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HARDY TYPE INEQUALITIES FOR FRACTIONAL AND q–FRACTIONAL INTEGRAL OPERATORS

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Abstract. Hardy type inequality for different fractional integral operators with sharp constants on finite intervals are given.

1. Introduction

The Hardy inequality is

$$\int_0^\infty x^{-\alpha p}\left|I^\alpha f(x)\right|^p dx \leqslant \left\{\frac{\Gamma(1/p')}{\Gamma(\alpha+1/p')}\right\}^p \int_0^\infty |f(x)|^p dx, \qquad (1.1)$$
$$1<p<\infty,\ 1/p+1/p'=1,$$

where the Riemann-Liouville transform I^α is defined by

$$I^\alpha f(x)=\int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} f(t)\,dt.$$

Several generalization of this inequality for I^α with sharp constants on finite and infinite intervals are given in $([6],[7],\ldots)$. This paper is devoted to give analogues of the previous inequality for different fractional integral operators of Erdelyi-Kober operator type, discrete transform and basic analogue of Erdelyi-Kober operator. The method used is the same as in [6].

2. Hardy type inequality for Erdelyi-Kober fractional integral operator

The following lemma is used hereafter to establish Hardy type inequalities.

LEMMA 2.1. *Let U, V be Hilbert spaces and assume $A: U \to V$ to be an infinite-dimensional linear compact operator. If there exists an orthonormal basis $\{u_n\}_{n\in\mathbb{N}}$ of*

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U, an orthonormal system (not necessary a basis) $\{v_n\}_{n\in\mathbb{N}}$ of V, a sequence of non-increasing positive numbers s_n such that

$$Au = \sum_{n\in\mathbb{N}} s_n (u,u_n)_U v_n,$$

for all $u \in U$, where $(.,.)_U$ is inner product in U. Then s_0 is the norm of the operator A.

The proof of this lemma can be found in [6].

We consider the Erdelyi-Kober operator [3]

$$T_{\lambda,\mu} f(x) = \frac{x^{-\lambda-\mu}\Gamma(\lambda+\mu+1)}{\Gamma(\lambda+1)\Gamma(\mu)} \int_0^x t^{\lambda} (x-t)^{\mu-1} f(t)\,dt,$$

where f is a locally integrable function and $\lambda > -1,\ \mu > 0$.

The Jacobi polynomials are defined by [12]

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!}\ {}_2F_1\left(\begin{matrix}-n, n+\alpha+\beta+1\\ \alpha+1\end{matrix}; \frac{1-x}{2}\right),$$

and satisfy the orthogonality relations

$$\int_{-1}^{1} P_m^{(\alpha,\beta)}(x) P_n(\alpha,\beta)(x)(1-x)^{\alpha}(1+x)^{\beta}dx$$
$$= \frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1}\frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+1)n!}\delta_{mn},$$

where $\alpha > -1$ and $\beta > -1$.

It will be convenient to use the slightly different Jacobi polynomials $J_n(x;\alpha,\beta)$, which are defined by

$$J_n(x;\alpha,\beta) = {}_2F_1\left(\begin{matrix}-n, n+\alpha+\beta+1\\ \alpha+1\end{matrix}; x\right).$$

THEOREM 2.1. *Let α, β and μ be a real numbers such that*

$$\min(\alpha,\ \alpha+\beta+1,\ \beta-\mu,\ \mu-1) > -1.$$

Then for all $f \in L^2\left((0,1),\ x^{\alpha}(1-x)^{\beta}\right)$ the following inequality

$$\int_0^1 x^{\alpha+\mu}(1-x)^{\beta-\mu}\left|T_{\alpha,\mu} f(x)\right|^2 dx \leqslant \frac{\Gamma(\alpha+\mu+1)\Gamma(\beta-\mu+1)}{\Gamma(\alpha+1)\Gamma(\beta+1)} \int_0^1 x^{\alpha}(1-x)^{\beta}|f(x)|^2 dx$$

is valid, with equality holds when $f(x) = 1$.

Proof. Let

$$u_n(x) = \sqrt{\frac{(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)\Gamma(n+\alpha+1)}{n!\Gamma(\alpha+1)^2\Gamma(n+\beta+1)}}\, J_n(x;\alpha,\beta),$$

$$v_n(x) = \sqrt{\frac{(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)\Gamma(n+\alpha+\mu+1)}{n!\Gamma(\alpha+\mu+1)^2\Gamma(n+\beta-\mu+1)}}\, J_n(x;\alpha+\mu,\beta-\mu),$$

$$s_n = \frac{\Gamma(\alpha+\mu+1)}{\Gamma(\alpha+1)}\sqrt{\frac{\Gamma(n+\alpha+1)\Gamma(n+\beta-\mu+1)}{\Gamma(n+\alpha+\mu+1)\Gamma(n+\beta+1)}}.$$

Then $\{u_n\}_{n\in\mathbb{N}}$ and $\{v_n\}_{n\in\mathbb{N}}$ are orthonormal basis in the spaces $L^2\left((0,1),x^{\alpha}(1-x)^{\beta}\right)$ and $L^2\left((0,1),x^{\alpha+\mu}(1-x)^{\beta-\mu}\right)$.

From the following asymptotic expansions:

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} \sim z^{a-b},\ as\ z\to\infty,\ |arg(z)|<\pi,$$

we obtain

$$s_n \sim \frac{\Gamma(\alpha+\mu+1)}{\Gamma(\alpha+1)} n^{-\mu},\ as\ n\to\infty.$$

On the other hand a simple calculation shows that

$$\left(\frac{s_{n+1}}{s_n}\right)^2 - 1 = -\mu\frac{2n+\alpha+\beta+1}{(n+\alpha+\mu+1)(n+\beta+1)} < 0 \quad (\mu>0).$$

Hence, the sequence $\{s_n\}_{n\in\mathbb{N}}$ is positive decreasing and approaching 0.

The beta integral yields

$$T_{\alpha,\mu}x^n = \frac{(\alpha)_n}{(\alpha+\mu+1)_n}x^n.$$

Therefore, we have

$$T_{\alpha,\mu}J_n(\,.\,;\alpha,\beta)(x) = J_n(x;\alpha+\mu,\beta-\mu),$$

and

$$T_{\beta,\mu}u_n = s_n v_n.$$

So the result is a consequence of Lemma 2.1. □

An equivalent inequality can be obtained for the fractional integral operators $S_{\lambda,\mu}$, which are defined by[3]

$$S_{\lambda,\mu}f(x) = \frac{x^{-\lambda-\mu}\Gamma(\lambda+\mu+1)}{\Gamma(\lambda+1)\Gamma(\mu)}\int_x^1 (1-t)^{\lambda}(x-t)^{\mu-1} f(1-t)\,dt,$$

where f is a locally integrable function and $\lambda > -1,\ \mu > 0$.

Put

$$u_n(x) = \sqrt{\frac{(2n+\alpha+\beta+1)\,\Gamma(n+\alpha+\beta+1)\,\Gamma(n+\alpha+1)}{n!\Gamma(\alpha+1)^2\Gamma(n+\beta+1)}} J_n(x;\alpha,\beta),$$

$$v_n(x) = \sqrt{\frac{(2n+\alpha+\beta+1)\,\Gamma(n+\alpha+\beta+1)\,\Gamma(n+\alpha-\mu+1)}{n!\Gamma(\alpha-\mu+1)^2\Gamma(n+\beta-\mu+1)}} J_n(1-x;\alpha+\mu,\beta-\mu),$$

$$s_n = \frac{\Gamma(\alpha-\mu+1)}{\Gamma(\alpha+1)}\sqrt{\frac{\Gamma(n+\alpha+1)\,\Gamma(n+\beta+\mu+1)}{\Gamma(n+\alpha-\mu+1)\,\Gamma(n+\beta+1)}},$$

and using the beta integral formula to obtain

$$S_{\alpha,\mu}u_n = s_n v_n.$$

Therefore, from the Lemma 2.1, we get under the following condition

$$\min(\alpha,\ \alpha+\beta+1,\ \beta-\mu, \mu-1) > -1,$$

the inequality for $S_{\alpha,\mu}$

$$\int_0^1 x^{\beta+\mu}(1-x)^{\alpha-\mu}\left|S_{\beta,\mu}f(x)\right|^2 dx \leqslant \frac{\Gamma(\alpha-\mu+1)\,\Gamma(\beta+\mu+1)}{\Gamma(\alpha+1)\,\Gamma(\beta+1)}\int_0^1 x^{\alpha}(1-x)^{\beta}\,|f(x)|^2\,dx.$$

Note that the equality holds when $f(x) = 1$.

3. Hardy type inequality for discrete transform

We consider the polynomials

$$\phi_N(x) = \sum_{k=0}^{N}\alpha_k x^k, \text{with } \alpha_k \neq 0,\ N = 0,1,\dots,$$

and we define a transform S_N on finite sequences $\{f(n),\ n = 0,\dots,N\}$ by

$$S_N[f;\phi_N,x] = \sum_{n=0}^{N}\frac{(-1)^n}{n!}\phi_N^{(n)}(x)\,f(n).$$

Then from the Taylor series it is easy to see

$$S_N[\binom{n}{j},\phi_N,x] = (-1)^j\alpha_j x^j,\ j = 0,\dots,N.$$

The transform $S_N[.;\phi_N,x]\,(3.1)$ with $\phi_N(x) = (1-x)^N$, has the property

$$S_N[Q_j(x,\alpha,\beta,N),(1-x)^N,x] = J_j(x;\beta,\alpha),\ j = 0,1,\dots,N$$

where $Q_j(x,\alpha,\beta,N)$ are the Hahn polynomials [12] given by

$$Q_j(x,\alpha,\beta,N) = {}_3F_2\left(\begin{matrix}-j,\, j+\alpha+\beta+1,\, -x\\ \alpha+1,\, -N\end{matrix};1\right),$$

and satisfying for $\alpha > -1$ and $\beta > -1$, the orthogonality relations

$$\sum_{x=0}^{N}\binom{\alpha+x}{x}\binom{\beta+N-\alpha}{N-x}Q_m(x,\alpha,\beta,N)\,Q_n(x,\alpha,\beta,N)$$
$$= \frac{(-1)^n(n+\alpha+\beta+1)_{N+1}(\beta+1)_n n!}{(2n+\alpha+\beta+1)(\alpha+1)_n(-N)_n N!}\delta_{mn}.$$

Let

$$u_j(x) = \sqrt{\frac{N!\,(2j+\alpha+\beta+1)\,(-N)_j\,(\alpha+1)_j}{(-1)^j\, j!\,(\beta+1)_j\,(j+\alpha+\beta+1)_{N+1}}}\; Q_j(x,\alpha,\beta,N)$$

$$v_j(x) = \sqrt{\frac{(2j+\alpha+\beta+1)\,\Gamma(j+\alpha+\beta+1)\,(\beta+1)_j}{j!\,\Gamma(\beta+1)\,\Gamma(j+\alpha+1)}}\; J_j(x,\beta,\alpha)$$

$$s_j = \sqrt{\frac{N!\,(-N)_j\,(\alpha+1)_j\,\Gamma(\beta+1)\,\Gamma(j+\alpha+1)}{(-1)^j\,(\beta+1)_j^2\,(j+\alpha+\beta+1)_{N+1}\,\Gamma(j+\alpha+\beta+1)}}.$$

For $j = 0,1,\ldots,N,$ we have

$$\frac{s_{j+1}}{s_j} = \sqrt{\frac{N-j}{N+\alpha+\beta+2+j}}\;\frac{\alpha+j+1}{\beta+j+1}$$
$$\leqslant \sqrt{\frac{N}{N+\alpha+\beta+2}}\left(1+\frac{\alpha-\beta}{\beta+j+1}\right).$$

Then, for $\beta \geqslant \alpha > -1,$ we see that the sequence $\{s_j\}_{j=0,\ldots,N}$ is decreasing and from Lemma 2.1, we obtain

$$\int_0^1 x^{\beta}\,(1-x)^{\alpha}\left|S_N[f,(1-x)^N,x]\right|^2 dx \leqslant \frac{N!\,\Gamma(\beta+1)\,\Gamma(\alpha+1)}{(\alpha+\beta+1)_{N+1}\,\Gamma(\alpha+\beta+1)}$$
$$\times \sum_{n=0}^{N}\binom{\alpha+n}{n}\binom{N+\beta-\alpha}{N-n}|f(n)|^2.$$

4. Hardy type inequality for q-fractional integral operator

Basic analogue of the Erdelyi-Kober fractional integral operator, defined by Al-Salam and Ismail (19.4.1, [8]) as

$$I^{\alpha,\eta}f(x) = \frac{\Gamma_q(\alpha+\eta)}{\Gamma_q(\alpha)\,\Gamma_q(\eta)}\int_0^1 t^{\alpha-1}\frac{(qt;q)_\infty}{(tq^{\eta};q)_\infty}f(xt)\,d_qt,\ \eta>0,\ \alpha>0,$$

where the q-integral of Jackson is defined by [4]

$$\int_0^a f(x)d_qx := (1-q)a\sum_{k=0}^{\infty} f(aq^k)q^k.$$

For complex a and $0<q<1$, the q-shift factorial are defined by [4]

$$(a;q)_0 := 1,\ \ (a;q)_n := \prod_{k=0}^{n-1}(1-aq^k),\ \ n=1,2,\dots,$$
$$(a;q)_\infty := \lim_{n\to\infty}(a;q)_n.$$

The q-gamma function is defined by

$$\Gamma_q(z) := \frac{(q;q)_\infty}{(q^z;q)_\infty}(1-q)^{1-z}.$$

The basic hypergeometric series are defined by [4]

$${}_r\varphi_s\left(\begin{matrix} a_1,a_2,\dots,a_r \\ b_1,b_2,\dots,b_s\end{matrix}\middle| q,z\right) := \sum_{k=0}^{+\infty}\frac{(a_1,a_2,\dots,a_r\,;q)_k}{(b_1,b_2,\dots,b_s\,;q)_k\,(q;\,q)_k}\left((-1)^k q^{\binom{k}{2}}\right)^{1+s-r} z^k,$$

where

$$(a_1,\dots,a_r\,;q)_n = (a_1;\,q)_n\,\dots(a_r;\,q)_n\,.$$

The little q-Jacobi polynomials are defined by

$$p_n(x;q^\alpha,q^\beta;q) := {}_2\phi_1\left(\begin{matrix} q^{-n},\ q^{\alpha+\beta+n+1} \\ q^{\alpha+1}\end{matrix}\middle| q,x\right),$$

which satisfy the orthogonality relations

$$\frac{(q^{\alpha+1},q^{\beta+1};q)_\infty}{(q^{\alpha+\beta+2},q;q)_\infty}\sum_{k=0}^{\infty} p_m(q^k;q^\alpha,q^\beta;q)p_n(q^k;q^\alpha,q^\beta;q)q^{k(\alpha+1)}\frac{(q^{k+1};q)_\infty}{(q^{\beta+k+1})_\infty}$$
$$= \frac{q^{n(\alpha+1)}(1-q^{\alpha+\beta+1})(q^{\beta+1},q;q)_n}{(1-q^{\alpha+\beta+2n+1})(q^{\alpha+1},q^{\alpha+\beta+1};q)_n}\delta_{m,n},$$

where $\alpha>-1$ and $\beta>-1$, see[12].

THEOREM 4.1. *Let* $\alpha,\ \beta$ *and* η *be real numbers such that*

$$\min(\alpha,\ \beta-\eta,\ \eta-1) > -1.$$

Then for all $f\in L^2\left((0,1)\,,x^\alpha\frac{(qx;q)_\infty}{\left(xq^\beta;q\right)_\infty}d_qx\right)$, *the inequality*

$$\int_0^1 x^{\alpha+\eta}\frac{(qx;q)_\infty}{\left(xq^{\beta-\eta};q\right)_\infty}\left|I^{\alpha+1,\eta}f(x)\right|^2 d_qx$$
$$\leqslant \frac{\left(q^{\alpha+1},q^{\beta+1};q\right)_\infty}{\left(q^{\alpha+\eta+1},q^{\beta-\eta+1};q\right)_\infty}\int_0^1 x^\alpha\frac{(qx;q)_\infty}{\left(xq^\beta;q\right)_\infty}\left|f(x)\right|^2 d_qx$$

is valid, with equality holds when $f(x)=1$.

Proof. When $f(x)=1$ we have $I^{\alpha+1,\eta}f(x)=1$ and the equality is easily verified. Let

$$u_n(x)=\sqrt{\frac{\left(q^{\alpha+1},q^{\beta+1};q\right)_\infty}{(1-q)\left(q,q^{\alpha+\beta+2};q\right)_\infty}\frac{1-q^{\alpha+\beta+2n+1}}{\left(1-q^{\alpha+\beta+1}\right)q^{n(\alpha+1)}}\frac{\left(q^{\alpha+1},q^{\alpha+\beta+1};q\right)_n}{\left(q,q^{\beta+1};q\right)_n}}$$
$$\times p_n\left(x;q^{\alpha},q^{\beta};q\right)$$
$$v_n(x)=\sqrt{\frac{\left(q^{\alpha+\eta+1},q^{\beta-\eta+1};q\right)_\infty}{(1-q)\left(q,q^{\alpha+\beta+2};q\right)_\infty}\frac{1-q^{\alpha+\beta+2n+1}}{\left(1-q^{\alpha+\beta+1}\right)q^{n(\alpha+\eta+1)}}\frac{\left(q^{\alpha+\eta+1},q^{\alpha+\beta+1};q\right)_n}{\left(q,q^{\beta-\eta+1};q\right)_n}}$$
$$\times p_n\left(x;q^{\alpha+\eta},q^{\beta-\eta};q\right)$$
$$s_n=\sqrt{\frac{\left(q^{\alpha+1},q^{\beta+1};q\right)_\infty q^{n\eta}}{\left(q^{\alpha+\eta+1},q^{\beta-\eta+1};q\right)_\infty}\frac{\left(q^{\alpha+1},q^{\beta-\eta+1};q\right)_n}{\left(q^{\beta+1},q^{\alpha+\eta+1};q\right)_n}}.$$

Using now the formula (19.4.24, [8])

$$I^{\alpha+1,\eta}\left(p_n\left(.;q^{\alpha},q^{\beta};q\right)\right)(x)=p_n\left(x;q^{\alpha+\eta},q^{\beta-\eta};q\right),$$

we get

$$I^{\alpha+1,\eta}u_n(x)=s_nv_n(x).$$

For $n=0,1,2,\ldots,$ we have

$$\frac{s_{n+1}}{s_n}=\sqrt{q^{\eta}\frac{(1-q^{\alpha+n+1})(1-q^{\beta-\eta+n+1})}{(1-q^{\beta+n+1})(1-q^{\alpha+\eta+n+1})}},$$

and

$$\lim_{n\to\infty}s_n=0.$$

Then, the condition

$$\min(\alpha,\ \beta-\eta,\ \eta-1)>-1,$$

shows that the sequence $\{s_n\}$ is positive, decreasing and tending to 0.

Hence, from the Lemma 2.1, we have $\sqrt{\frac{\left(q^{\alpha+1},q^{\beta+1};q\right)_\infty}{\left(q^{\alpha+\eta+1},q^{\beta-\eta+1};q\right)_\infty}}$ is the norm of the operator $I^{\alpha+1,\eta}$. □

Consider the family of operators, see ([8], 19.5.1)

$$S_r(f)(\cos\theta)=\frac{(q,r^2;q)_\infty}{2\pi}\int_0^{\pi}\frac{(e^{2i\phi},e^{-2i\phi};q)_\infty f(\cos\phi)}{(re^{i(\phi+\theta)},re^{-i(\phi+\theta)},re^{i(\phi-\theta)},re^{-i(\phi+\theta)};q)_\infty}d\phi,\ r\in(0,1).$$

The operators S_r satisfy the semigroup property [8]

$$S_r\circ S_t=S_{rt}\text{ for }r,t,rt\in(0,1).$$

Let

$$u_n(x) = \sqrt{\frac{(q,t_1t_2;q)_\infty (t_1t_2;q)_n}{2\pi (q;q)_n t_1^{2n}}} \frac{p_n(x;t_1,t_2 \mid q)}{(t_1e^{i\theta},t_1e^{-i\theta};q)_\infty}, \ x = \cos\theta$$

$$v_n(x) = \sqrt{\frac{(q,t_1t_2;q)_\infty (t_1t_2;q)_n}{2\pi (q;q)_n (rt_1)^{2n}}} \frac{p_n(x;rt_1,t_2/r \mid q)}{(rt_1e^{i\theta},rt_1e^{-i\theta};q)_\infty}$$

$$s_n = r^n,$$

where $p_n(x;t_1,t_2 \mid q)$ is the Al-Salam-Chihara polynomials defined by

$$p_n(x;t_1,t_2;q) := {}_3\phi_2\left(\begin{matrix} q^{-n}, \ t_1e^{i\theta}, t_2e^{-i\theta} \\ t_1t_2, \ 0 \end{matrix} \middle| q,q \right).$$

If $\max(|t_1|,|t_2|) < 1$, then we have the orthogonality relations [12]

$$\frac{1}{2\pi}\int_{-1}^{1} \frac{w(x;t_1,t_2|q)}{\sqrt{1-x^2}} p_m(x;t_1,t_2 \mid q)\, p_n(x;t_1,t_2 \mid q)\, dx = \frac{t_1^{2n}}{(t_1t_2;q)_n (q^{n+1},t_1t_2;q)_\infty}\delta_{mn},$$

where

$$w(x;t_1,t_2|q) := \frac{(e^{2i\theta},e^{-2i\theta};q)_\infty}{(t_1e^{i\theta},t_1e^{-i\theta},t_2e^{i\theta},t_2e^{-i\theta};q)_\infty}, \ x = cos\theta.$$

Using Theorem 19.5.1, [8], we obtain

$$S_r\left(\frac{p_n(x;t_1,t_2 \mid q)}{(t_1e^{i\theta},t_1e^{-i\theta};q)_\infty}\right) = \frac{p_n(x;rt_1,t_2/r \mid q)}{(rt_1e^{i\theta},rt_1e^{-i\theta};q)_\infty},$$

and

$$S_r(u_n)(x) = s_n v_n(x).$$

Let

$$0 < r < 1, \ \text{and} \ \max(|t_1|,|t_2|) < 1.$$

By Lemma 2.1, we get

$$\int_0^\pi \frac{(e^{2i\theta},e^{-2i\theta},t_1re^{i\theta},t_1re^{-i\theta};q)_\infty}{(t_2/re^{i\theta},t_2/re^{-i\theta};q)_\infty} |S_r(f)(\cos\theta)|^2\, d\theta$$
$$\leqslant \int_0^\pi \frac{(e^{2i\theta},e^{-2i\theta},t_1e^{i\theta},t_1e^{-i\theta};q)_\infty}{(t_2e^{i\theta},t_2e^{-i\theta};q)_\infty} |f(\cos\theta)|^2\, d\theta.$$

Note that, the equality holds when

$$f(x) = \frac{1}{(t_1e^{i\theta},t_1e^{-i\theta};q)_\infty}.$$

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