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ON MONOTONICITY PROPERTIES OF THE L_p–CENTROID BODIES

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Abstract. Monotonicity properties for the quermassintegrals and dual quermassintegrals of the L_p-centroid bodies are proved and equality cases are investigated.

1. Introduction

Let $\mathscr{K}^n$ denote the set of convex bodies (compact, convex subsets with non-empty interiors) in Euclidean space $\mathbb{R}^n$ and let $\mathscr{K}_o^n$ denote the set of convex bodies containing the origin in their interiors in $\mathbb{R}^n$. Denote by $\mathscr{S}_o^n$ the set of star bodies (about the origin) in $\mathbb{R}^n$ and by S^{n-1} the unit sphere in $\mathbb{R}^n$. By $V(K)$ we denote the n-dimensional volume of body K and for the standard unit ball B in $\mathbb{R}^n$ we set $\omega_n = V(B)$.

The notion of classical centroid body was given by Petty (see [13] or books [3] and [14]) and it is one of the most important notions in the Brunn-Minkowski theory. For further study see Petty's seminal work [13]. During the past two decades, a number of important results for the classical centroid bodies have been obtained by Milman and Pajor [11, 12], Lutwak [5, 6], Zhang [19, 20, 21] (see also books [3] and [14]).

In 1997, Lutwak and Zhang ([10]) extended the notion of classical centroid body. They introduced the L_p-centroid body as follows: for each compact star-shaped (about the origin) $K \subset \mathbb{R}^n$ and for real number $p \geqslant 1$, the L_p-centroid body $\Gamma_p K$ of K is the origin-symmetric convex body whose support function is defined by ([10])

$$h^p_{\Gamma_p K}(u) = \frac{1}{c_{n,p} V(K)} \int_K |\, u \cdot x \,|^p \, dx, \tag{1.1}$$

for all $u \in S^{n-1}$, where the integration is with respect to the Lebesgue measure, $u \cdot v$ denotes the standard inner product of u and v, and

$$c_{n,p} = \omega_{n+p} / \omega_2 \omega_n \omega_{p-1}, \tag{1.2}$$

with

$$\omega_n = \pi^{\frac{n}{2}} / \Gamma\big(1 + \frac{n}{2}\big).$$

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Associated with the L_p-centroid bodies, Lutwak and Zhang ([10]) established the L_p-centro-affine inequality. Afterwards, Lutwak, Yang and Zhang ([9]) proved the L_p-Busemann-Petty centroid inequality. For further study of L_p-centroid bodies, except articles [9, 10], see also [1, 2, 15, 18]. In particular, in [15] authors presented a type of monotonicity properties for the volumes of L_p-centroid body $\Gamma_p K$ and its polar $\Gamma_p^* K$ as follows:

THEOREM A. *Suppose* $K,L \in \mathscr{S}_o^n$, $p \geqslant 1$. *If for any* $Q \in \mathscr{S}_o^n$, $\widetilde{V}_{-p}(K,Q) \leqslant \widetilde{V}_{-p}(L,Q)$, *then*

$$\frac{V(\Gamma_p K)^{-\frac{p}{n}}}{V(K)} \geqslant \frac{V(\Gamma_p L)^{-\frac{p}{n}}}{V(L)},$$

with equality if and only if $K = L$.

THEOREM B. *Suppose* $K,L \in \mathscr{S}_o^n$, $p \geqslant 1$. *If for any* $Q \in \mathscr{S}_o^n$, $\widetilde{V}_{-p}(K,Q) \leqslant \widetilde{V}_{-p}(L,Q)$, *then*

$$\frac{V(\Gamma_p^* K)^{\frac{p}{n}}}{V(K)} \geqslant \frac{V(\Gamma_p^* L)^{\frac{p}{n}}}{V(L)},$$

with equality if and only if $K = L$.

In Theorem A and Theorem B, $\widetilde{V}_{-p}(M,N)$ denotes the L_p-dual mixed volume of $M,N \in \mathscr{S}_o^n$ which is defined by ([8])

$$\widetilde{V}_{-p}(M,N) = \frac{1}{n}\int_{S^{n-1}} \rho_M^{n+p}(u)\rho_N^{-p}(u)dS(u). \tag{1.3}$$

For $K \in \mathscr{K}^n$, $i = 0,1,\cdots,n-1$, the quermassintegrals $W_i(K)$ of K are given by ([3, 14])

$$W_i(K) = \frac{1}{n}\int_{S^{n-1}} h(K,u)dS_i(K,u), \tag{1.4}$$

where $S_i(K,\cdot)$ $(i = 0,1,\cdots,n-1)$ denotes the mixed surface area measure of K. From definition (1.4), we easily see that $W_0(K) = V(K)$.

For $K \in \mathscr{S}_o^n$ and any real i, the dual quermassintegrals $\widetilde{W}_i(K)$ of K are defined by ([3, 14])

$$\widetilde{W}_i(K) = \frac{1}{n}\int_{S^{n-1}} \rho(K,u)^{n-i}dS(u), \tag{1.5}$$

where the integration is with respect to the spherical Lebesgue measure S on S^{n-1}. Obviously, $\widetilde{W}_0(K) = V(K)$.

Associated with the notion of quermassintegrals, Wang and Leng ([18]) extended Theorem A as follows:

THEOREM C. *Suppose* $K, L \in \mathscr{S}_o^n$, $p \geqslant 1$, $i = 0, 1, \cdots, n-1$. *If for any* $Q \in \mathscr{S}_o^n$,

$$\widetilde{V}_{-p}(K,Q) \leqslant \widetilde{V}_{-p}(L,Q),$$

then

$$\frac{W_i(\Gamma_p K)^{-\frac{p}{n-i}}}{V(K)} \geqslant \frac{W_i(\Gamma_p L)^{-\frac{p}{n-i}}}{V(L)},$$

with equality if and only if $K = L$.

In this paper, we give an extension of Theorem B on dual quermassintegrals and obtain the following dual form of Theorem C:

THEOREM 1.1. *Suppose* $K, L \in \mathscr{S}_o^n$, $p \geqslant 1$, i *is any real and* $i \neq n$. *If for any* $Q \in \mathscr{S}_o^n$,

$$\widetilde{V}_{-p}(K,Q) \leqslant \widetilde{V}_{-p}(L,Q),$$

then

$$\frac{\widetilde{W}_i(\Gamma_p^* K)^{\frac{p}{n-i}}}{V(K)} \geqslant \frac{\widetilde{W}_i(\Gamma_p^* L)^{\frac{p}{n-i}}}{V(L)},$$

with equality if and only if $K = L$.

From Theorem 1.1 and (1.3), we know that if $K \subseteq L$, then $\widetilde{V}_{-p}(K,Q) \leqslant \widetilde{V}_{-p}(L,Q)$, for any $Q \in \mathscr{S}_o^n$. So we get that

COROLLARY 1.1. *Suppose* $K, L \in \mathscr{S}_o^n$, $p \geqslant 1$, i *is any real and* $i \neq n$. *If* $K \subseteq L$, *then*

$$\frac{\widetilde{W}_i(\Gamma_p^* K)^{\frac{p}{n-i}}}{V(K)} \geqslant \frac{\widetilde{W}_i(\Gamma_p^* L)^{\frac{p}{n-i}}}{V(L)},$$

with equality if and only if $K = L$.

We give generalizations of Theorem C and Theorem 1.1 in Section 3.

2. Some notions

2.1. Support function, radial function and polar body

If $K \in \mathscr{K}^n$, then its support function $h_K = h(K,\cdot): \mathbb{R}^n \to (-\infty, \infty)$ is defined by ([3])

$$h(K,x) = \max\{x \cdot y : y \in K\}, \qquad x \in \mathbb{R}^n,$$

where $x \cdot y$ denotes the standard inner product of x and y.

If K is compact star-shaped (about the origin) in $\mathbb{R}^n$, its radial function $\rho_K = \rho(K,\cdot) : \mathbb{R}^n \backslash \{0\} \longrightarrow [0, +\infty)$ is defined by ([3])

$$\rho(K,x) = \max\{\lambda \geqslant 0 : \lambda x \in K\}, \qquad x \in \mathbb{R}^n \backslash \{0\}.$$

If ρ_K is positive and continuous, K will be called a star body (about the origin). Two star bodies K and L are said to be dilates (of one another) if $\rho_K(u)/\rho_L(u)$ is independent of $u \in S^{n-1}$.

If $K \in \mathscr{K}_o^n$, the polar body K^* of K is defined by ([3])

$$K^* = \{x \in \mathbb{R}^n : x \cdot y \leqslant 1, y \in K\}. \tag{2.1}$$

From definition (2.1), we know that:

$$h_{K^*} = \frac{1}{\rho_K} \qquad \text{and} \qquad \rho_{K^*} = \frac{1}{h_K}. \tag{2.2}$$

2.2. L_p-mixed quermassintegrals

In [7], Lutwak introduced the notion of the L_p-mixed quermassintegrals. For $K, L \in \mathscr{K}_o^n$, $p \geqslant 1$ and $i = 0, 1, \cdots, n-1$, the L_p-mixed quermassintegrals $W_{p,i}(K,L)$ of K and L have the following integral representation

$$W_{p,i}(K,L) = \frac{1}{n} \int_{S^{n-1}} h_L^p(v) dS_{p,i}(K,v), \tag{2.3}$$

where the measure $S_{p,i}(K,\cdot)$ is the L_p-mixed surface area measure of K.

From (1.4) and (2.3), it follows immediately that for each $K \in \mathscr{K}_o^n$ and $p \geqslant 1$,

$$W_{p,i}(K,K) = W_i(K). \tag{2.4}$$

The Minkowski inequality for the L_p-mixed quermassintegrals $W_{p,i}$ can be stated ([7]):

For $K, L \in \mathscr{K}_o^n$, $p \geqslant 1$ and $i = 0, 1, \cdots, n-1$, then

$$W_{p,i}(K,L)^{n-i} \geqslant W_i(K)^{n-i-p} W_i(L)^p, \tag{2.5}$$

with equality for $p > 1$ if and only if K and L are dilates; for $p = 1$ and $0 \leqslant i < n-1$ if and only if K and L are homothetic; for $p = 1$ and $i = n-1$, (2.5) is an identity.

2.3. L_p-dual mixed quermassintegrals

The notion of L_p-dual mixed quermassintegrals was defined by Wang and Leng ([16]). For K, $L \in \mathscr{S}_o^n$, $p \geqslant 1$ and real $i \neq n$, the L_p-dual mixed quermassintegrals $\widetilde{W}_{-p,i}(K,L)$ of K and L are given by

$$\widetilde{W}_{-p,i}(K,L) = \frac{1}{n} \int_{S^{n-1}} \rho_K^{n+p-i}(u) \rho_L^{-p}(u) dS(u). \tag{2.6}$$

From (2.6) and (1.3), we immediately obtain that:

$$\widetilde{W}_{-p,0}(K,L) = \widetilde{V}_{-p}(K,L). \tag{2.7}$$

Together with (1.5) and (2.6), we easily get

$$\widetilde{W}_{-p,i}(K,K)=\widetilde{W}_i(K). \tag{2.8}$$

In particular, from (2.7)

$$\widetilde{V}_{-p}(K,K)=V(K). \tag{2.9}$$

The Minkowski inequality for the L_p-dual mixed quermassintegrals can be stated ([16]): *If $K,L\in\mathscr{S}_o^n$, $p\geqslant 1$ and i is any real, then for $i<n$ or $i>n+p$,*

$$\widetilde{W}_{-p,i}(K,L)^{n-i}\geqslant\widetilde{W}_i(K)^{n+p-i}\widetilde{W}_i(L)^{-p}; \tag{2.10}$$

for $n<i<n+p$,

$$\widetilde{W}_{-p,i}(K,L)^{n-i}\leqslant\widetilde{W}_i(K)^{n+p-i}\widetilde{W}_i(L)^{-p}. \tag{2.11}$$

Equality holds in (2.10) and (2.11) if and only if K and L are dilates.

Taking $i=0$ in inequality (2.10), we have that (see [8]): *If $K,L\in\mathscr{S}_o^n$, $p\geqslant 1$, then*

$$\widetilde{V}_{-p}(K,L)^n\geqslant V(K)^{n+p}V(L)^{-p}, \tag{2.12}$$

with equality if and only if K and L are dilates.

2.4. L_p-mixed projection bodies

Lutwak, Yang and Zhang in [9] introduced the notion of L_p-projection body. For each $K\in\mathscr{K}_o^n$ and $p\geqslant 1$, the L_p-projection body Π_pK of K is the origin-symmetric convex body whose support function is defined by

$$h^p_{\Pi_pK}(u)=\frac{1}{n\omega_nc_{n-2,p}}\int_{S^{n-1}}|\,u\cdot v\,|^p\,dS_p(K,v), \tag{2.13}$$

for all $u\in S^{n-1}$. Here $S_p(K,\cdot)$ is the L_p-surface area measure of K.

Motivated by (2.13), Wang and Leng ([17]) defined the following notion of L_p-mixed projection body. For each $K\in\mathscr{K}_o^n$, $p\geqslant 1$ and $i=0,1,\cdots,n-1$, the L_p-mixed projection body $\Pi_{p,i}K$ of K is the origin-symmetric convex body whose support function is defined by

$$h^p_{\Pi_{p,i}K}(u)=\frac{1}{nc_{n-2,p}\omega_n}\int_{S^{n-1}}|\,u\cdot v\,|^p dS_{p,i}(K,v), \tag{2.14}$$

for all $u\in S^{n-1}$.

Obviously, from (2.13) and (2.14) for $i=0$ we have $\Pi_{p,0}K=\Pi_pK$.

2.5. L_p-mixed centroid bodies

From definition (1.1), we easily obtain

$$h^p_{\Gamma_pK}(u)=\frac{1}{(n+p)c_{n,p}V(K)}\int_{S^{n-1}}|\,u\cdot v\,|^p\,\rho_K(v)^{n+p}dS(v). \tag{2.15}$$

Now we define a new geometric body as follows: For each compact star-shaped (about the origin) $K \subset \mathbb{R}^n$, real $p \geqslant 1$ and any real i, the L_p-mixed centroid body $\Gamma_{p,i}K$ of K is the origin-symmetric body whose support function is given by

$$h^p_{\Gamma_{p,i}K}(u) = \frac{1}{(n+p)c_{n,p}V(K)} \int_{S^{n-1}} |\, u \cdot v \,|^p \, \rho_K(v)^{n+p-i} dS(v), \tag{2.16}$$

for all $u \in S^{n-1}$.

According to the integral form of Minkowski inequality (see [4]), we know that for $p \geqslant 1$, the L_p-mixed centroid body $\Gamma_{p,i}K$ is a convex body.

Obviously, from (2.15) and (2.16) for $i = 0$ we have

$$\Gamma_{p,0}K = \Gamma_p K. \tag{2.17}$$

3. Proofs of Theorems

In this section, using the L_p-mixed centroid bodies, we give general forms of Theorem C and Theorem 1.1 as follows:

THEOREM 3.1. *Suppose $K, L \in \mathscr{S}^n_o$, $p \geqslant 1$, $i = 0, 1, \cdots, n-1$ and real $j \neq n, n+p$. If for any $Q \in \mathscr{S}^n_o$,*

$$\widetilde{W}_{-p,j}(K,Q) \leqslant \widetilde{W}_{-p,j}(L,Q),$$

then

$$\frac{W_i(\Gamma_{p,j}K)^{-\frac{p}{n-i}}}{V(K)} \geqslant \frac{W_i(\Gamma_{p,j}L)^{-\frac{p}{n-i}}}{V(L)}, \tag{3.1}$$

with equality if and only if $K = L$.

THEOREM 3.2. *Suppose $K, L \in \mathscr{S}^n_o$, $p \geqslant 1$, real $i \neq n$ and real $j \neq n$, $n+p$. If for any $Q \in \mathscr{S}^n_o$,*

$$\widetilde{W}_{-p,j}(K,Q) \leqslant \widetilde{W}_{-p,j}(L,Q),$$

then

$$\frac{\widetilde{W}_i(\Gamma^*_{p,j}K)^{\frac{p}{n-i}}}{V(K)} \geqslant \frac{\widetilde{W}_i(\Gamma^*_{p,j}L)^{\frac{p}{n-i}}}{V(L)}, \tag{3.2}$$

with equality if and only if $K = L$.

LEMMA 3.1. *If $K, L \in \mathscr{S}^n_o$, $p \geqslant 1$ and real $j \neq n$, then for all $Q \in \mathscr{S}^n_o$,*

$$\widetilde{W}_{-p,j}(K,Q) = \widetilde{W}_{-p,j}(L,Q) \tag{3.3}$$

if and only if $K = L$.

Proof. If (3.3) is true, taking $Q = K$ in (3.3), and using (2.8), we have that $\widetilde{W}_j(K) = \widetilde{W}_{-p,j}(L,K)$. Now inequality (2.10) and (2.11) give that $\widetilde{W}_j(K) \geqslant \widetilde{W}_j(L)$ for

$j<n$ and $\widetilde{W}_j(K)\leqslant\widetilde{W}_j(L)$ for $n<j<n+p$ or $j>n+p$, and equality holds if and only if K and L are dilates. Let $Q=L$ in (3.3). Then we have $\widetilde{W}_j(K)\leqslant\widetilde{W}_j(L)$ for $j<n$ and $\widetilde{W}_j(K)\geqslant\widetilde{W}_j(L)$ for $n<j<n+p$ or $j>n+p$, and equality holds if and only if K and L are dilates. Hence, for real $j\neq n,n+p$, we have that $\widetilde{W}_j(K)=\widetilde{W}_j(L)$ and K and L must be dilates. Thus $K=L$.

Conversely, if $K=L$ then (3.3) obviously is true. □

LEMMA 3.2. *If $K\in\mathscr{S}_o^n$, $p\geqslant 1$, $i=0,1,\cdots,n-1$ and real $j\neq n$, then*

$$W_{p,i}(M,\Gamma_{p,j}K)=\frac{\omega_n}{V(K)}\widetilde{W}_{-p,j}(K,\Pi^*_{p,i}M), \tag{3.4}$$

for any $M\in\mathscr{K}_o^n$.

Here $\Pi^*_{p,i}M$ denotes the polar of $\Pi_{p,i}M$.

Proof. According to (2.3) and (2.16), we have

$$\begin{aligned}W_{p,i}(M,\Gamma_{p,j}K)&=\frac{1}{n}\int_{S^{n-1}}h^p_{\Gamma_{p,j}K}(u)dS_{p,i}(M,u)\\&=\frac{1}{n(n+p)c_{n,p}V(K)}\int_{S^{n-1}}\int_{S^{n-1}}|\,u\cdot v\,|^p\,\rho_K^{n+p-j}(v)dS(v)dS_{p,i}(M,u).\end{aligned}$$

From (1.2) it follows

$$nc_{n-2,p}=(n+p)c_{n,p},$$

and this, together with formulas (2.2), (2.6) and definition (2.14), gives

$$\begin{aligned}W_{p,i}(M,\Gamma_{p,j}K)&=\frac{\omega_n}{nV(K)}\int_{S^{n-1}}h^p_{\Pi_{p,i}M}(v)\rho_K^{n+p-j}(v)dS(v)\\&=\frac{\omega_n}{nV(K)}\int_{S^{n-1}}\rho_K^{n+p-j}(v)\rho^{-p}_{\Pi^*_{p,i}M}(v)dS(v)\\&=\frac{\omega_n}{V(K)}\widetilde{W}_{-p,j}(K,\Pi^*_{p,i}M).\quad\square\end{aligned}$$

Proof of Theorem 3.1. Since K, $L\in\mathscr{S}_o^n$, and for real $j\neq n$ and any $Q\in\mathscr{S}_o^n$,

$$\widetilde{W}_{-p,j}(K,Q)\leqslant\widetilde{W}_{-p,j}(L,Q), \tag{3.5}$$

with equality in (3.5) if and only if $K=L$ by Lemma 3.1.

Taking $Q=\Pi^*_{p,i}M$ for any $M\in\mathscr{K}_o^n$ in (3.5), we get that

$$\widetilde{W}_{-p,j}(K,\Pi^*_{p,i}M)\leqslant\widetilde{W}_{-p,j}(L,\Pi^*_{p,i}M).$$

Hence, using (3.4) we obtain that

$$V(K)W_{p,i}(M,\Gamma_{p,j}K)\leqslant V(L)W_{p,i}(M,\Gamma_{p,j}L),$$

taking $M=\Gamma_{p,j}L$, and together with equality (2.4) and inequality (2.5), we get that

$$\begin{aligned} V(L)W_i(\Gamma_{p,j}L) &\geqslant V(K)W_{p,i}(\Gamma_{p,j}L,\Gamma_{p,j}K) \\ &\geqslant V(K)W_i(\Gamma_{p,j}L)^{\frac{n-i-p}{n-i}}W_i(\Gamma_{p,j}K)^{\frac{p}{n-i}}. \end{aligned} \tag{3.6}$$

According to the condition of equality in (2.5), equality holds in the second inequality of (3.6) for $p>1$ if and only if $\Gamma_{p,j}K$ and $\Gamma_{p,j}L$ are dilates, for $p=1$ and $0\leqslant i<n-1$ if and only if $\Gamma_{p,j}K$ and $\Gamma_{p,j}L$ are homothetic. For $p=1$ and $i=n-1$, the second inequality of (3.6) is an identity.

From inequality (3.6), we have that

$$\frac{W_i(\Gamma_{p,j}K)^{-\frac{p}{n-i}}}{V(K)} \geqslant \frac{W_i(\Gamma_{p,j}L)^{-\frac{p}{n-i}}}{V(L)},$$

which is inequality (3.1).

By the equality conditions of inequality (3.5) and the second inequality of (3.6), we know that the equality holds in inequality (3.1) if and only if $K=L$. Theorem 3.1 is proven. □

Taking $j=0$ in Theorem 3.1, and using (2.7) and (2.17), we easily obtain Theorem C.

The proof of Theorem 3.2 requires the following lemma.

LEMMA 3.3. *If $K\in\mathscr{S}_o^n$, $p\geqslant 1$, $i,j\in\mathbb{R}$, $i,j\neq n$, then*

$$\frac{\widetilde{W}_{-p,j}(K,\Gamma^*_{p,i}M)}{V(K)} = \frac{\widetilde{W}_{-p,i}(M,\Gamma^*_{p,j}K)}{V(M)}, \tag{3.7}$$

for any $M\in\mathscr{S}_o^n$.

Proof. According to (2.2) and definition (2.16), we get

$$\rho^{-p}_{\Gamma^*_{p,i}M}(u) = \frac{1}{(n+p)c_{n,p}V(M)}\int_{S^{n-1}} |\, u\cdot v\,|^p \rho_M(v)^{n+p-i}dS(v), \tag{3.8}$$

for all $u\in S^{n-1}$.

Using (3.8) and (2.6), respectively, we have that

$$\begin{aligned} &\widetilde{W}_{-p,j}(K,\Gamma^*_{p,i}M) \\ &= \frac{1}{n}\int_{S^{n-1}} \rho_K(u)^{n+p-j}\rho^{-p}_{\Gamma^*_{p,i}M}(u)dS(u) \\ &= \frac{1}{n(n+p)c_{n,p}V(M)}\int_{S^{n-1}}\int_{S^{n-1}} |\, u\cdot v\,|^p \rho_K(u)^{n+p-j}\rho_M(v)^{n+p-i}dS(v)dS(u) \\ &= \frac{V(K)}{nV(M)}\int_{S^{n-1}} \rho_M(v)^{n+p-i}\rho^{-p}_{\Gamma^*_{p,j}K}(v)dS(v) \\ &= \frac{V(K)}{V(M)}\widetilde{W}_{-p,i}(M,\Gamma^*_{p,j}K). \quad \square \end{aligned}$$

Proof of Theorem 3.2. Since $K,L \in \mathscr{S}_o^n$, and for real $j \neq n$ and any $Q \in \mathscr{S}_o^n$,

$$\widetilde{W}_{-p,j}(K,Q) \leqslant \widetilde{W}_{-p,j}(L,Q), \tag{3.9}$$

with equality if and only if $K = L$ by Lemma 3.1.

Taking $Q = \Gamma_{p,i}^* M$ in (3.9) for any $M \in \mathscr{S}_o^n$ and real $i \neq n$, then

$$\widetilde{W}_{-p,j}(K,\Gamma_{p,i}^* M) \leqslant \widetilde{W}_{-p,j}(L,\Gamma_{p,i}^* M),$$

using (3.7), we have that

$$V(K)\widetilde{W}_{-p,i}(M,\Gamma_{p,j}^* K) \leqslant V(L)\widetilde{W}_{-p,i}(M,\Gamma_{p,j}^* L), \tag{3.10}$$

with equality if and only if $K = L$.

For $i < n$, let $M = \Gamma_{p,j}^* L$ in (3.10), and together with (2.8), we get that

$$V(L)\widetilde{W}_i(\Gamma_{p,j}^* L) \geqslant V(K)\widetilde{W}_{-p,i}(\Gamma_{p,j}^* L,\Gamma_{p,j}^* K). \tag{3.11}$$

But according to inequality (2.10), we know that

$$\widetilde{W}_{-p,i}(\Gamma_{p,j}^* L,\Gamma_{p,j}^* K) \geqslant \widetilde{W}_i(\Gamma_{p,j}^* L)^{\frac{n+p-i}{n-i}}\widetilde{W}_i(\Gamma_{p,j}^* K)^{-\frac{p}{n-i}}, \tag{3.12}$$

with equality if and only if $\Gamma_{p,j}^* K$ and $\Gamma_{p,j}^* L$ are dilates in inequality (3.12).

From (3.11) and (3.12), we obtain that

$$\frac{\widetilde{W}_i(\Gamma_{p,j}^* K)^{\frac{p}{n-i}}}{V(K)} \geqslant \frac{\widetilde{W}_i(\Gamma_{p,j}^* L)^{\frac{p}{n-i}}}{V(L)},$$

this is just inequality (3.2).

Because of $\Gamma_{p,j}^* K = \Gamma_{p,j}^* L$ when $K = L$, this implies $\Gamma_{p,j}^* K$ and $\Gamma_{p,j}^* L$ are dilates. Thus, according to the conditions of equality in (3.9) and (3.12), equality holds in (3.2) if and only if $K = L$.

For $n < i < n+p$, taking $M = \Gamma_{p,j}^* L$ in (3.10) and using (2.8), we obtain inequality (3.11) again. Since $n - i < 0$, using inequality (2.11), inequality (3.12) follows. This proves inequality (3.2), and equality holds under same conditions as in the case $i < n$.

For $i > n+p$, let $M = \Gamma_{p,j}^* K$ in (3.10). Using (2.8), we have

$$V(K)\widetilde{W}_i(\Gamma_{p,j}^* K) \leqslant V(L)\widetilde{W}_{-p,i}(\Gamma_{p,j}^* K,\Gamma_{p,j}^* L). \tag{3.13}$$

This together with inequality (2.10), and noting $n - i < -p < 0$, gives

$$\widetilde{W}_{-p,i}(\Gamma_{p,j}^* K,\Gamma_{p,j}^* L) \leqslant \widetilde{W}_i(\Gamma_{p,j}^* K)^{\frac{n+p-i}{n-i}}\widetilde{W}_i(\Gamma_{p,j}^* L)^{-\frac{p}{n-i}}. \tag{3.14}$$

Equality in inequality (3.14) holds if and only if $\Gamma_{p,j}^* K$ and $\Gamma_{p,j}^* L$ are dilates.

From (3.13) and (3.14), we get inequality (3.2), and equality holds similarly as in the case $i < n$.

For $i=n+p$, using formulas (2.6) and (1.5), it follows

$$\widetilde{W}_{-p,n+p}(M,\Gamma^*_{p,j}K)=\widetilde{W}_{n+p}(\Gamma^*_{p,j}K),$$

for any $M\in\mathscr{S}^n_o$. Hence, from inequality (3.10), we have

$$V(K)\widetilde{W}_{n+p}(\Gamma^*_{p,j}K)\leqslant V(L)\widetilde{W}_{n+p}(\Gamma^*_{p,j}L),$$

i.e.

$$\frac{\widetilde{W}_{n+p}(\Gamma^*_{p,j}K)^{-1}}{V(K)}\geqslant\frac{\widetilde{W}_{n+p}(\Gamma^*_{p,j}L)^{-1}}{V(L)}. \tag{3.15}$$

This is just the case $i=n+p$ of inequality (3.2). According to the condition of equality in (3.10), equality holds in (3.15) if and only if $K=L$. The proof of Theorem 3.2 is now complete. □

By letting $j=0$ in Theorem 3.2, together with (2.7) and (2.17), we immediately get Theorem 1.1.

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