-VI) problem of finding $x^* \in K$, $p^* \in T(x^*)$ such that

$$\langle A(x^*,p^*), y-x^*\rangle + F(y) - F(x^*) \notin -\operatorname{int} C(x^*) \quad \forall\, y \in K. \tag{1.1}$$

We say that (x^*,p^*) is a solution of (SVVF-VI).

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Now we define the following setvalued vector F-complementarity problem (SVVF-CP) which consists of finding $x^* \in K$, $p^* \in T(x^*)$ such that

$$\langle A(x^*,p^*),x^*\rangle + F(x^*) \notin \operatorname{int} C(x^*) \text{ and } \langle A(x^*,p^*),y\rangle + F(y) \notin -\operatorname{int} C(x^*)\ \forall\, y \in K. \tag{1.2}$$

We say that (x^*,p^*) is a solution of (SVVF-CP).

In this paper, we study a new type of setvalued vector F-variational inequality and setvalued vector F-complementarity problem in Hausdorff topological vector spaces. We establish an equivalence between SVVF-VI and SVVF-CP under certain conditions. We consider the concept of weak C-pseudo-monotonicity to prove the existence of solutions for setvalued vector F-variational inequality using the continuous selection theorem due to Ding, Kim and Tan [4]. We obtain some new existence theorems of solutions for SVVF-VI and SVVF-CP respectively.

The vector variational inequality theory initiated by Giannessi [7] has emerged as a powerful tool for a wide class of vector optimization problems and vector equilibrium problems, see [2, 7, 8, 12].

In 2000, Chen and Hou [3] summarized existence results of solutions for vector variational inequalities and pointed out that most of the existence results in this area touch upon a weak version of vector variational inequality and its generalizations. The existence of solutions for strong vector variational inequality is still an open problem. Recently, by using the combination of demicontinuity and pseudomonotonicity, Fang and Huang [6, 9] initiated a new class of vector F-complementarity problem with demi-pseudomonotone mappings in Banach spaces. They also presented the solvability of this class of vector F-complementarity problems and demi-pseudomonotone mappings and finite dimensional continuous mappings in reflexive Banach spaces, see [4, 8, 11, 13, 14].

Special Cases

(i) If we take $A(x^*,p^*) = A(p^*)$, then (1.1) is one of the variant form of the problem considered by Y. C. Lin [16] for finding $x^* \in K$, $p^* \in T(x^*)$ such that

$$\langle A(p^*),y-x^*\rangle + F(y) - F(x^*) \notin -\operatorname{int} C(x^*)\ \ \forall\, y \in K. \tag{1.3}$$

(ii) If $F = 0$, then SVVF-VI (1.1) reduces to the generalized vector variational inequality of finding $x^* \in K$, $p^* \in T(x^*)$ such that

$$\langle A(x^*,p^*),y-x^*\rangle \notin -\operatorname{int} C(x^*)\ \ \forall\, y \in K. \tag{1.4}$$

(iii) Again if $A(x^*,p^*) = p^*$, then (1.4) collapses to the problem of finding $x^* \in K$ and $p^* \in T(x^*)$ such that

$$\langle p^*,y-x^*\rangle \notin -\operatorname{int} C(x^*)\ \ \forall\, y \in K, \tag{1.5}$$

which was studied by Yang and Yao [18].

(iv) If T is a single valued mapping, then (1.5) reduces to the problem of finding $x^* \in K$ such that

$$\langle T(x^*), y - x^* \rangle \notin -\mathrm{int}\, C(x^*) \;\; \forall\, y \in K, \tag{1.6}$$

which was studied by Chen [1], Yu and Yao [20].

Furthermore if $X = \mathbb{R}^n$, $L(X,Y) = X^*$ and $C(x) = R_+ = [0, +\infty)$ for all $x \in K$, then (1.6) reduces to the classical variational inequalities: Find $x^* \in K$ such that

$$\langle T(x^*), y - x^* \rangle \geqslant 0 \;\; \forall\, y \in K, \tag{1.7}$$

considered and studied by Stampacchia [17].

(v) If A is a single valued mapping, then (1.1) reduces to the problem of finding $x^* \in K$, $p^* \in T(x^*)$ such that

$$\langle p^*, y - x^* \rangle + F(y) - F(x^*) \notin -\mathrm{int}\, C(x^*) \;\; \forall\, y \in K, \tag{1.8}$$

considered by Huang et al. [12].

(vi) If T is a single valued mapping, then (1.8) reduces to the vector F-variational inequality: Finding $x^* \in K$ such that

$$\langle T(x^*), y - x^* \rangle + F(y) - F(x^*) \notin -\mathrm{int}\, C(x^*) \;\; \forall\, y \in K, \tag{1.9}$$

which was considered by Li and Huang [15] with $C(x) = C$ for all $x \in K$, where C is a pointed closed and convex cone in Y.

(vii) We remark that if $F = 0$, then (1.2) SVVF-CP collapses to the (GVCP) generalized vector complementarity problems of finding $x^* \in K$ and $p^* \in T(x^*)$ such that

$$\langle A(x^*, p^*), x^* \rangle \notin \mathrm{int}\, C(x^*) \text{ and } \langle A(x^*, p^*), y \rangle \notin -\mathrm{int}\, C(x^*) \; \forall\, y \in K. \tag{1.10}$$

(viii) We note that if $A(x^*, p^*) = p^*$, then problem (1.10) collapses to the problem of finding $x^* \in K$ and $p^* \in T(x^*)$ such that

$$\langle p^*, x^* \rangle \notin \mathrm{int}\, C(x^*) \text{ and } \langle p^*, y \rangle \notin -\mathrm{int}\, C(x^*) \; \forall\, y \in K, \tag{1.11}$$

considered by Huang and Guo [10].

(ix) Let X be a real Banach space, $L(X,Y) = X^*$ and $C(x) = R_+ = [0, +\infty)$ for all $x \in K$, then the generalized vector complementarity problem reduces to a problem of finding $x^* \in K$ and $p^* \in T(x^*)$ such that

$$\langle A(x^*, p^*), x^* \rangle + F(x^*) = 0 \text{ and } \langle A(x^*, p^*), y \rangle + F(y) \geqslant 0 \; \forall\, y \in K, \tag{1.12}$$

is a variant form considered and studied by Lee et al. [14].

(x) If $A(x^*, p^*) = p^*$ then (1.12) reduces to a problem of finding $x^* \in K$ and $p^* \in T(x^*)$ such that

$$\langle p^*, x^* \rangle + F(x^*) = 0 \text{ and } \langle p^*, y \rangle + F(y) \geqslant 0 \ \forall \ y \in K, \tag{1.13}$$

is a variant form of L.C. Zeng, Y.C. Lin and J.C. Yao [21].

(xi) If T is a single valued mapping, then (1.13) reduces to the problem of finding $x^* \in K$ such that

$$\langle T(x^*), x^* \rangle + F(x^*) = 0 \text{ and } \langle T(x^*), y \rangle + F(y) \geqslant 0 \ \forall \ y \in K, \tag{1.14}$$

which was the problem introduced by Yin, Xu and Zhang [19].

(xii) We note that if $F = 0$, then (1.14) collapses to the classical complementarity problem of finding $x^* \in K$ such that

$$\langle T(x^*), x^* \rangle = 0 \text{ and } \langle T(x^*), y \rangle \geqslant 0 \ \forall \ y \in K. \tag{1.15}$$

2. Preliminaries

LEMMA 2.1. [12] *If F is a selection of T on K, then every solution of VF-CP (vector F-complementarity problem) is a solution of GVF-CP (generalized vector F-complementarity problem) (respectively, every solutions of VF-VI (vector F-variational inequality) is a solution of GVF-VI (generalized vector F-variational inequality)).*

DEFINITION 2.1. T is said to be

(i) weakly C-pseudomonotone on K with respect to F, if for every pair of points $x, y \in K$ and for all $u \in T(x)$, $v \in T(y)$ such that

$$\langle A(x,u), y - x \rangle + F(y) - F(x) \in -\mathrm{int}C(x)$$

implies that

$$\langle A(y,v), x - y \rangle + F(x) - F(y) \notin \mathrm{int}C(y);$$

(ii) C-pseudomonotone on K with respect to F, if for every pair of points $x, y \in K$ and for all $u \in T(x)$, $v \in T(y)$

$$\langle A(x,u), y - x \rangle + F(y) - F(x) \notin -\mathrm{int}C(x)$$

implies that

$$\langle A(y,v), x - y \rangle + F(x) - F(y) \in -C(y);$$

(iii) strictly C-pseudomonotone on K with respect to F, if for every pair of points $x, y \in K$ and for all $u \in T(x)$, $v \in T(y)$ such that

$$\langle A(x,u), y - x \rangle + F(y) - F(x) \notin -\mathrm{int}C(x)$$

implies that

$$\langle A(y,v), x - y \rangle + F(x) - F(y) \in -\mathrm{int}C(y).$$

REMARK 2.1. We remark that the strict C-pseudomonotonicity implies C-pseudomonotonicity and C-pseudomonotonicity implies the weak C-pseudomonotonicity. But the converse is not necessarily true.

LEMMA 2.2. *Let f be a selection of T on K. If T is weakly C-pseudomonotone (respectively C-pseudomonotone and strictly C-pseudomonotone) on K with respect to F, then f is also weakly C-pseudomonotone (respectively C-pseudomonotone and strictly C-pseudomonotone) on K with respect to F.*

DEFINITION 2.2. Let K be a nonempty subset of topological vector space X. A setvalued mapping $T : K \to 2^X$ is called KKM-mapping if for every finite subset $\{x_1, x_2, \cdots, x_n\}$ of K $\mathrm{co}\{x_1, x_2, \cdots, x_n\}$ is contained in $\bigcup_{i=1}^{n} T(x_i)$, where co denotes the convex hull.

LEMMA 2.3. [5] *Let K be a nonempty subset of Hausdorff topological vector space X. Let $G : K \to 2^X$ be a KKM-mapping such that for any $y \in K$, $G(y)$ is closed and $G(y^*)$ is compact for some $y^* \in K$. Then there exists $x^* \in K$ such that $x^* \in G(y)$ for all $y \in K$, i.e.,*

$$\bigcap_{y \in K} G(y) \neq \emptyset.$$

LEMMA 2.4. *Let Y be a topological vector space with a pointed closed and convex cone C such that $\mathrm{int}C \neq \emptyset$. Then for all $x, y \in Y$*

(i) $x - y \in -C$ and $x \notin -\mathrm{int}C \Rightarrow y \notin -\mathrm{int}C$;

(ii) $x \in -\mathrm{int}C$ and $y \notin \mathrm{int}C \Rightarrow x + y \notin C$.

Proof. (i) If $y \in -\mathrm{int}C$, then $x = x - y + y \in -C - \mathrm{int}C \subseteq -\mathrm{int}C$ which contradicts the assumption $x \notin -\mathrm{int}C$.

(ii) If $x + y \in C$. Then $y = x + y - x \in C + \mathrm{int}C \subseteq \mathrm{int}C$, which contradicts the assumption $y \notin \mathrm{int}C$. □

DEFINITION 2.3. We say that

(i) F is C-convex if

$$F(\alpha x_1 + (1-\alpha)x_2) \in \alpha F(x_1) + (1-\alpha)F(x_2) - C \;\forall\, x_1, x_2 \in K,\; \alpha \in [0,1]$$

where C is a closed and convex cone of Y with $\mathrm{int}C \neq \emptyset$;

(ii) F is sublinear if

(a) (positive homogenity) $F(\alpha x) = \alpha F(x)$ for all $\alpha \geqslant 0$ and $x \in K$,

(b) (subadditivity) $F(x_1 + x_2) \in \{F(x_1) + F(x_2)\} - C$ for all $x_1, x_2 \in K$, where C is a closed and convex cone of Y with $\mathrm{int}C \neq \emptyset$;

(iii) F is hemicontinuous on K if for all $x,y \in K$, the function

$$\Omega(\alpha) = \langle F(x+\alpha(y-x)),\, y-x\rangle$$

is continuous on $[0,1]$.

REMARK 2.2. We note that F is C-convex if and only if

$$F\Big(\sum_{i=1}^{n} \alpha_i x_i\Big) \in \sum_{i=1}^{n} \alpha_i F(x_i) - C$$

for all $x_i \in K$ and $\alpha_i \in [0,1]$, $(i = 1,2,\cdots,n)$ with $\sum_{i=1}^{n} \alpha_i = 1$. If F is positive homogeneous then F is C-convex if and only if F is subadditive. In fact if F is positive homogeneous and C-convex, then we have

$$\begin{aligned} F(x_1+x_2) &= F\Big(\frac{1}{2}(2x_1)+\frac{1}{2}(2x_2)\Big) \\ &\in \frac{1}{2}F(2x_1)+\frac{1}{2}F(2x_2) - C \\ &= F(x_1)+F(x_2) - C \ \text{ for all } x_1,x_2 \in K. \end{aligned}$$

This means that if F is subadditive, then it follows that

$$\begin{aligned} F(\alpha x_1 + (1-\alpha)x_2) &\in F(\alpha x_1) + F((1-\alpha)x_2) - C \\ &= \alpha F(x_1) + (1-\alpha)F(x_2) - C \end{aligned}$$

for all $x_1,x_2 \in K$ and $\alpha \in [0,1]$. Thus F is convex.

3. Main results

THEOREM 3.1. *Let* $\bigcap_{x\in K} C(x) = C$ *with* $\text{int}\, C \neq \emptyset$.

(i) If (x^*,p^*) *solves SVVF-CP and there exists* $z_0 \in K$ *such that*

$$\langle A(x^*,p^*),\, z_0 + x^*\rangle + F(z_0+x^*) \in -C$$

and

$$F(z_0+y) = F(z_0)+F(y) \ \ \forall\, y \in K,$$

then (x^*,p^*) *solves SVVF-VI;*

(ii) Let F *be sublinear. If* (x^*,p^*) *solves SVVF-VI, then* (x^*,p^*) *solves SVVF-CP.*

Proof. (i) Let (x^*,p^*) be a solution of SVVF-CP. Then $x^* \in K$ and $p^* \in T(x^*)$ such that

$$\langle A(x^*,p^*),\, y\rangle + F(y) \notin -\text{int}\, C(x^*) \ \ \forall\, y \in K.$$

From the assumptions, we have

$$\{\langle A(x^*,p^*),\ y+z_0\rangle + F(y+z_0)\} - \{\langle A(x^*,p^*), y-x^*\rangle + F(y) - F(x^*)\}$$
$$= \langle A(x^*,p^*),\ z_0+x^*\rangle + F(z_0+x^*) \in -C \subseteq -C(x^*) \ \ \forall\, y \in K.$$

By Lemma 2.4(i), we get

$$\langle A(x^*,p^*),\ y-x^*\rangle + F(y) - F(x^*) \notin -\mathrm{int}\,C(x^*) \ \ \forall\, y \in K.$$

Thus (x^*,p^*) is a solution of SVVF-VI.

(ii) Let (x^*,p^*) be a solution of SVVF-VI, then $x^* \in K$ and $p^* \in T(x^*)$ such that

$$\langle A(x^*,p^*),\ y-x^*\rangle + F(y) - F(x^*) \notin -\mathrm{int}\,C(x^*) \ \ \forall\, y \in K. \tag{3.1}$$

Since $F: K \to Y$ is positively homogeneous and K a convex cone, letting $y = \frac{1}{2}x^*$ in (3.1), we have

$$\langle A(x^*,p^*),\ x^*\rangle + F(x^*) \notin \mathrm{int}\,C(x^*).$$

Given that F is subadditive, we get

$$\{\langle A(x^*,p^*),\ y+x^*-x^*\rangle + F(y+x^*) - F(x^*)\} - \{\langle A(x^*,p^*),\ y\rangle + F(y)\}$$
$$= F(y+x^*) + F(x^*) - F(y) \in -C \subseteq -C(x^*) \ \ \forall\, y \in K. \tag{3.2}$$

By Lemma 2.4(i) it follows from (3.1) and (3.2),

$$\langle A(x^*,p^*),\ y\rangle + F(y) \notin -\mathrm{int}\,C(x^*) \ \ \forall\, y \in K$$

which shows that (x^*,p^*) solves SVVF-CP. This completes the proof. □

If T is a single valued mapping and $A(x^*,T(x^*)) = T(x^*)$, then from Theorem 3.1, we obtain the following:

COROLLARY 3.1. [12] *Let* $\bigcap_{x\in K} C(x) = C$ *with* $intC \neq \emptyset$.

(i) If x^* *solves SVVF-CP and there exists* $z_0 \in K$ *such that*

$$\langle T(x^*),\ z_0+x^*\rangle + F(z_0+x^*) \in -C(x^*)$$

and

$$F(y+z_0) = F(y) + F(z_0) \ \ \forall\, y \in K$$

then x^* *solves SVVF-VI.*

(ii) Let F *be sublinear. If* x^* *solves SVVF-VI, then* x^* *solves SVVF-CP.*

THEOREM 3.2. *Let T be strictly C-pseudomonotone on K with respect to F. If SVVF-VI is solvable, then the solution of SVVF-VI is unique.*

Proof. Suppose that SVVF-VI has two distinct solution x_1^* and x_2^*. Then $x_1^*, x_2^* \in K$, $p_1^* \in T(x_1^*)$ and $p_2^* \in T(x_2^*)$ such that

$$\langle A(x_1^*, p_1^*),\ x_2^* - x_1^* \rangle + F(x_2^*) - F(x_1^*) \notin -\mathrm{int}\, C(x_1^*) \tag{3.3}$$

and

$$\langle A(x_2^*, p_2^*),\ x_1^* - x_2^* \rangle + F(x_1^*) - F(x_2^*) \notin -\mathrm{int}\, C(x_2^*). \tag{3.4}$$

Since T is strictly C-pseudomonotone on K with respect to F, then from (3.3) that

$$\langle A(x_2^*, p_2^*),\ x_1^* - x_2^* \rangle + F(x_1^*) - F(x_2^*) \in -\mathrm{int}\, C(x_2^*),$$

which contradicts (3.4), completing the proof. □

THEOREM 3.3. *Let K be a nonempty weakly compact and convex subset of X and $\bigcap_{x \in K} C(x) = C$ with $intC \neq \emptyset$.*

Assume that the following conditions hold:

(i) F is C-convex and continuous;

(ii) T is hemicontinuous and weakly C-pseudo monotone on K with respect to F;

(iii) the graph W of $W: K \to 2^Y$ weakly closed in $K \times Y$, where $W(x) = Y \setminus (-intC(x))$ for all $x \in K$;

(iv) A is continuous and affine.

Then SVVF-VI (1.1) has a solution. Further if F is positive homogeneous, then SVVF-CP (1.2) has a solution.

In order to prove Theorem 3.3, we first show the following lemma.

LEMMA 3.1. *If all assumptions in Theorem 3.3 hold, then SVVF-VI (1.1) is equivalent to the following problem: Find $x^* \in K$, $q \in T(y)$ such that*

$$\langle A(y,q),\ x^* - y \rangle + F(x^*) - F(y) \notin -intC(y), \ \ \forall\, y \in K. \tag{3.5}$$

Proof. Since T is C-pseudomonotone, it is easy to see that every solutions of SVVF-VI (1.1) is also a solution of problem (3.5). Conversely, let $x^* \in K$ be a solution of problem (3.5). Then

$$\langle A(y,q),\ x^* - y \rangle + F(x^*) - F(y) \notin \mathrm{int}\, C(y), \ \ \forall\, y \in K,\ q \in T(y).$$

Since $y \in K$, $q \in T(y)$ and $\alpha \in (0,1)$ set $y_\alpha = (1-\alpha)x^* + \alpha y$. We have

$$\langle A(y_\alpha, q_\alpha),\ x^* - y_\alpha \rangle + F(x^*) - F(y_\alpha) \notin \mathrm{int}\, C(y_\alpha). \tag{3.6}$$

We now prove that

$$\langle A(y_\alpha,q_\alpha),\ y-y_\alpha\rangle + F(y) - F(y_\alpha) \notin -\mathrm{int}\,C(y_\alpha). \tag{3.7}$$

Suppose (3.7) is not true. Then

$$\langle A(y_\alpha,q_\alpha),\ y-y_\alpha\rangle + F(y) - F(y_\alpha) \in \mathrm{int}\,C(y_\alpha). \tag{3.8}$$

Since F is convex and $C \subseteq C(x)$ for all $x \in K$,

$$\begin{aligned}
0 &= \langle A(y_\alpha,q_\alpha),\ y_\alpha - y_\alpha\rangle + F(y_\alpha) - F(y_\alpha)\\
&\in \alpha\{\langle A(y,q_\alpha),\ y-y_\alpha\rangle + F(y) - F(y_\alpha)\}\\
&\quad + (1-\alpha)\{\langle A(x^*,q_\alpha),\ x^*-y_\alpha\rangle + F(x^*) - F(y_\alpha)\} - C\\
&\subseteq \alpha\{\langle A(y,q_\alpha),\ y-y_\alpha\rangle + F(y) - F(y_\alpha)\}\\
&\quad + (1-\alpha)\{\langle A(x^*,q_\alpha), x^*-y_\alpha)\rangle + F(x^*) - F(y_\alpha)\} - C(y_\alpha)
\end{aligned}$$

that is

$$\begin{aligned}
&\alpha\{\langle A(y,q_\alpha),\ y-y_\alpha\rangle + F(y) - F(y_\alpha)\}\\
&\quad +(1-\alpha)\{\langle A(x^*,q_\alpha),\ x^*-y_\alpha\rangle + F(x^*) - F(y_\alpha)\} \in C(y_\alpha).
\end{aligned} \tag{3.9}$$

Since $C(y_\alpha)$ is a convex cone, from Lemma 2.4 (ii), (3.6) and (3.8), we have

$$\begin{aligned}
&\alpha\{\langle A(y,q_\alpha),\ y-y_\alpha\rangle + F(y) - F(y_\alpha)\}\\
&\quad +(1-\alpha)\{\langle A(x^*,q_\alpha),\ x^*-y_\alpha\rangle + F(x^*) - F(y_\alpha)\} \notin C(y_\alpha),
\end{aligned}$$

which is a contradiction with (3.9). Therefore (3.7) is true. Then

$$\langle A(y,q_\alpha),\ y-y_\alpha\rangle + F(y) - F(y_\alpha)\} \in W(y_\alpha). \tag{3.10}$$

Since F is continuous, T is hemicontinuous on K and W is weakly closed in $K \times Y$, from (3.10), we obtain

$$\begin{aligned}
&(y_\alpha, \langle A(y,q_\alpha),\ y-y_\alpha\rangle + F(y) - F(y_\alpha)) \to\\
&\qquad (x^*, \langle A(x^*,p),\ y-x^*\rangle + F(y) - F(x^*)),\ \text{ as } \alpha \to 0,
\end{aligned}$$

and

$$(x^*, \langle A(x^*,p^*),\ y-x^*\rangle + F(y) - F(x^*)) \in \mathrm{Graph}(W).$$

This shows that

$$\langle A(x^*,p^*),\ y-x^*\rangle + F(y) - F(x^*) \notin -\mathrm{int}\,C(x^*),\ \ p^* \in T(x^*)$$

and so x^* is a solution of SVVF-VI (1.1), this completes the proof. □

Proof of Theorem 3.3. Define the mapping P, $P^c : K \to 2^Y$ by

$$P(y) = \{x \in K,\ p \in T(x) \,|\, \langle A(x,p),\ y-x\rangle + F(y) - F(x) \notin -\mathrm{int}\,C(x)\}$$

and

$$P^c(y) = \{x \in K,\ q \in T(y) | \langle A(y,q),\ x-y\rangle + F(x) - F(y) \notin -\mathrm{int}C(y)\}$$

for each $y \in K$.

The proof is divided in four steps.

Step 1. We prove that P is a KKM-mapping. For this consider $P(y) \neq \emptyset$ for each $y \in K$, since $y \in P(y)$. Let z be in the convex hull of any finite subset $\{y_1, y_2, \cdots, y_n\}$ of K. Then $z = \sum_{i=1}^{n} \lambda_i\ y_i \in K$ for some nonnegative λ_i, $1 \leqslant i \leqslant n$, with $\sum_{i=1}^{n} \lambda_i = 1$. Suppose that $z \notin \bigcap_{i=1}^{n} P(y_i)$. Then $z \notin P(y_i)$ for all $i = 1, 2, \cdots, n$ and thus

$$\langle A(z,p'),\ y_i - z\rangle + F(y_i) - F(z) \in -\mathrm{int}C(z),\ p' \in T(z)\ \forall\ i = 1, 2, \cdots, n\ .$$

Since $C(z)$ is a convex cone, we obtain

$$\sum_{i=1}^{n} \lambda_i \{\langle A(z,p'),\ y_i - z\rangle + F(y_i) - F(z)\} \in -\mathrm{int}C(z).$$

Since F is C-convex, we have

$$\begin{aligned}
0 &= \langle A(z,p'),\ z - z\rangle + F(z) - F(z) \\
&= \langle A(z,p'),\ \sum_{i=1}^{n} \lambda_i y_i - z\rangle + F(\sum_{i=1}^{n} \lambda_i y_i) - F(z) \\
&\in \sum_{i=1}^{n} \lambda_i \langle A(z,p'),\ y_i - z\rangle + \sum_{i=1}^{n} \lambda_i F(y_i) - F(z) - C \\
&= \sum_{i=1}^{n} \lambda_i \{\langle A(z,p'),\ y_i - z\rangle + F(y_i) - F(z)\} - C \\
&\subseteq -\mathrm{int}C(z) - C(z) \\
&\subseteq -\mathrm{int}C(z).
\end{aligned}$$

Thus $0 \in -\mathrm{int}C(z)$, which is a contradiction. Therefore, P is a KKM mapping.

Step 2. Since T is C-pseudomonotone on K with respect to F, it follows that $P(y) \subseteq P^c(y)$ for all $y \in K$ and hence P^c is also a KKM-mapping.

Step 3. We show that $P^c(y)$ is a weakly closed and weakly compact for all $y \in K$ $\bigcap_{y \in K} P^c(y) \neq \emptyset$. In fact, let $\{x_\alpha\}$ be a net of $P^c(y)$ such that x_α converges weakly to $x_0 \in K$. For each α, since $x_\alpha \in P^c(y)$, we obtain

$$\langle A(y,q),\ x_\alpha - y\rangle + F(x_\alpha) - F(y) \notin -\mathrm{int}C(y)$$

and so

$$\langle A(y,q),\ x_\alpha - y\rangle + F(x_\alpha) - F(y) \in Y \backslash \mathrm{int}C(y).$$

Suppose that $q \in T(y) = 2^{L(X,Y)}$ is a continuous. Since F is a continuous we have

$$\langle A(y,q),\ x_\alpha - y\rangle + F(x_\alpha) - F(y) \to \langle A(y,q),\ x_0 - y\rangle + F(x_0) - F(y).$$

Since $Y\backslash \mathrm{int}C(y)$ is closed , we have

$$\langle A(y,q),\ x_0 - y\rangle + F(x_0) - F(y) \in Y\backslash \mathrm{int}C(y).$$

Hence

$$\langle A(y,q),\ x_0 - y\rangle + F(x_0) - F(y) \notin Y\backslash \mathrm{int}C(y)$$

and $P^c(y)$ is weakly closed.

Since K is weakly compact, $P^c(y)$ is also weakly compact for all $y \in K$. By Step 2, we know that P^c is a KKM-mapping. Therefore from Lemma 2.3, we have

$$\bigcap_{y\in K} P^c(y) \neq \emptyset.$$

Step 4. We prove that SVVF-VI has a solution. From Lemma 3.1, we have $\bigcap_{y\in K} P^c(y) \neq \emptyset$ and by Step 3, we obtain $\bigcap_{y\in K} P^c(y) \neq \emptyset$. Then $\bigcap_{y\in K} P(y) \neq \emptyset$ and so SVVF-VI (1.1) has a solution. Furthermore, if F is positive homogeneous, then from Remark 2.2 and Corollary 3.1, SVVF-CP (1.2) has a solution. This completes the proof. □

THEOREM 3.4. *Let K be a nonempty closed and convex subset of X and $\bigcap_{x\in K} C(x) = C$ with $intC \neq \emptyset$. Assume that conditions (i)-(iv) in Theorem 3.3 hold. If the following coercive condition on K is satisfied, then there exists a weakly compact subset D of X and $y_0 \in D\cap K$ such that for all $x \in K\backslash D$*

$$\langle A(y_0,q_0),\ y_0 - x\rangle + F(y_0) - F(x) \in -intC(y_0).$$

Then SVVF-VI (1.1) has a solution. Furthermore if F is positive homogeneous, then SVVF-CP (1.2) has a solution.

Proof. As the proof in Theorem 3.3, we only need to prove that $P^c(y_0)$ is weakly compact. From the coercive condition, it is clear that $P^c(y_0) \subseteq D$. Consider Step 3 in the proof of Theorem 3.3, $P^c(y_0)$ is weakly closed. Since D is weakly compact, $P^c(y_0)$ is also weakly compact. This completes the proof. □

THEOREM 3.5. *Let K be a nonempty weakly compact convex subset of X and $\bigcap_{x\in K} C(x) = C$ with $intC \neq \emptyset$. Assume that assumptions (i)-(iv) in Theorem 3.3 hold and the following conditions are satisfied:*

(i) T is weakly C-pseudomonotone on K with respect to F;

(ii) there is a continuous selection f of T on K.

Then SVVF-VI (1.1) has a solution. Furthermore if F is positive homogeneous, then SVVF-CP (1.2) has a solution.

Proof. By the assumption, there is a continuous selection $f : K \to 2^{L(X,Y)}$ such that

$$f(x) \in T(x) = 2^{L(X,Y)} \ \forall x \in K.$$

It follows from Lemma 2.2, that F is also weakly C-pseudomonotone. Then all conditions in Theorem 3.3 are satisfied. Thus there exists a solution (x^*, p^*) of SVVF-VI (1.1). By Lemma 2.1, (x^*, p^*) solves SVVF-VI (1.1).

Furthermore, if F is positive homogeneous, then it follows from Remark 2.2 and Theorem 3.1 that SVVF-CP (1.2) has a solution. This completes the proof. □

THEOREM 3.6. *Let K be a nonempty closed convex subset of X and $\bigcap_{x \in K} C(x) = C$ with $\operatorname{int} C \neq \emptyset$. Assume that all conditions in Theorem 3.5 hold and the continuous selection f of T satisfies the coercive condition on K defined in Theorem 3.4. Then SVVF-VI (1.1) has a solution.*

Furthermore, if F is positive homogeneous, then SVVF-CP (1.2) has a solution.

Proof. It follows from Theorems 3.4 and 3.5, that the condition holds. This completes the proof. □

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