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A NEW PROOF OF SHAPIRO INEQUALITY

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Abstract. We present a new proof of Shapiro cyclic inequality. Especially, we treat the case $n = 23$ precisely.

1. Introduction

Let $n \geqslant 3$ be an integer, $x_1, x_2, \dots, x_n$ be positive real numbers, and let

$$E_n(x_1,\dots,x_n) := \sum_{i=1}^{n} \frac{x_i}{x_{i+1}+x_{i+2}},$$

here we regard $x_{i+n} = x_i$ for $i \in \mathbb{Z}$. In this article, we present a new proof of the following theorem:

THEOREM 1.1.
(1) If n is an odd integer with $3 \leqslant n \leqslant 23$, then

$$E_n(x_1,\dots,x_n) \geqslant n/2. \qquad (P_n)$$

Moreover, $E_n(x_1,\dots,x_n) = n/2$ holds only if $x_1 = x_2 = \cdots = x_n$.

(2) If n is an even integer with $4 \leqslant n \leqslant 12$, then (P_n) holds. Moreover, the equality holds only if $(x_1,\dots,x_n) = (a, b, a, b, \dots, a, b)$ $(\exists a > 0, \exists b > 0)$.

(3) If n is an even integer with $n \geqslant 14$ or an odd integer with $n \geqslant 25$, then there exists $x_1 > 0, \dots, x_n > 0$ such that $E_n(x_1,\dots,x_n) < n/2$.

(3) was proved by [4] in 1979. It is said that (1) was proved by [6] in 1989. (2) was proved by [2] in 2002. Note that [2] treats (1) to be an open problem. The author also thinks we should give a more agreeable proof of (1). In this article, we give more precise proof of (1) than [6].

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2. Basic Facts

Throughout this article, we use the following notations:

$$\partial_i E_n(\mathbf{x}) := \frac{\partial}{\partial x_i} E_n(\mathbf{x}) = \frac{1}{x_{i+1}+x_{i+2}} - \frac{x_{i-2}}{(x_{i-1}+x_i)^2} - \frac{x_{i-1}}{(x_i+x_{i+1})^2}$$

$$\overline{K_n} := \left\{ (x_1,\ldots,x_n) \in \mathbb{R}^n \;\middle|\; x_1 \geqslant 0, \ldots, x_n \geqslant 0 \right\}$$

$$K_n^\circ := \left\{ (x_1,\ldots,x_n) \in \mathbb{R}^n \;\middle|\; x_1 > 0, \ldots, x_n > 0 \right\}$$

$$K_n^\bullet := \left\{ (x_1,\ldots,x_n) \in \overline{K_n} \;\middle|\; \begin{array}{l} (x_1,\ldots,x_n) \notin K_n^\circ, \\ (x_i,x_{i+1}) \neq (0,0) \text{ for any } i \in \mathbb{Z}. \end{array} \right\}$$

$$K_n := K_n^\circ \cup K_n^\bullet$$

It is easy to see that there exists $\mathbf{a} \in K_n^\bullet$ such that

$$\inf_{\mathbf{x} \in K_n^\circ} E_n(\mathbf{x}) = E_n(\mathbf{a}).$$

Thus, we consider $E_n(\mathbf{x})$ to be a continious function on $K_n^\bullet$.

PROPOSITION 2.1. ([3])
(1) If (P_n) is false, then (P_{n+2}) is also false.
(2) If (P_n) is false for an odd integer $n \geqslant 3$, then (P_{n+1}) is also false.

Proof. Assume that there exists positive real numbers $a_1,\ldots, a_n$ such that $E_n(a_1,\ldots, a_n) < n/2$.

(1) Since, $E_{n+2}(a_1,\ldots,a_n,a_1,a_2) = 1 + E_n(a_1,\ldots,a_n) < \dfrac{n+2}{2}$, (P_{n+2}) is false.

(2) Note that

$$\begin{aligned} & E_{n+1}(a_1,\ldots,a_{r-1},a_r,a_r,a_{r+1},\ldots,a_n) - E_n(a_1,\ldots,a_n) - \frac{1}{2} \\ & = \frac{a_{r-1}}{a_r+a_r} + \frac{a_r}{a_r+a_{r+1}} - \frac{a_{r-1}}{a_r+a_{r+1}} - \frac{1}{2} \\ & = \frac{(a_r-a_{r-1})(a_r-a_{r+1})}{2a_r(a_r+a_{r+1})} \end{aligned}$$

for $1 \leqslant r \leqslant n$. Thus, it is sufficient to show that there exists r such that $(a_r - a_{r-1})(a_r - a_{r+1}) \leqslant 0$.

Assume that $(a_r - a_{r-1})(a_r - a_{r+1}) > 0$ for all $1 \leqslant r \leqslant n$. Since n is odd,

$$\prod_{r=1}^{n} (a_r - a_{r+1})^2 = \prod_{r=1}^{n} (a_{r-1} - a_r)(a_r - a_{r+1}) < 0.$$

This is a contradiction. □

PROPOSITION 2.2. ([4])

(1) $E_{14}(42,\ 2,\ 42,\ 4,\ 41,\ 5,\ 39,\ 4,\ 38,\ 2,\ 38,\ 0,\ 40,\ 0) < 7$. *Thus* (P_{14}) *is false.*

(2) $E_{25}(34,\ 5,\ 35,\ 13,\ 30,\ 17,\ 24,\ 18,\ 18,\ 17,\ 13,\ 16,\ 9,\ 16,\ 5,\ 16,\ 2,\ 18,\ 0,\ 21,\ 0,\ 25,\ 0,\ 29,\ 0) < 25/2$. *Thus* (P_{25}) *is false.*

Thus, Theorem 1.1 (3) is proved by Proposition 2.1 and 2.2. It is essential to show (P_{12}) and (P_{23}) for a proof of Theorem 1.1 (2) and (3).

DEFINITION 2.3. We say that $\mathbf{x} = (x_1, \dots, x_n) \in K_n$ and $\mathbf{y} = (y_1, \dots, y_n) \in K_n$ belong to the *same component* if "$x_i = 0 \iff y_i = 0$" for all $i = 1, \dots, n$.

Let $\mathbf{x} = (x_1, \dots, x_n) \in K_n^\bullet$. If $x_{i-1} = 0$, $x_i \neq 0$, $x_{i+1} \neq 0, \dots, x_j \neq 0$, and $x_{j+1} = 0$ for $i < j \in \mathbb{Z}$, then we call $(x_i, \dots, x_j)$ to be a *segment* of $\mathbf{x}$, and we define $j - i + 1$ to be the *length* of this segment. A segment of length l is called l*-semgent*.

For a segment $\mathbf{s} := (x_i, \dots, x_j)$ of $\mathbf{x}$, we denote

$$S(\mathbf{s}) := \sum_{k=i}^{j-1} \frac{x_k}{x_{k+1} + x_{k+2}}, \quad \mathrm{Head}(\mathbf{s}) := x_i, \quad \mathrm{Tail}(\mathbf{s}) := x_j.$$

Here we define $S(\mathbf{s}) = 0$, if the length of $\mathbf{s}$ is 1.

Let $\mathbf{s}_1, \dots, \mathbf{s}_r$ be all the segments of $\mathbf{x}$ in this order. Let l_k be the length of $\mathbf{s}_k$. Then $(l_1, \dots, l_r)$ is called the *index* of $\mathbf{x}$. Note that

$$E_n(\mathbf{x}) = \sum_{k=1}^{r} S(\mathbf{s}_k) + \sum_{k=1}^{r} \frac{\mathrm{Tail}(\mathbf{s}_{k-1})}{\mathrm{Head}(\mathbf{s}_k)}.$$

Here we regard $\mathbf{s}_{k+r} = \mathbf{s}_k$ for $k \in \mathbb{Z}$.

THEOREM 2.4. *Assume that* $\min_{\mathbf{x} \in K_n^\bullet} E_n(\mathbf{x}) = E_n(\mathbf{a})$ *at* $\mathbf{a} = (a_1, \dots, a_n) \in K_n^\bullet$. *Let* $\mathbf{s}_1, \dots, \mathbf{s}_r$ *be all the segments of* $\mathbf{a}$ *in this order, and let* l_k *be the length of* $\mathbf{s}_k$. *Then the followings hold.*

(1) $\dfrac{\mathrm{Tail}(\mathbf{s}_1)}{\mathrm{Head}(\mathbf{s}_2)} = \dfrac{\mathrm{Tail}(\mathbf{s}_2)}{\mathrm{Head}(\mathbf{s}_3)} = \cdots = \dfrac{\mathrm{Tail}(\mathbf{s}_{r-1})}{\mathrm{Head}(\mathbf{s}_r)} = \dfrac{\mathrm{Tail}(\mathbf{s}_r)}{\mathrm{Head}(\mathbf{s}_1)}$.

(2) Assume that $\mathbf{a} = (\mathbf{s}_1,\ 0,\ \mathbf{s}_2,\ 0, \dots,\ \mathbf{s}_r,\ 0)$, *and let* σ *be a permutation of* $\{1, 2, \dots, r\}$. *Then there exist real numbers* $t_1 > 0$, $t_2 > 0, \dots, t_r > 0$ *such that*

$$\mathbf{b} := \left(t_1 \mathbf{s}_{\sigma(\mathbf{1})}, 0, t_2 \mathbf{s}_{\sigma(\mathbf{2})}, 0, \dots, t_r \mathbf{s}_{\sigma(\mathbf{r})}, 0\right)$$

satisfies $E_n(\mathbf{b}) = E_n(\mathbf{a})$.

Proof. (1) Since $E_n(a_{1+k}, a_{2+k}, \dots, a_{n+k}) = E_n(a_1, a_2, \dots, a_n)$, we may assume $\mathbf{a} = (\mathbf{s}_1,\ 0,\ \mathbf{s}_2,\ 0, \dots,\ \mathbf{s}_r,\ 0)$. Let $x_i := \mathrm{Head}(\mathbf{s}_i)$, $y_i := \mathrm{Tail}(\mathbf{s}_i)$. Define $t_1, \dots, t_r$ by $t_1 := 1$ and

$$t_j := \frac{y_1 y_2 \cdots y_{j-1}}{x_2 x_3 \cdots x_j} \cdot \left(\frac{x_1 x_2 \cdots x_r}{y_1 y_2 \cdots y_r}\right)^{\frac{j-1}{r}}$$

for $j = 2, 3, \dots, r$. It is easy to see that

$$\frac{t_{j-1} y_{j-1}}{t_j x_j} = \sqrt[r]{\frac{y_1 \cdots y_r}{x_1 \cdots x_r}} = \frac{t_r y_r}{t_1 x_1}.$$

Let

$$\mathbf{c} = (t_1\mathbf{s}_1, 0, t_2\mathbf{s}_2, 0, \ldots, t_r\mathbf{s}_r, 0).$$

Note that $S(t_i\mathbf{s}_i) = S(\mathbf{s}_i)$. By AM-GM inequality,

$$\begin{aligned} E_n(\mathbf{a}) &= \sum_{i=1}^{r} S(\mathbf{s}_i) + \sum_{i=1}^{r} \frac{y_{i-1}}{x_i} \\ &\geqslant \sum_{i=1}^{r} S(\mathbf{s}_i) + r \cdot \sqrt[r]{\frac{y_1 \cdots y_r}{x_1 \cdots x_r}} \\ &= \sum_{i=1}^{r} S(t_i\mathbf{s}_i) + \sum_{i=1}^{r} \frac{t_{i-1}y_{i-1}}{t_i x_i} = E_n(\mathbf{c}). \end{aligned}$$

Since $E_n(\mathbf{a})$ is the minimum, we have $E_n(\mathbf{a}) = E_n(\mathbf{c})$. By the equality condition of AM-GM inequality, we have $t_1 = t_2 = \cdots = t_r = 1$. Thus

$$\frac{y_{j-1}}{x_j} = \sqrt[r]{\frac{y_1 \cdots y_r}{x_1 \cdots x_r}},$$

and we have (1).

(2) By the same argument as (1), we conclude that there exists positive integers $t'_1, \ldots, t'_r$ such that

$$\mathbf{b} := (t'_1\mathbf{s}_{\sigma(1)}, 0, t'_2\mathbf{s}_{\sigma(2)}, 0, \ldots, t'_r\mathbf{s}_{\sigma(r)}, 0)$$

satisfies

$$E_n(\mathbf{b}) = \sum_{i=1}^{r} S(\mathbf{s}_i) + r \cdot \sqrt[r]{\frac{y_1 \cdots y_r}{x_1 \cdots x_r}}.$$

Thus $E_n(\mathbf{b}) = E_n(\mathbf{a})$. □

REMARK 2.5. By the above theorem, we may assume that the index $(l_1, \ldots, l_r)$ of $\mathbf{a}$ satisfies $l_1 \geqslant l_2 \geqslant \cdots \geqslant l_r$, if $\min_{\mathbf{x} \in K_n^\bullet} E_n(\mathbf{x}) = E_n(\mathbf{a})$. Thus, we always write the index of such $\mathbf{a}$ in descending order.

DEFINITION 2.6. Assume that $\mathbf{a} \in K_n^\bullet$ satisfies the condition of the above theorem. Then we define $U(\mathbf{a})$ to be

$$U(\mathbf{a}) := \frac{\mathrm{Tail}(\mathbf{s}_1)}{\mathrm{Head}(\mathbf{s}_2)} = \frac{\mathrm{Tail}(\mathbf{s}_2)}{\mathrm{Head}(\mathbf{s}_3)} = \cdots = \frac{\mathrm{Tail}(\mathbf{s}_{r-1})}{\mathrm{Head}(\mathbf{s}_r)} = \frac{\mathrm{Tail}(\mathbf{s}_r)}{\mathrm{Head}(\mathbf{s}_1)}.$$

Note that $E_n(\mathbf{a}) = rU(\mathbf{a}) + \sum_{k=1}^{r} S(\mathbf{s}_k)$, for $\mathbf{a} = (\mathbf{s}_1, 0, \mathbf{s}_2, 0, \ldots, \mathbf{s}_r, 0)$.

3. Bushell Theorem

We survey and improve the results of [1]. In this section, we denote

$$\begin{aligned}
A_i(\mathbf{x}) &:= \frac{x_i}{x_{i+1}+x_{i+2}}\\
B(\mathbf{x}) &:= \big(x_2+x_3, x_3+x_4, \ldots, x_n+x_1, x_1+x_2\big)\\
R(\mathbf{x}) &:= \left(\frac{1}{x_n}, \frac{1}{x_{n-1}}, \frac{1}{x_{n-2}}, \ldots, \frac{1}{x_1}\right)\\
T(\mathbf{x}) &= \left(\frac{x_n}{(x_1+x_2)^2}, \ldots, \frac{x_{n+1-i}}{(x_{n+2-i}+x_{n+3-i})^2}, \ldots, \frac{x_1}{(x_2+x_3)^2}\right)
\end{aligned}$$

for $\mathbf{x} = (x_1, \ldots, x_n)$. We also denote the i-th element of $B(\mathbf{x})$ by $B(\mathbf{x})_i = x_{i+1}+x_{i+2}$. $R(\mathbf{x})_i$ and $T(\mathbf{x})_i$ are also defined similarly. The symbol $T(\mathbf{x})$ are used throughout this article.

LEMMA 3.1. ([1] Lemma 3.2, 4.2) *The above functions satisfy the followings.*

(1) $\partial_i E_n(\mathbf{x}) = (R(B(\mathbf{x}))_{n+1-i} - (B(T(\mathbf{x})))_{n+1-i}$.

(2) $(T^2(\mathbf{x}))_i = \dfrac{x_i}{\big(1-(B(\mathbf{x}))_i\partial_i E_n(\mathbf{x})\big)^2}$.

(3) $E_n(T(\mathbf{x})) - E_n(\mathbf{x}) = \displaystyle\sum_{i=1}^{n} \frac{x_i\big(\partial_i E_n(\mathbf{x})\big)^2}{(B(T(\mathbf{x})))_{n+1-i}}$.

(4)
$$\begin{aligned}
&E_n(\mathbf{x})+E_n(\mathbf{y})\\
&= E_n(\mathbf{x}+\mathbf{y}) + E_n(T(\mathbf{x})+T(\mathbf{y}))\\
&\quad - \sum_{i=1}^{n} \frac{(T(\mathbf{x})+T(\mathbf{y}))_{n+1-i}\big(\partial_i E_n(\mathbf{x})+\partial_i E_n(\mathbf{y})\big)}{\big(R(B(\mathbf{x}))+R(B(\mathbf{y}))\big)_{n+1-i}\cdot(B(T(\mathbf{x})+T(\mathbf{y})))_{n+1-i}}.
\end{aligned}$$

Proof. (1)
$$\begin{aligned}
\partial_i E_n(\mathbf{x}) &= \frac{1}{x_{i+1}+x_{i+2}} - \left(\frac{x_{i-2}}{(x_{i-1}+x_i)^2} + \frac{x_{i-1}}{(x_i+x_{i+1})^2}\right)\\
&= (R(B(\mathbf{x}))_{n+1-i} - (B(T(\mathbf{x})))_{n+1-i}.
\end{aligned}$$

(2) $(T(\mathbf{x}))_i = \dfrac{x_{n+1-i}}{(B(\mathbf{x}))^2_{n+1-i}}$. Combine this with (1), we obtain

$$(T^2(\mathbf{x}))_i = \frac{(T(\mathbf{x}))_{n+1-i}}{(B(T(\mathbf{x})))^2_{n+1-i}} = \frac{x_i/(B(\mathbf{x}))_i^2}{\big((R(B(\mathbf{x})))_{n+1-i} - \partial_i E_n(\mathbf{x})\big)^2}. \tag{3.1.1}$$

Since $(B(\mathbf{x}))_i \cdot (R(B(\mathbf{x})))_{n+1-i} = 1$, we obtain (2).

(3) By the similar calculation as above, we obtain

$$\begin{aligned}
E_n(T(\mathbf{x})) - E_n(\mathbf{x}) &= \sum_{i=1}^{n} \frac{(T(\mathbf{x}))_i}{(B(T(\mathbf{x})))_i} - \sum_{i=1}^{n} \frac{x_i}{(B(\mathbf{x}))_i}\\
&= \sum_{i=1}^{n} \left(\frac{(T(\mathbf{x}))_{n+1-i}}{(B(T(\mathbf{x})))_{n+1-i}} - \frac{x_i}{(B(\mathbf{x}))_i}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{n}\left(\frac{x_i}{(B(\mathbf{x}))_i\big(1-(B(\mathbf{x}))_i\partial_i E_n(\mathbf{x})\big)}-\frac{x_i}{(B(\mathbf{x}))_i}\right)\\
&= \sum_{i=1}^{n}\frac{x_i\partial_i E_n(\mathbf{x})}{1-(B(\mathbf{x}))_i\partial_i E_n(\mathbf{x})}.
\end{aligned}
$$

Since,

$$
\begin{aligned}
\sum_{i=1}^{n} x_i\partial_i E_n(\mathbf{x}) &= \sum_{i=1}^{n}\frac{x_i}{x_{i+1}+x_{i+2}}-\sum_{i=1}^{n}\frac{x_{i-2}x_i}{(x_{i-1}+x_i)^2}-\sum_{i=1}^{n}\frac{x_{i-1}x_i}{(x_i+x_{i+1})^2}\\
&= \sum_{i=1}^{n}\frac{x_{i-1}(x_i+x_{i+1})}{(x_i+x_{i+1})^2}-\sum_{i=1}^{n}\frac{x_{i-1}x_{i+1}}{(x_i+x_{i+1})^2}-\sum_{i=1}^{n}\frac{x_{i-1}x_i}{(x_i+x_{i+1})^2}=0,
\end{aligned}
$$

we obtain

$$
\begin{aligned}
E_n(T(\mathbf{x}))-E_n(\mathbf{x}) &= \sum_{i=1}^{n} x_i\partial_i E_n(\mathbf{x})\left(\frac{1}{1-(B(\mathbf{x}))_i\partial_i E_n(\mathbf{x})}-1\right)\\
&= \sum_{i=1}^{n}\frac{x_i\big(\partial_i E_n(\mathbf{x})\big)^2}{(B(T(\mathbf{x})))_{n+1-i}}.
\end{aligned}
$$

(4) Let $a := x_i$, $b := x_{i+1}+x_{i+2} = (B(\mathbf{x}))_i$, $c := y_i$, $d := (B(\mathbf{y}))_i$.

$$
\begin{aligned}
&\frac{x_i+y_i}{(B(\mathbf{x}+\mathbf{y}))_i}+\frac{(T(\mathbf{x})+T(\mathbf{y}))_{n+1-i}}{\big(R(B(\mathbf{x}))+R(B(\mathbf{y}))\big)_{n+1-i}} \qquad (3.1.2)\\
&= \frac{a+c}{b+d}+\frac{a/b^2+c/d^2}{1/b+1/d}=\frac{a}{b}+\frac{c}{d}=A_i(\mathbf{x})+A_i(\mathbf{y})
\end{aligned}
$$

By (1), we have

$$
\begin{aligned}
&\frac{(T(\mathbf{x})+T(\mathbf{y}))_{n+1-i}}{(B(T(\mathbf{x})+T(\mathbf{y})))_{n+1-i}}-\frac{(T(\mathbf{x})+T(\mathbf{y}))_{n+1-i}}{\big(R(B(\mathbf{x}))+R(B(\mathbf{y}))\big)_{n+1-i}}\\
&= \frac{(T(\mathbf{x})+T(\mathbf{y}))_{n+1-i}\big(\partial_i E_n(\mathbf{x})+\partial_i E_n(\mathbf{y})\big)}{\big(R(B(\mathbf{x}))+R(B(\mathbf{y}))\big)_{n+1-i}\cdot(B(T(\mathbf{x})+T(\mathbf{y})))_{n+1-i}}. \qquad (3.1.3)
\end{aligned}
$$

Take $\sum_{i=1}^{n}$ of (3.1.2) and (3.1.3), we obtain (4). □

THEOREM 3.2. ([1] Theorem 3.3)

(1) $E_n(T(\mathbf{x})) \geqslant E_n(\mathbf{x})$ holds for $\mathbf{x}\in K_n$. Moreover, if $E_n(T(\mathbf{x})) = E_n(\mathbf{x})$, then $T^2(\mathbf{x}) = \mathbf{x}$ holds.

(2) If $\min_{\mathbf{x}\in K_n^\bullet} E_n(\mathbf{x}) = E_n(\mathbf{a})$ at $\mathbf{a}\in K_n$, then the following holds.

$$
T^2(\mathbf{a})=\mathbf{a},\quad E_n(T(\mathbf{a}))=E_n(\mathbf{a}).
$$

Proof. (1) $E_n(T(\mathbf{x})) \geqslant E_n(\mathbf{x})$ follows from Lemma 3.1 (3). Assume that $E_n(T(\mathbf{x})) = E_n(\mathbf{x})$. Then $x_i\big(\partial_i E_n(\mathbf{x})\big)^2 = 0$ ($\forall i = 1, \ldots, n$), by Lemma 3.1 (3). Thus $x_i = 0$ or $\partial_i E_n(\mathbf{x}) = 0$. By Lemma 3.1 (2), we obtain $(T^2(\mathbf{x}))_i = x_i$.

(2) If E_n is minimum at $\mathbf{a}$, then $a_i = 0$ or $\partial_i E_n(\mathbf{a}) = 0$. By Lemma 3.1 (2), we have $(T^2(\mathbf{a}))_i = a_i$. We also have $E_n(T(\mathbf{a})) = E_n(\mathbf{a})$ by Lemma 3.1 (3). □

LEMMA 3.3. ([1] Lemma 4.3) *Let a, b, c, d, e be positive real numbers, and p, q be real numbers. Assume that*

$$p\frac{1+\lambda a}{(1+\lambda c)^2} + q\frac{1+\lambda b}{(1+\lambda d)^2} = \frac{1}{1+\lambda e} \tag{3.3.1}$$

for all real numbers $\lambda \geqslant 0$. Then the followings hold.

(1) If $p = 0$, then $q = 1$ and $b = d = e$.
(2) If $q = 0$, then $p = 1$ and $a = c = e$.
(3) If $p \neq 0$ and $q \neq 0$, then $c = d = e$.

Proof. (1) Substitute $\lambda = 0$, $p = 0$ for (3.3.1), we have $q = 1$. In this case, (3.3.1) is equivalent to

$$(1+\lambda b)(1+\lambda e) = (1+\lambda d)^2.$$

As an equality of a polynomial in λ, we have $b = d = e$.

(2) can be proved similarly as (1).

(3) Let

$$\begin{aligned} g(\lambda) := {} & p(1+\lambda a)(1+\lambda d)^2(1+\lambda e) \\ & + q(1+\lambda b)(1+\lambda c)^2(1+\lambda e) - (1+\lambda c)^2(1+\lambda d)^2. \end{aligned} \tag{3.3.2}$$

$g(\lambda) = 0$ as a polynomial in λ. Thus

$$0 = g\left(-\frac{1}{e}\right) = -\left(1-\frac{c}{e}\right)^2\left(1-\frac{d}{e}\right)^2,$$

and we have $c = e$ or $d = e$.

Assume that $d \neq e$. Then $c = e$. From (3.3.2), we obtain

$$p(1+\lambda a)(1+\lambda d)^2 + q(1+\lambda b)(1+\lambda e)^2 - (1+\lambda e)(1+\lambda d)^2 = 0. \tag{3.3.3}$$

Substitute $\lambda = -1/e$ for (3.3.3), we obtain $p(1-a/e)(1-d/e)^2 = 0$. Thus $a = e$. Then

$$p(1+\lambda d)^2 + q(1+\lambda b)(1+\lambda e) - (1+\lambda d)^2 = 0. \tag{3.3.4}$$

Substitute $\lambda = -1/e$ for (3.3.4), we have $d = e$. A contradiction. Thus $d = e$.

Similarly, we have $c = e$. □

THEOREM 3.4.

(1) Assume that $\min_{\mathbf{x}\in K_n} E_n(\mathbf{x}) = E_n(\mathbf{a}) = E_n(\mathbf{b})$ *at* $\mathbf{a},\ \mathbf{b}\in K_n^{\bullet}$ *and that* $\mathbf{a}$ *and* $\mathbf{b}$ *belong to the same component. Then, there exists a real number* $\mu > 0$ *such that* $\mathbf{a} = \mu\mathbf{b}$.

(2) Assume that $\min_{\mathbf{x}\in K_n} E_n(\mathbf{x}) = E_n(\mathbf{a})$ *at* $\mathbf{a}\in K_n^{\circ}$. *Then* $E_n(\mathbf{a}) = n/2$. *Moreover* $\mathbf{a} = (a,\ a,\ a,\dots,\ a)$ *(*$\exists a > 0$*), or* $\mathbf{a} = (a,\ b,\ a,\ b,\ \dots,\ a,\ b)$ *(*$\exists a > 0,\ b > 0$*).*

Proof. Assume that $\min_{\mathbf{x}\in K_n} E_n(\mathbf{x}) = E_n(\mathbf{a}) = E_n(\mathbf{b})$ for $\mathbf{a},\ \mathbf{b}\in K_n$, and that $\mathbf{a}$ and $\mathbf{b}$ belong to the same component. Let $\lambda > 0$ be any real number.

If $a_i \neq 0$, then $\partial_i E_n(\mathbf{a}) = \partial_i E_n(\lambda\mathbf{b}) = 0$. If $a_i = 0$, then $b_i = 0$ and $(T(\mathbf{a}))_{n+1-i} = 0,\ (T(\lambda\mathbf{b}))_{n+1-i} = 0$. Thus we have

$$(T(\mathbf{a}) + T(\lambda\mathbf{b}))_{n+1-i}\cdot\big(\partial_i E_n(\mathbf{a}) + \partial_i E_n(\lambda\mathbf{b})\big) = 0$$

($\forall i\in\mathbb{Z}$). We use the Lemma 3.1 (4) with $\mathbf{x} = T(\mathbf{a}),\ \mathbf{y} = \lambda\mathbf{b}$. Since the numerators of the fractions in $\sum$ in Lemma 3.1 (4) are zero, we have

$$E_n(\mathbf{a}) + E_n(\lambda\mathbf{b}) = E_n(\mathbf{a}+\lambda\mathbf{b}) + E_n(T(\mathbf{a}) + T(\lambda\mathbf{b})).$$

Since $E_n(\lambda\mathbf{b}) = E_n(\mathbf{b}) = E_n(\mathbf{a})$ is minimum, we have

$$E_n(\mathbf{a}+\lambda\mathbf{b}) = E_n(T(\mathbf{a}) + T(\lambda\mathbf{b})) = E_n(\mathbf{a}).$$

Since $E_n(\mathbf{x})$ is minimum at $\mathbf{x} = \mathbf{a}+\lambda\mathbf{b}$ for any $\lambda > 0$, we have

$$0 = \partial_i E_n(\mathbf{a}+\lambda\mathbf{b}) = \frac{1}{(B(\mathbf{a}+\lambda\mathbf{b}))_i} - \frac{a_{i-2}+\lambda b_{i-2}}{(B(\mathbf{a}+\lambda\mathbf{b}))_{i-2}^2} - \frac{a_{i-1}+\lambda b_{i-1}}{(B(\mathbf{a}+\lambda\mathbf{b}))_{i-1}^2} \qquad (3.4.1)$$

when $a_i \neq 0$. Let

$$a := \frac{b_{i-2}}{a_{i-2}},\quad b := \frac{b_{i-1}}{a_{i-1}},\quad c := \frac{(B(\mathbf{b}))_{i-2}}{(B(\mathbf{a}))_{i-2}},\quad d := \frac{(B(\mathbf{b}))_{i-1}}{(B(\mathbf{a}))_{i-1}},$$
$$e := \frac{(B(\mathbf{b}))_i}{(B(\mathbf{a}))_i},\quad p := \frac{a_{i-2}(B(\mathbf{a}))_i}{(B(\mathbf{a}))_{i-2}^2},\quad q := \frac{a_{i-1}(B(\mathbf{a}))_i}{(B(\mathbf{a}))_{i-1}^2}.$$

Then, (3.4.1) become (3.3.1). It is easy to see that the cases (1) and (2) of Lemma 3.3 do not occur. Lemma 3.3 (3) implies

$$\frac{(B(\mathbf{b}))_{i-2}}{(B(\mathbf{a}))_{i-2}} = \frac{(B(\mathbf{b}))_{i-1}}{(B(\mathbf{a}))_{i-1}} = \frac{(B(\mathbf{b}))_i}{(B(\mathbf{a}))_i} =: \frac{1}{\mu} > 0.$$

Thus

$$a_{i+1} + a_{i+2} = B(\mathbf{u}) = \mu B(\mathbf{v}) = \mu(b_{i+1} + b_{i+2}) \qquad (3.4.2)$$

($\forall i\in\mathbb{Z}$). If n is odd, then $a_i = \mu b_i$ ($\forall i\in\mathbb{Z}$) from (3.4.2). Thus $\mathbf{a} = \mu\mathbf{b}$.

We treat the case n is even. Let $\mathbf{w} = (1,\ -1,\ 1,\ -1,\dots,\ -1)\in\mathbb{R}^n$. By elementary linear algebra, we conclude that the solutions of the system of equations (3.4.2) is of the form

$$\mathbf{a} - \mu\mathbf{b} = \nu\mathbf{w} \quad (\exists\nu\in\mathbb{R}).$$

If $\mathbf{a} \in K_n^\bullet$, then $\mathbf{a}$ and $\mathbf{b}$ have zeros at the same places. Thus, ν must be zero. Thus we obtain (1).

We shall prove (2). Apply above argument to $\mathbf{b} = (a_2, a_3, \ldots, a_n, a_1)$. If n is odd, then $\mathbf{a} = \mu\mathbf{b}$. Thus $\mu = 1$, and $a_1 = a_2 = \cdots = a_n$. In this case, $E_n(\mathbf{a}) = n/2$.

If n is even, $\mathbf{a} - \mu\mathbf{b} = \nu\mathbf{w}$. Thus $\mathbf{a} = (a_1, a_2, a_1, a_2, \ldots, a_1, a_2)$. Then $E_n(\mathbf{a}) = n/2$. □

COROLLARY 3.5. *Assume that* $\min_{\mathbf{x}\in K_n} E_n(\mathbf{x}) = E_n(\mathbf{a})$ *at* $\mathbf{a} \in K_n^\bullet$. *Let* $\mathbf{s}$ *and* $\mathbf{t}$ *be segments of* $\mathbf{a}$ *with the same length* l. *Then, there exists a real number* $c > 0$ *such that* $\mathbf{s} = c\mathbf{t}$.

Proof. We construct a vector $\mathbf{b}$ as in the proof of Theorem 2.4 (2), where σ is the transposition of $\mathbf{s}$ and $\mathbf{t}$. Then $E_n(\mathbf{a}) = E_n(\mathbf{b})$. By Theorem 3.4, $\mathbf{a} = \mu\mathbf{b}$ $(\exists\mu > 0)$. Thus $\mathbf{s} = c\mathbf{t}$ $(\exists c > 0)$. □

COROLLARY 3.6. *Assume that* $\min_{\mathbf{x}\in K_n} E_n(\mathbf{x}) = E_n(\mathbf{a})$ *at* $\mathbf{a} \in K_n^\bullet$. *Let* $\mathbf{s} = (a_1, \ldots, a_l)$ *be a* l*-segment of* $\mathbf{a}$ *with* $l \geqslant 2$. *Let* $U := U(\mathbf{a})$. *Then there exists a real number* $\mu > 0$ *such that*

$$\left(\frac{U^2}{a_l}, \frac{a_{l-1}}{a_l^2}, \frac{a_{l-2}}{(a_{l-1}+a_l)^2}, \frac{a_{l-3}}{(a_{l-2}+a_{l-1})^2}, \ldots, \frac{a_2}{(a_3+a_4)^2}, \frac{a_1}{(a_2+a_3)^2}\right)$$
$$= \mu(a_1, a_2, a_3, a_4, \ldots, a_{l-1}, a_l). \tag{3.6.1}$$

Proof. We may assume that $\mathbf{a} = (\mathbf{s}, 0, \ldots)$. Rotate the elements of $T(\mathbf{a})$ so that the segment corresponding to $\mathbf{s}$ comes to be the same place with $\mathbf{s}$, and we denote this vector by $\mathbf{b}$. Then the top segment of $\mathbf{b}$ is

$$\left(\frac{a_l}{a_{l+2}^2}, \frac{a_{l-1}}{a_l^2}, \frac{a_{l-2}}{(a_{l-1}+a_l)^2}, \frac{a_{l-3}}{(a_{l-2}+a_{l-1})^2}, \ldots, \frac{a_2}{(a_3+a_4)^2}, \frac{a_1}{(a_2+a_3)^2}\right).$$

By Theorem 3.2 (2), $E_n(\mathbf{b}) = E_n(T(\mathbf{a})) = E_n(\mathbf{a})$. By Theorem 3.4, $\mathbf{b} = \mu\mathbf{a}$ $(\exists\mu > 0)$. Since $U = a_l/a_{l+2}$, $a_l/a_{l+2}^2 = U^2/a_l$. Thus, we have (3.6.1). □

4. Bushell-McLead Theorem

The aim of this section is to explain Theorem 4.3, according to [2]. In This section, we denote

$$K_n^\triangle := \left\{(x_1, \ldots, x_n) \in K_n^\bullet \;\middle|\; x_{n-1} = 1,\, x_n = 0\right\}$$
$$y_i := \frac{x_i}{x_{i+1} + x_{i+2}} = A_i(\mathbf{x}).$$

Note that $y_n = 0$, $y_{n-1} = x_{n-1}/x_1$, and $y_{n-2} = x_{n-2}$ for $\mathbf{x} = (x_1, \ldots, x_n) \in K_n^\triangle$. The map $\Phi\colon K_n^\triangle \to \Phi(K_n^\triangle)$ defined by $\Phi(x_1, \ldots, x_n) = (y_1, \ldots, y_n)$ is bijective. The inverse map Φ^{-1} is obtained as the solution of the system of equations $y_i(x_{i+1} + x_{i+2}) -$

$x_i = 0$ $(i = 1, \ldots, n-2)$. Let

$$P_k(z_1, z_2, \ldots, z_k) := \begin{vmatrix} z_1 & z_1 & & & & & & \\ -1 & z_2 & z_2 & & & & & \\ & -1 & z_3 & z_3 & & & & \\ & & \ddots & \ddots & \ddots & & & \\ & & & \ddots & \ddots & \ddots & & \\ & & & & -1 & z_{k-2} & z_{k-2} & \\ & & & & & -1 & z_{k-1} & z_{k-1} \\ & & & & & & -1 & z_k \end{vmatrix}.$$

Inductively, we can prove that $x_i = P_{n-i-1}(y_i,\ y_{i+1}, \ldots,\ y_{n-2})$. By the properties of determinant, we can prove the following lemma.

LEMMA 4.1. ([2] Lemma 3.1) *The followings hold. Here we put* $P_0 := 1$ *and* $P_{-1} = 1$.

(1) $P_k(z_1, \ldots,\ z_k) = z_k P_{k-1}(z_1, \ldots,\ z_{k-1}) + z_{k-1} P_{k-2}(z_1, \ldots,\ z_{k-2})$.

(2) For $1 \leqslant j < k$,

$$\begin{aligned} P_k(z_1, \ldots, z_k) &= P_j(z_1, \ldots, z_j) P_{k-j}(z_{j+1}, \ldots, z_k) \\ &\quad + z_j P_{j-1}(z_1, \ldots, z_{j-1}) P_{k-j-1}(z_{j+2}, \ldots, z_k). \end{aligned}$$

LEMMA 4.2. ([2] Lemma 3.2) *Let* $\mathbf{x} = (x_1, \ldots,\ x_n) \in K_n^{\triangle}$, *and* $(y_1, \ldots,\ y_n) = \Phi(x_1, \ldots,\ x_n)$. *Assume that* $x_i \partial_i E_n(\mathbf{x}) = 0$ *for all* $i = 1,\ 2, \ldots,\ n$. *Then the followings hold.*

(1) $y_i = y_1^2 P_{i-1}(y_1, \ldots,\ y_{i-1}) P_{n-i-1}(y_i, \ldots,\ y_{n-2})$

(2) $y_1 - y_i = y_1^2 y_{i-1} P_{i-2}(y_1, \ldots,\ y_{i-2}) P_{n-i-2}(y_{i+1}, \ldots,\ y_{n-2})$

Proof. Put $p_i := P_i(y_1, \ldots,\ y_i)$. Then (1), (2) can be written as (1) $y_i = y_1^2 p_{i-1} x_i$, and (2) $y_1 - y_i = y_1^2 y_{i-1} p_{i-2} x_{i+1}$.

(1) As a formal rational function

$$\begin{aligned} x_i \partial_i E_n(\mathbf{x}) &= \frac{x_i}{x_{i+1} + x_{i+2}} - \frac{x_{i-2} x_i}{(x_{i-1} + x_i)^2} - \frac{x_{i-1} x_i}{(x_i + x_{i+1})^2} \\ &= y_i - \frac{y_{i-2}^2 x_i}{x_{i-2}} - \frac{y_{i-1}^2 x_i}{x_{i-1}}. \end{aligned}$$

So, the condition $x_i \partial_i E_n(\mathbf{x}) = 0$ can be represented as

$$\frac{y_i}{x_i} = \frac{y_{i-2}^2}{x_{i-2}} + \frac{y_{i-1}^2}{x_{i-1}} \tag{4.2.1}$$

as an equation in the field $\mathbb{R}\,(x_1, \ldots,\ x_{n-2})$. Here, we regard $x_0 = x_n = 0,\ x_{-1} = x_{n-1} = 1$, $y_0 = y_n = 0$, and $y_{-1} = y_{n-1} = 1/x_1$. It is enough to show

$$\frac{y_i}{x_i} = y_1^2 p_{i-1} \tag{4.2.2}$$

in $\mathbb{R}(x_1,\dots,x_{n-2})$.

Consider the case $i=1$. Then, $p_0=1$. (4.2.1) can be written as $y_1/x_1=1/x_1^2$. Multiply $x_1^2y_1$, then we have (4.2.2).

Consider the case $i=2$. By (4.2.1) and $x_1y_1=1$, $y_1=P_1(y_1)=p_1$, we have

$$\frac{y_2}{x_2}=\frac{y_1^2}{x_1}=y_1^3=y_1^2p_1.$$

Thus we obtain (4.2.2).

Consider the case $i\geqslant 3$. We shall prove (4.2.2) by induction on i. By the induction assumption, $y_j/x_j=y_1^2p_{j-1}$ for $1\leqslant j<i$. By Lemma 4.1 (1), $p_{i-1}=y_{i-1}p_{i-2}+y_{i-2}p_{i-3}$. Thus

$$\frac{y_i}{x_i}=\frac{y_{i-2}^2}{x_{i-2}}+\frac{y_{i-1}^2}{x_{i-1}}=y_1^2(y_{i-2}p_{i-3}+y_{i-1}p_{i-2})=y_1^2p_{i-1}.$$

(2) Apply Lemma 4.1 (5) with $k=n-2$, $j=i-1$, then we obtain $x_1=p_{i-1}x_i+y_{i-1}p_{i-2}x_{i+1}$. Since $x_1=1/y_1$, after multiplying y_1^2 to the both hand sides, we obtain $y_1=y_1^2p_{i-1}x_i+y_1^2y_{i-1}p_{i-2}x_{i+1}$. By (1),

$$y_1-y_i=y_1-y_1^2p_{i-1}x_i=y_1^2y_{i-1}p_{i-2}x_{i+1}.$$

Thus we obtain (2). □

THEOREM 4.3. ([2] Proposition 3.3) *If* $\min_{\mathbf{x}\in K_n}E_n(\mathbf{x})=E_n(\mathbf{a})$ *at* $\mathbf{a}\in K_n^\bullet$, *then* $U(\mathbf{a})\geqslant 1/2$.

Proof. We may assume $\mathbf{a}=(x_1,\dots,x_n)\in K_n^\triangle$. By Lemma 4.2 (1), (2), we have $0\leqslant x_i/(x_{i+1}+x_{i+2})=y_i\leqslant y_1=1/x_1=U(\mathbf{a})$ $(i=1,\dots,n)$. Assume that $U(\mathbf{a})<1/2$. Then $x_1>2$, and $2x_i\leqslant x_{i+1}+x_{i+2}$. Take $\sum$, we obtain

$$2\sum_{i=1}^n x_i<\sum_{i=1}^n(x_{i+1}+x_{i+2})=2\sum_{i=1}^n x_i.$$

A contradiction. □

5. Short segments

The following Theorem is an extenstion of [2] Lemma 4.1, [5] §4, §5 and [6] §5.

THEOREM 5.1. *Assume that* $\min_{\mathbf{x}\in K_n}E_n(\mathbf{x})=E_n(\mathbf{a})$ *at* $\mathbf{a}\in K_n^\bullet$. *Then* $\mathbf{a}$ *does not contain segments of length* $2,\ 3,\ 4,\ 5,\ 7$, *or* 9.

Proof. Let $\mathbf{s}=(a_1,\dots,a_l)$ be a l-segment of $\mathbf{a}$ $(l\geqslant 2)$. Put $U:=U(\mathbf{a})$, $V:=\dfrac{a_{l-1}+a_l}{a_l}>1$. Note that $a_{l+1}=0$, $a_{l+2}=a_l/U$ by Theorem 2.4 (1). By Theorem 4.3, $U\geqslant 1/2$.

Since $a_{l+2}+a_{l+3} \geqslant a_{l+2} = a_l/U$, we have

$$0 \leqslant \partial_{l+1}E_n(\mathbf{a}) = \frac{1}{a_{l+2}+a_{l+3}} - \frac{a_{l-1}}{a_l^2} - \frac{a_l}{a_{l+2}^2} \leqslant \frac{1}{a_l}\big(U-(V-1)-U^2\big).$$

Thus, we have $V \leqslant 1+U-U^2$. Since $1 < V \leqslant 1+U-U^2$, we have $U<1$ and $1 < V \leqslant \frac{5}{4} - \left(U-\frac{1}{2}\right)^2 \leqslant \frac{5}{4}$. Thus (U, V) is included in the set

$$D := \left\{(u,v) \in \mathbb{R}^2 \;\middle|\; 1/2 \leqslant u < 1,\ 1 < v \leqslant 1+u-u^2\right\}.$$

By (3.6.1), $\dfrac{a_1a_l}{U^2} = \dfrac{1}{\mu} = \dfrac{a_2a_l^2}{a_{l-1}}$. Thus we have

$$a_2 = \frac{a_1a_{l-1}}{a_lU^2} = \frac{V-1}{U^2}a_1.$$

Since $\partial_{i-2}E_n(\mathbf{a}) = 0$ $(i = 3, 4, \ldots, l+2)$, we have

$$a_i = \frac{1}{\dfrac{a_{i-4}}{(a_{i-3}+a_{i-2})^2} + \dfrac{a_{i-3}}{(a_{i-2}+a_{i-1})^2}} - a_{i-1}.$$

Here $a_{-1} = a_{n-1} = Ua_1$ and $a_0 = a_n = 0$. Inductively, we obtain

$$a_3 = \frac{1}{a_{n-1}/a_1^2} - a_2 = \frac{U-V+1}{U^2}a_1 \qquad (\text{if } l \geqslant 3)$$
$$a_4 = \frac{V-U}{U^2}a_1 \qquad (\text{if } l \geqslant 4)$$
$$a_5 = \frac{1+UV-V^2}{U^2V}a_1 \qquad (\text{if } l \geqslant 5).$$

Thus, we define a series of rational functions by

$$f_1(u,v) := 1, \quad f_2(u,v) := \frac{v-1}{u^2}, \quad f_3(u,v) := \frac{u-v+1}{u^2}, \quad f_4(u,v) := \frac{v-u}{u^2}$$
$$f_i(u,v) := \frac{1}{\dfrac{f_{i-4}(u,v)}{(f_{i-3}(u,v)+f_{i-2}(u,v))^2} + \dfrac{f_{i-3}(u,v)}{(f_{i-2}(u,v)+f_{i-1}(u,v))^2}} - f_{i-1}(u,v)$$

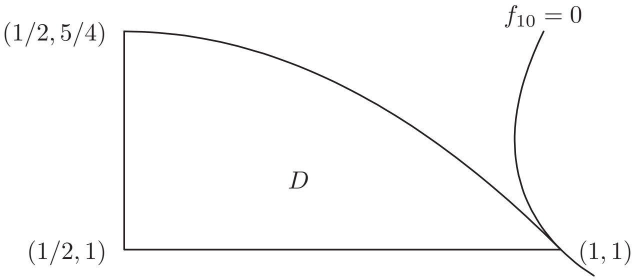
$(i \geqslant 5)$. Then, $a_i = f_i(U, V)a_1$ for $1 \leqslant i \leqslant l+2$. Especially, $f_{l+1}(U, V) = a_{l+1}/a_1 = 0$.

Since $u-v+1>0$, $v-u>0$, $1+uv-v^2>0$ on D, we obtain $f_i(u, v) > 0$ on D for $i = 3, 4, 5$. Thus $a_{l+1} \neq 0$ for $l = 2, 3, 4$. Therefore, $\mathbf{a}$ does not contain segments of length $2, 3$, or 4.

Similarly, $f_i(u, v) > 0$ on D for $i = 6, 8, 10$. We need numerical analysis to prove this. If you have 'Mathematica', execute the following.

```
<< Graphics‘ImplicitPlot‘;
fi[i_, u_, v_] := (a = 1; b = (v-1)/u^2;
   c = (1+u-v)/u^2; d = (v-u)/u^2;
   Do[(e=1/(a/(b+c)^2 + b/(c+d)^2) - d; a=b; b=c; c=d; d = e),
   {k, 5, i, 1}]; e)
G1[i_]:=(Plot3D[fi[i, u, v], {u, 1/2, 1}, {v, 1, 1 + u - u^2}])
G2[i_]:=(ImplicitPlot[(u^2 - u + v - 1) fi[i, u, v] == 0,
   {u, 1/2, 1}, {v, 1, 5/4}])
```

For example, you can observe the graph of $f_{10}(u,\ v)$ by `G1[10]`. You can also draw the graph of $f_{10}(u,\ v)=0$ by `G2[10]`.



$f_{10}(u,\ 1+u-u^2)$ have a zero of the order 2 at $u=1$. Thus, as the above figure, the graph of $f_{10}(u,\ v)=0$ tangents to the parabola $v=1+u-u^2$ at $(1,\ 1)$, but have no common point with D. Thus we know that $f_{10}(u,\ v)>0$ on D.

We know also $f_8(u,\ v)>0$ on D similarly.

It is possible to prove $f_6(u,\ v)>0$ on D directly. $f_6(u,\ v)$ can be written as $f_6(u,v)=\dfrac{f_{6,1}(u,v)f_{6,2}(u,v)}{u^2vf_{6,3}(u,v)}$, here

$$\begin{aligned} f_{6,1}(u,v) &:= 1-v+v^3-uv^2 \\ f_{6,2}(u,v) &:= (1+v-v^2)+uv \\ f_{6,3}(u,v) &:= -1+v+v^3-v^3+uv^2. \end{aligned}$$

It is easy too see that $f_{6,1}(u,\ v)>0,\ f_{6,2}(u,\ v)>0,\ f_{6,3}(u,\ v)>0$ on D. Thus $f_6(u,\ v)>0$ on D. Since $f_6(u,\ v)>0,\ f_8(u,\ v)>0$ and $f_{10}(u,\ v)>0$ on D, we conclude that $\mathbf{a}$ does not contain segments of length $5,\ 7$, or 9. $\square$

COROLLARY 5.2. *Assume that* $\min_{\mathbf{x}\in K_n} E_n(\mathbf{x})=E_n(\mathbf{a})$ *at* $\mathbf{a}\in K_n^{\bullet}$.

(1) If $n=12$*, then the index of* **a** *must be* (11).

(2) If $n=23$*, then the index of* **a** *must be one of the following 17 indexes:* (22), $(20,\ 1)$, $(18,\ 1,\ 1)$, $(16,\ 1,\ 1,\ 1)$, $(15,\ 6)$, $(14,\ 1,\ 1,\ 1,\ 1)$, $(13,\ 8)$, $(13,\ 6,\ 1)$, $(12,\ 1,\ 1,\ 1,\ 1,\ 1)$, $(11,\ 10)$, $(11,\ 8,\ 1)$, $(11,\ 6,\ 1,\ 1)$, $(10,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1)$, $(8,\ 6,\ 6)$, $(8,\ 1,\ 1\ ,1,\ 1,\ 1,\ 1,\ 1)$, $(6,\ 6,\ 6,\ 1)$, $(6,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1)$.

DEFINITION 5.3. Assume that $\min_{\mathbf{x}\in K_n} E_n(\mathbf{x}) = E_n(\mathbf{a})$ at $\mathbf{a} \in K_n^\bullet$, and that $\mathbf{s} = (s_1, s_2, \dots, s_l)$ is a l-segment of $\mathbf{a}$ with $l \geqslant 2$. Then, we define

$$V_l(\mathbf{a}) := 1 + \frac{s_{l-1}}{s_l},$$
$$R_l(\mathbf{a}) := \frac{s_1}{s_l} = \frac{\mathrm{Head}(\mathbf{s})}{\mathrm{Tail}(\mathbf{s})}.$$

If there are no segment of length l in $\mathbf{a}$, we define $R_l(\mathbf{a}) := 1$. Moreover we define $R_1(\mathbf{a}) := 1$. By Corollary 3.5, $V_l(\mathbf{a})$ and $R_l(\mathbf{a})$ do not depend the choice of $\mathbf{s}$.

THEOREM 5.4. *Assume that* $\min_{\mathbf{x}\in K_n} E_n(\mathbf{x}) = E_n(\mathbf{a})$ *at* $\mathbf{a} \in K_n^\bullet$.

(1) If $\mathbf{a}$ *contains segment of length* 6, *then the following holds.*

$$1/2 \leqslant U(\mathbf{a}) < 0.63894, \quad R_6(\mathbf{a}) < 1/2$$

(2) If $\mathbf{a}$ *contains a segment of length* 8, *then the following holds.*

$$1/2 \leqslant U(\mathbf{a}) < 0.73254, \quad R_8(\mathbf{a}) < 0.65994$$

(3) If $\mathbf{a}$ *contains a segment of length* 10, *then the following holds.*

$$0.63893 < U(\mathbf{a}) < 0.78332, \quad R_{10}(\mathbf{a}) < 0.90213$$

(4) If $\mathbf{a}$ *contains a segment of length* 11, *then the following holds.*

$$0.94197 < U(\mathbf{a}) < 1$$

(5) If $\mathbf{a}$ *contains a segment of length* 12, *then the following holds.*

$$0.73253 < U(\mathbf{a}) < 0.81295, \quad R_{12}(\mathbf{a}) < 1.20768$$

(6) If $\mathbf{a}$ *contains a segment of length* 13, *then the following holds.*

$$0.90868 < U(\mathbf{a}) < 1$$

(7) If $\mathbf{a}$ *contains a segment of length* 14, *then the following holds.*

$$0.78331 < U(\mathbf{a}) < 0.83098, \quad R_{14}(\mathbf{a}) < 1.61530$$

(8) If $\mathbf{a}$ *contains a segment of length* 15, *then the following holds.*

$$1/2 \leqslant U(\mathbf{a}) < 0.63894 \quad \text{or} \quad 0.88942 < U(\mathbf{a}) < 0.94198$$

(9) If $\mathbf{a}$ *contains a segment of length* 16, *then the following holds.*

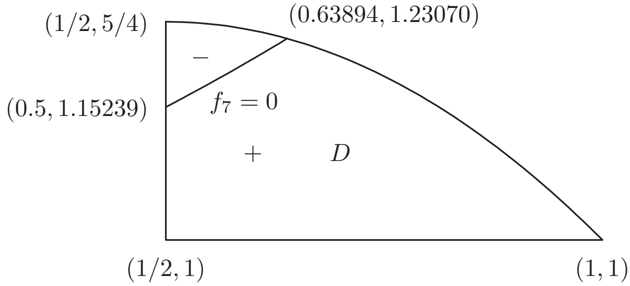
$$0.81294 < U(\mathbf{a}) < 0.84220, \quad R_{16}(\mathbf{a}) < 2.20409$$

Proof. We use the same notation with the proof of Theorem 5.1. Moreover put $U := U(\mathbf{a})$, $V := V_l(\mathbf{a})$, and

$$D'_i := \{(u,v) \in D \mid f_i(u,v) > 0\},$$
$$D_i := D'_2 \cap D'_3 \cap D'_4 \cap \cdots \cap D'_i.$$

Note that $D'_2 = D'_3 = D'_4 = D'_5 = D'_6 = D'_8 = D'_{10} = D$.

(1) Consider the case $l = 6$. The graph Γ_7 of $f_7(u,v) = 0$ on D is as following.



This curve Γ_7 is the hyper elliptic curve defined by

$$(2v - 2v^2 - v^3 + v^4) + u(-1 + 2v + v^2 - 2v^3) + u^2v^2 = 0.$$

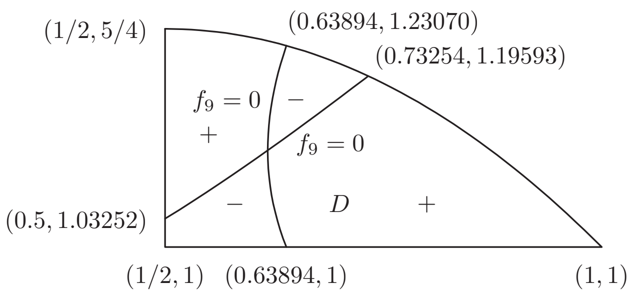
Thus, we put

$$f_{7,1}(v) := \frac{(v^2-1)(2v-1) + \sqrt{(v-1)(v^3+v^2+3v-1)}}{2v^2}.$$

We obtain the intersection of Γ_7 and the parabola $v = 1 + u - u^2$ on D by solving $f_7(u,\ 1 + u - u^2) = 0$. This root is $u \sim 0.6389355101$ (rounded up). If $\mathbf{a}$ has a 6-segment, then $f_7(U,\ V) = 0$. Thus $1/2 \leqslant U < 0.6389355101$. Since $f_6(f_{7,1}(v),\ v)$ is monotonically increasing on $1.15239 < v < 1.23070$, we have

$$R_6(\mathbf{a}) \leqslant 1/f_6(f_{7,1}(1.23070), 1.23070) < 0.42657 < 1/2$$

(2) Consider the case $l = 8$. The graph Γ_9 of $f_9(u,v) = 0$ on D is as following.



We can calculate the root of $f_9(u,\ 1+u-u^2)=0$ with $1/2 \leqslant u < 1$ by

```
FindRoot[fi[9, u, 1+u-u^2] == 0, {u, 0.7}]
```

and we have $u \sim 0.7325361425$ (rounded up). Thus $1/2 \leqslant U < 0.7325361425$. Execute

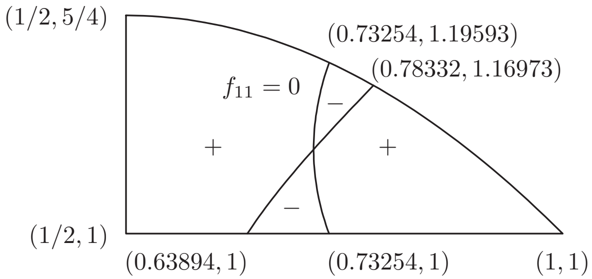
```
Plot3D[1/fi[8, u, v], {u, 1/2, 0.7325361425}, {v, 1, 1 + u - u^2}]
Maximize[{1/fi[8, 0.7325361425, v], 1<v <= 5/4}, v] // N
```

and we conclude that

$$\frac{1}{f_8(u,v)} < \frac{1}{f_8(0.73254, 1.10735)} < 0.65994$$

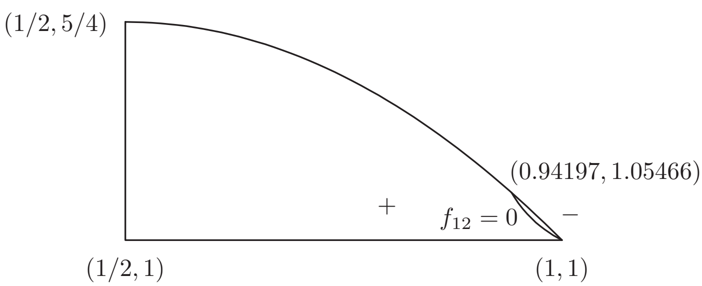
on $\Gamma_9 \cap D$. Thus $R_8(\mathbf{a}) < 0.65994$.

(3) Consider the case $l=10$. The graph Γ_{11} of $f_{11}(u,v)=0$ on D is as following.



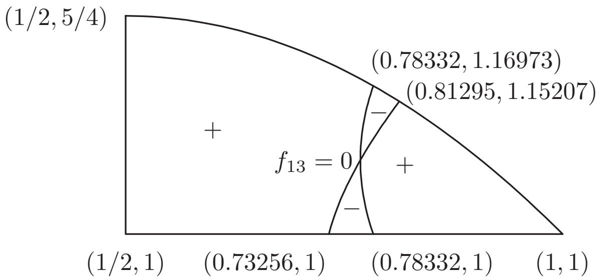
Thus $0.6389355100 < U < 0.7833151924$. Since $1/f_{10} < 1/f_{10}(0.78332, 1.09863) < 0.90213$ on $\Gamma_{11} \cap D$, we have $R_{10}(\mathbf{a}) < 0.90213$.

(4) Consider the case $l=11$. The graph of $f_{12}(u,v)=0$ on D is a curve connecting $(1,\ 1)$ and $(0.94197,\ 1.05466)$ as following.



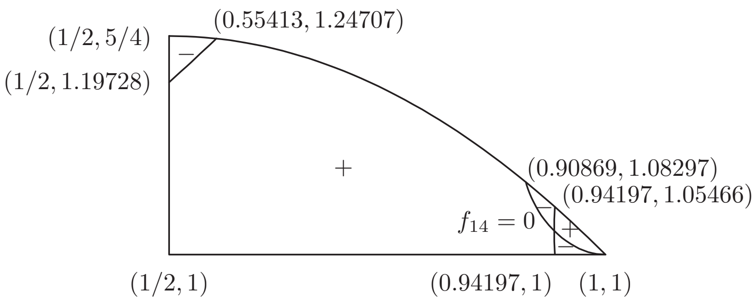
Thus, $0.9419748741 < U < 1$.

(5) Consider the case $l = 12$. The graph Γ_{13} of $f_{13}(u,v) = 0$ on D is as following.



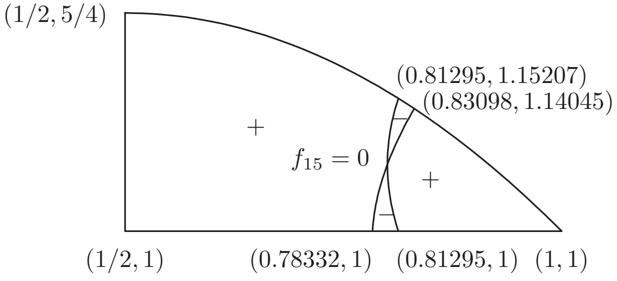
Thus, $0.7325361424 < U < 0.8129451277$. Since $1/f_{13}(u,\ v) < 1/f_{13}(0.81295,\ 1.08843) < 1.20768$ on $\Gamma_{13} \cap D$, we have $R_{12}(\mathbf{a}) < 1.20768$.

(6) Consider the case $l = 13$. The graph of $f_{14}(u,v) = 0$ on D is as following. But the curve connecting $(1/2,\ 1.19728)$ and $(0.55413,\ 1.24707)$ is included in $D - D_6'$ on which $a_6 < 0$. Thus, we omit this curve.



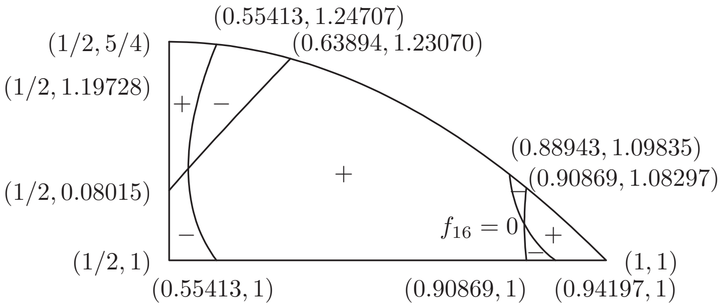
Thus we have $0.9086897811 < U < 1$.

(7) Consider the case $l = 14$. The graph Γ_{15} of $f_{15}(u,v) = 0$ on D is as following.



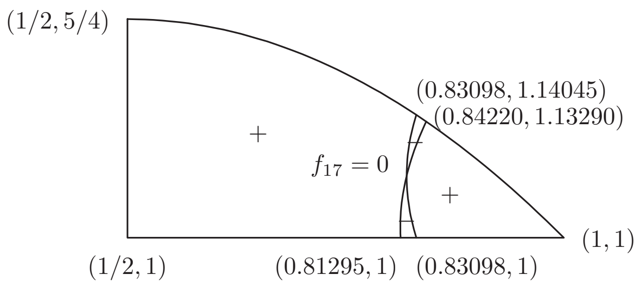
Thus, $0.7833151923 < U < 0.8309779815$. Since $1/f_{14}(u,\ v) < 1/f_{14}(0.83098,\ 1.08039) < 1.61530$, we have $R_{14}(\mathbf{a}) < 1.61530$.

(8) Consider the case $l=15$. The graph Γ_{16} of $f_{16}(u,v)=0$ on D is as following.



Thus, $1/2 \leqslant U < 0.6389355101$ or $0.8894259160 < U < 0.9419748742$.

(9) Consider the case $l=16$. The graph Γ_{17} of $f_{17}(u,v)=0$ on D is as following.



Thus, $0.8129451276 < U < 0.8421985095$. Since $1/f_{16}(u,\ v) < 1/f_{16}(0.84220, 1.07460) < 2.20409$ on $\Gamma_{17} \cap D$, we have $R_{16}(\mathbf{a}) < 2.20409$. □

6. Proof of Theorem 1.1

THEOREM 6.1. *Assume that* $\min_{\mathbf{x}\in K_{23}} E_{23}(\mathbf{x}) = E_{23}(\mathbf{a})$ *at* $\mathbf{a} \in K_{23}^{\bullet}$. *Then the index of* $\mathbf{a}$ *can not be any of the following values.*

(1) $(6,\ 6,\ 6,\ 1)$, $(6,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1)$.
(2) $(8,\ 6,\ 6)$, $(8,\ 1,\ 1\ ,1,\ 1,\ 1,\ 1,\ 1)$.
(3) $(10,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1)$.
(4) $(11,\ 10)$, $(11,\ 8,\ 1)$, $(11,\ 6,\ 1,\ 1)$.
(5) $(13,\ 8)$, $(13,\ 6,\ 1)$.
(6) $(15,\ 6)$.
(7) $(12,\ 1,\ 1,\ 1,\ 1,\ 1)$.
(8) $(14,\ 1,\ 1,\ 1,\ 1)$.
(9) $(16,\ 1,\ 1,\ 1)$.

Proof. We use the same notation with the proof of Theorem 5.1. Let $U := U(\mathbf{a})$, $R_l := R_l(\mathbf{a})$, and let m_i be the number of l_i-segments in $\mathbf{a}$ ($i=1,\ldots,\ q$), and let

$r := m_1 + m_2 + \cdots + m_q$ be the number of segments in $\mathbf{a}$. Then,

$$U^r R_{l_1}^{m_1} \cdots R_{l_q}^{m_q} = 1. \tag{6.1.1}$$

(1) In these cases, $U < 1$, $R_6 < 1$ by Theorem 5.4 (1). Thus (6.1.1) can not hold.

(2) In these cases, $U < 1$, $R_6 < 1$, $R_8 < 1$ by Theorem 5.4 (1), (2). Thus (6.1.1) can not hold.

(3) In this case, $U < 1$, $R_{10} < 1$ by Theorem 5.4 (3). Thus (6.1.1) can not hold.

(4) In these cases, $0.94197 < U < 1$ by Theorem 5.4 (4). But if $\mathbf{a}$ have a segment of length 10, 8 or 6, then $0.63893 < U < 0.78332$, $1/2 \leqslant U < 0.73254$, $1/2 \leqslant U < 0.63894$ respectively. There exists no such U.

(5) is similar to (4).

(6) Consider the case $(15,\ 6)$. $1/2 \leqslant U < 0.63894$ and $R_6(\mathbf{a}) < 1/2$ by Theorem 5.4 (1), (8). Execute

```
Plot3D[Ri[15, u, v], {u, 1/2, 0.6389355101}, {v, 1, 1 + u - u^2}]
Maximize[{Ri[15, 0.6389355101, V], 1 <= V <= 5/4}, V] // N
```

Thus we have $1/f_{15}(u,\ v) < 1/f_{15}(0.63894,\ 1.09583) < 0.08952$ on the set $\Gamma_{16} \cap \{(u,v) \in D \mid 1/2 \leqslant u \leqslant 0.63894\}$. Thus $R_{15} < 0.08952$ and (6.1.1) can not hold.

(7) In this case, $1 = U^6 R_{12} < 0.81295^6 \times 1.20768 < 1$. A contradiction.

(8) In this case, $1 = U^5 R_{14} < 0.83098^5 \times 1.61530 < 1$. A contradiction.

(9) In this case, $1 = U^4 R_{16} < 0.84220^4 \times 2.20409 < 1$. A contradiction. $\square$

The left cases are (11) when $n = 12$, and (22), $(20,\ 1)$, $(18,\ 1,\ 1)$ when $n = 23$.

THEOREM 6.2.

(1) Assume that $\min_{\mathbf{x} \in K_{12}} E_{12}(\mathbf{x}) = E_{12}(\mathbf{a})$ *at* $\mathbf{a} \in K_{12}^{\bullet}$. *Then the index of* $\mathbf{a}$ *can not be* (11). *Thus, Theorem 1.1 (2) holds.*

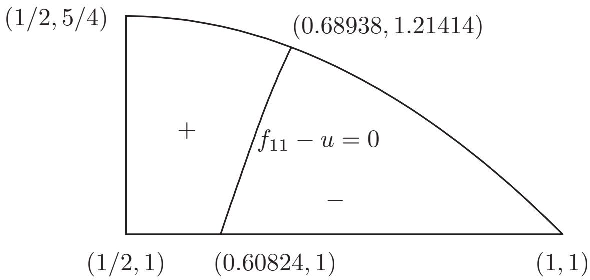
(2) Assume that $\min_{\mathbf{x} \in K_{23}} E_{23}(\mathbf{x}) = E_{23}(\mathbf{a})$ *at* $\mathbf{a} \in K_{23}^{\bullet}$. *Then the index of* $\mathbf{a}$ *can not be* (22).

Proof. We use the same notation with the proof of Theorem 6.1.

(1) We may assume $\mathbf{a} = (1,\ a_2, \ldots,\ a_{11},\ 0)$. Note that $a_{11} = U a_1 = U$. We draw the graph of $f_{11}(u,\ v) - u = 0$ on D. Execute

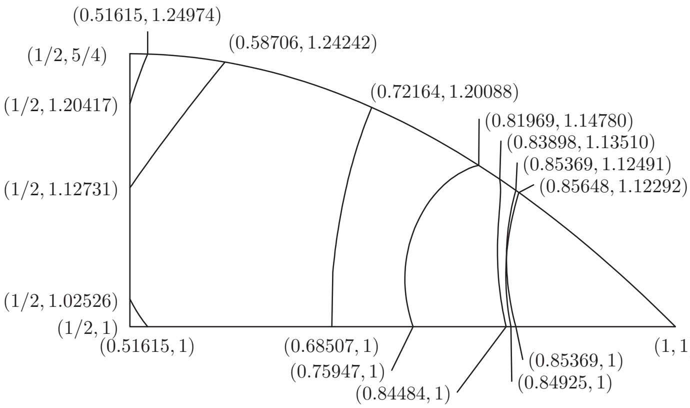
```
Plot3D[Ai[11,u,v]-u, {u, 0.5, 1}, {v, 1, 1.25}]
ImplicitPlot[(u^2-u+v-1) (Ai[11,u,v]-u)==0, {u, 0.5, 1}, {v, 1, 1.25}]
```

We obtain the following.



Thus $0.6082388995 < U < 0.6893774937$. But $0.94197 < U < 1$ by Theorem 5.4 (4). Thus the index (11) can not occur.

(2) We may assume $\mathbf{a} = (1, a_2, \ldots, a_{21}, 0)$, here $a_{21} = U$. The graph of $f_{23}(u, v) = 0$ and the graph of $f_{22}(u, v) - u = 0$ on D are as following.



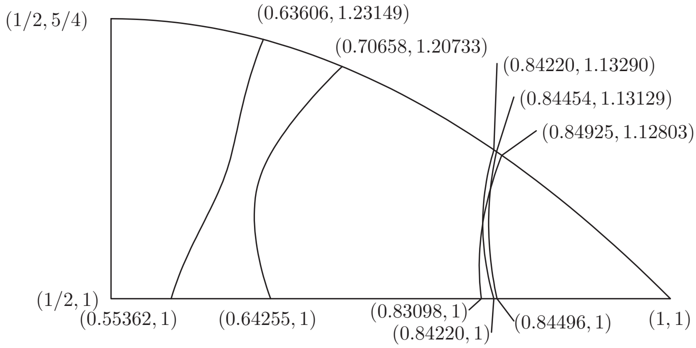
The graph Γ_{23} of $f_{23}(u, v) = 0$ consists of five parts. The first is the curve connecting $(1/2, 1.20417)$ and $(0.51615, 1.24974)$, the second is $(1/2, 1.12731)$ – $(0.58706, 1.24242)$, the third is $(1/2, 1.02526)$ – $(0.51615, 1)$, the fourth is $(0.84925, 1)$ – $(0.85648, 1.12292)$, and the fifth is $(0.85369, 1)$ – $(0.85369, 1.12491)$. The graph Γ'_{22} of $f_{22}(u, v) - u = 0$ consists of three parts. The first is $(0.68507, 1)$ – $(0.72164, 1.20088)$, the second is $(0.75947, 1)$ – $(0.81969, 1.14780)$, and the third is $(0.84484, 1)$ – $(0.83898, 1.13510)$. As the above figure, $\Gamma_{23} \cap \Gamma'_{22} \cap D = \emptyset$. Thus, (U, V_{23}) can not exists if the index of $\mathbf{a}$ is (23). □

THEOREM 6.3. *Assume that* $\min_{\mathbf{x} \in K_{23}} E_{23}(\mathbf{x}) = E_{23}(\mathbf{a})$ *at* $\mathbf{a} \in K^{\bullet}_{23}$. *Then, the index of* $\mathbf{a}$ *can not be any of the following values. Thus, Theorem 1.1 (1) holds.*

(1) $(18, 1, 1)$.
(2) $(20, 1)$.

Proof. (1) We may assume that $\mathbf{a} = (1, a_2, \ldots, a_{18}, 0, a_{20}, 0, a_{22}, 0)$. Let $U := U(\mathbf{a})$ and $V := V_{18}(\mathbf{a})$. Then, $a_{22} = U$, $a_{20} = U^2$, $a_{18} = U^3$, $f_{19}(U, V) = 0$ and $f_{18}(U, V) = U^3$.

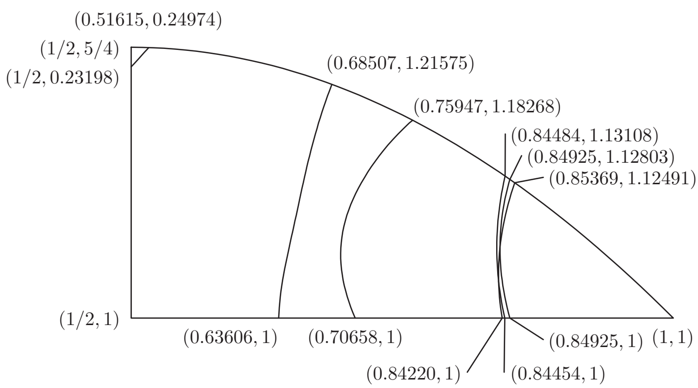
The graph of $f_{19}(u,\ v) = 0$ and the graph of $f_{18}(u,\ v) - u^3 = 0$ on D are as following.



The graph Γ_{19} of $f_{19}(u,\ v) = 0$ consists of two parts. The first is the curve C_1 connecting $(0.83098,\ 1)$ and $(0.84925,\ 1.12803)$, and the second is $(0.84220,\ 1)$ – $(0.84220,\ 1.13290)$. The graph Γ'_{18} of $f_{18}(u,\ v) - u^3 = 0$ consists of three parts. The first is $(0.55362,\ 1)$ – $(0.63606,\ 1.23149)$, the second is $(0.64255,\ 1)$ – $(0.70658,\ 1.20733)$, and the third is the curve C_2 connecting $(0.84496,\ 1)$ and $(0.84454,\ 1.13129)$. As the above figure, $\Gamma_{19} \cap \Gamma'_{18} \cap D = C_1 \cap C_2 \sim (0.8391429974,$ $1.0981287467)$. Thus $U \sim 0.8391429974$ and $V \sim 1.0981287467$. In this case $E_{23}(\mathbf{a})$ $> 11.511 > 23/2 = E_{23}(1,\ 1, \ldots,\ 1)$. So, $E_{23}(\mathbf{a})$ can not be minimum.

(2) We may assume $\mathbf{a} = (1,\ a_2, \ldots,\ a_{20},\ 0,\ a_{22},\ 0)$. Let $U := U(\mathbf{a})$ and $V := V_{18}(\mathbf{a})$. Then $a_{22} = U$, $a_{20} = U^2$, $f_{21}(U,\ V) = 0$ and $f_{20}(U,\ V) = U^3$.

The graph of $f_{21}(u,\ v) = 0$ and the graph of $f_{20}(u,\ v) - u^2 = 0$ on D are as following.



The graph Γ_{21} of $f_{21}(u,\ v) = 0$ consists of three parts. The first is $(1/2,\ 1.23198)$ – $(0.51615,\ 1.24974)$, the second is the curve C_3 connecting $(0.84220,\ 1)$ and $(0.85369,$ $1.12491)$, and the third is $(0.84925,\ 1)$ – $(0.84925,\ 1.12803)$. The graph Γ'_{20} of $f_{20}(u,\ v) - u^2 = 0$ consists of three parts. The first is $(0.63606,\ 1)$ – $(0.68507,$

1.21575), the second is (0.70658, 1) – (0.75947, 1.18268), and the third is the curve C_4 connecting (0.84454, 1) and (0.84484, 1.13108). As the above figure, $\Gamma_{21} \cap \Gamma'_{20} \cap D = C_3 \cap C_4 \sim (0.8388196493, 1.0346467269)$. Thus $U \sim 0.8388196493$, and $V \sim 1.0346467269$. Then $E_{23}(\mathbf{a}) > 11.512 > 23/2 = E_{23}(1, \ldots, 1)$. Thus $E_{23}(\mathbf{a})$ can not be minimum. □

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