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COMPOSITION IDENTITIES FOR THE CAPUTO FRACTIONAL DERIVATIVES AND APPLICATIONS TO OPIAL–TYPE INEQUALITIES

M. ANDRIĆ, J. PEČARIĆ AND I. PERIĆ

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Abstract. This paper gives improvements of a composition rule for the left-sided and the right-sided Caputo fractional derivatives. As application, improvements of Opial-type inequalities involving the Caputo fractional derivatives are also given.

1. Introduction and preliminaries

This paper is motivated by the book of Anastassiou [2] and papers [3] and [4] and presents improvements of a composition rule for the Caputo fractional derivatives (compare Theorem 1.4 and Theorem 2.1).

First we follow [7] and survey some facts about fractional derivatives needed in this paper.

By $C^m[a,b]$ we denote the space of all functions on $[a,b]$ which have continuous derivatives up to order m, and $AC[a,b]$ is the space of all absolutely continuous functions on $[a,b]$. By $AC^m[a,b]$ we denote the space of all functions $g \in C^{m-1}[a,b]$ with $g^{(m-1)} \in AC[a,b]$.

By $L^p[a,b]$, $1 \leqslant p < \infty$, we denote the space of all Lebesgue measurable functions f for which $|f|^p$ is Lebesgue integrable on $[a,b]$, and by $L^\infty[a,b]$ the set of all functions measurable and essentially bounded on $[a,b]$. Clearly, $L^\infty[a,b] \subset L^p[a,b]$ for all $p \geqslant 1$.

Let $f \in L^1[a,b]$, $\nu > 0$, $n = [\nu]+1$ ($[\cdot]$ is the integral part) and Γ is the gamma function $\Gamma(\nu) = \int_0^\infty e^{-t} t^{\nu-1} dt$. The **Riemann-Liouville fractional integrals** $J^\nu_{a+}f$ (left-sided) and $J^\nu_{b-}f$ (right-sided) of order ν are defined by

$$J^\nu_{a+}f(x) = \frac{1}{\Gamma(\nu)} \int_a^x (x-t)^{\nu-1} f(t)\,dt, \quad x \in (a,b], \tag{1.1}$$

$$J^\nu_{b-}f(x) = \frac{1}{\Gamma(\nu)} \int_x^b (t-x)^{\nu-1} f(t)\,dt, \quad x \in [a,b). \tag{1.2}$$

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LEMMA 1.1. [7, Lemma 2.3] *Let* $1 \leqslant p \leqslant \infty$, $f \in L^p[a,b]$ *and* $\alpha, \beta > 0$. *Then the equations*

$$J_{a+}^{\alpha} J_{a+}^{\beta} f(x) = J_{a+}^{\alpha+\beta} f(x), \qquad J_{b-}^{\alpha} J_{b-}^{\beta} f(x) = J_{b-}^{\alpha+\beta} f(x) \tag{1.3}$$

are satisfied at almost every point $x \in [a,b]$. *If* $\alpha + \beta > 1$, *then the relations in (1.3) hold at any point of* $[a,b]$.

For $\nu \geqslant 0$, $n = [\nu] + 1$, $f \in AC^m[a,b]$, where $m = n$ if $\nu \notin \mathbb{N}_0$ and $m = n-1$ if $\nu \in \mathbb{N}_0$, **the Caputo fractional derivatives of order** ν $D_{a+}^{\nu} f$ (left-sided) and $D_{b-}^{\nu} f$ (right-sided) are defined by

$$D_{a+}^{\nu} f(x) = \left(\tilde{D}_{a+}^{\nu} \left[f - T_{m-1}(f;a)\right]\right)(x),$$

$$D_{b-}^{\nu} f(x) = \left(\tilde{D}_{b-}^{\nu} \left[f - \tilde{T}_{m-1}(f;b)\right]\right)(x),$$

where $T_{m-1}(f;a)(t) = \sum_{k=0}^{m-1} \frac{f^{(k)}(a)}{k!}(t-a)^k$, $T_{m-1}(f;b)(t) = \sum_{k=0}^{m-1} \frac{f^{(k)}(b)}{k!}(b-t)^k$ and $\tilde{D}_{a+}^{\nu}$, $\tilde{D}_{b-}^{\nu}$ denote the left-sided and the right-sided Riemann-Liouville fractional derivatives (see [7, p.91]).

The following theorem appears to be more convenient for us in upcoming results.

THEOREM 1.2. [7, Theorem 2.1] *Let* $\nu \geqslant 0$, $n = [\nu] + 1$. *If* $f \in AC^m[a,b]$, $m = n$ *if* $\nu \notin \mathbb{N}_0$ *and* $m = n-1$ *if* $\nu \in \mathbb{N}_0$, *then the* ***Caputo fractional derivatives*** $D_{a+}^{\nu} f$ *(left-sided) and* $D_{b-}^{\nu} f$ *(right-sided) exist almost everywhere on* $[a,b]$.

(a) If $\nu \notin \mathbb{N}_0$, *then* $D_{a+}^{\nu} f$ *and* $D_{b-}^{\nu} f$ *are represented by*

$$D_{a+}^{\nu} f(x) = \frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} f^{(n)}(t)\,dt =: J_{a+}^{n-\nu} f^{(n)}(x) \tag{1.4}$$

and

$$D_{b-}^{\nu} f(x) = \frac{(-1)^n}{\Gamma(n-\nu)} \int_x^b (t-x)^{n-\nu-1} f^{(n)}(t)\,dt =: (-1)^n J_{b-}^{n-\nu} f^{(n)}(x). \tag{1.5}$$

In particular, when $0 < \nu < 1$ *and* $f \in AC[a,b]$,

$$D_{a+}^{\nu} f(x) = \frac{1}{\Gamma(1-\nu)} \int_a^x (x-t)^{-\nu} f'(t)\,dt =: J_{a+}^{1-\nu} f'(x) \tag{1.6}$$

and

$$D_{b-}^{\nu} f(x) = -\frac{1}{\Gamma(1-\nu)} \int_x^b (t-x)^{-\nu} f'(t)\,dt =: -J_{b-}^{1-\nu} f'(x). \tag{1.7}$$

(b) If $\nu = n \in \mathbb{N}_0$, *then* $D_{a+}^{n} f$ *and* $D_{b-}^{n} f$ *are represented by*

$$D_{a+}^{n} f(x) = f^{(n)}(x), \quad D_{b-}^{n} f(x) = (-1)^n f^{(n)}(x). \tag{1.8}$$

In particular,

$$D_{a+}^{0} f(x) = D_{b-}^{0} f(x) = f(x). \tag{1.9}$$

The following statement is analogous to that of Theorem 1.2 for functions $f \in C^n[a,b]$.

THEOREM 1.3. [7, Theorem 2.2] *Let* $\nu \geqslant 0$, $n = [\nu] + 1$ *and let* $f \in C^n[a,b]$. *Then the Caputo fractional derivatives* $D^\nu_{a+}f$ *and* $D^\nu_{b-}f$ *are continuous on* $[a,b]$.

(a) If $\nu \notin \mathbb{N}_0$, *then* $D^\nu_{a+}f$ *and* $D^\nu_{b-}f$ *are represented by (1.4) and (1.5) respectively. Moreover,*

$$D^\nu_{a+}f(a) = D^\nu_{b-}f(b) = 0. \tag{1.10}$$

In particular, they have the forms (1.6) and (1.7) for $0 < \nu < 1$, *respectively.*

(b) If $\nu = n \in \mathbb{N}_0$, *then* $D^n_{a+}f$ *and* $D^n_{b-}f$ *have representations given in (1.8). In particular, the relations in (1.9) hold.*

The following identities are instructive for results in this paper (for more details see [7, Lemma 2.5, Lemma 2.6]):

$$J^n_{a+}f^{(n)}(x) = f(x) - \sum_{i=0}^{n-1} \frac{f^{(i)}(a)}{i!}(x-a)^i,$$

$$J^n_{b-}f^{(n)}(x) = f(x) - \sum_{i=0}^{n-1} \frac{(-1)^i f^{(i)}(b)}{i!}(b-x)^i,\ n \in \mathbb{N}.$$

Our first goal is to improve composition identity for the left-sided Caputo fractional derivatives in the following theorem, identity (1.12). We will prove that it is not necessary to assume that $f^{(k)}(a) = 0$ for $k < m$ and that condition $\nu - \gamma \geqslant 1$ can be relaxed. This will be used in all presented Opial-type inequalities involving the Caputo fractional derivatives.

THEOREM 1.4. [2, Theorem 16.16] *Let* $\nu \geqslant \gamma + 1$, $\gamma \geqslant 0$. *Call* $n = [\nu] + 1$, $m = [\gamma] + 1$. *Assume* $f \in AC^n[a,b]$ *such that* $f^{(i)}(a) = 0$ *for* $i = 0, 1, \ldots, n-1$, *and* $D^\nu_{a+}f \in L_\infty(a,b)$. *Then*

$$D^\gamma_{a+}f \in C[a,b], \quad D^\gamma_{a+}f(x) = J^{m-\gamma}_{a+}f^{(m)}(x), \tag{1.11}$$

and

$$D^\gamma_{a+}f(x) = \frac{1}{\Gamma(\nu-\gamma)} \int_a^x (x-t)^{\nu-\gamma-1} D^\nu_{a+}f(t)\,dt \tag{1.12}$$

for all $x \in [a,b]$.

Also we will give composition identity for the right-sided Caputo fractional derivatives (see Theorem 2.2).

2. Main result

We relax some conditions in composition identity for the left-sided Caputo fractional derivative given in Theorem 1.4.

THEOREM 2.1. *Let* $\nu > \gamma \geqslant 0$, $n = [\nu] + 1$, $m = [\gamma] + 1$ *and* $f \in AC^k[a,b]$, $k = n$ *if* $\nu \notin \mathbb{N}_0$ *and* $k = n-1$ *if* $\nu \in \mathbb{N}_0$. *Let* $D^{\nu}_{a+}f, D^{\gamma}_{a+}f \in L^1[a,b]$. *Suppose that one of the following conditions holds:*

(a) $\nu,\ \gamma \notin \mathbb{N}_0$ *and* $f^{(i)}(a) = 0$ *for* $i = m, \dots, n-1$.

(b) $\nu \in \mathbb{N},\ \gamma \notin \mathbb{N}_0$ *and* $f^{(i)}(a) = 0$ *for* $i = m, \dots, n-2$.

(c) $\nu \notin \mathbb{N},\ \gamma \in \mathbb{N}_0$ *and* $f^{(i)}(a) = 0$ *for* $i = m-1, \dots, n-1$.

(d) $\nu \in \mathbb{N},\ \gamma \in \mathbb{N}_0$ *and* $f^{(i)}(a) = 0$ *for* $i = m-1, \dots, n-2$.

Then

$$D^{\gamma}_{a+}f(x) = \frac{1}{\Gamma(\nu-\gamma)} \int_a^x (x-t)^{\nu-\gamma-1} D^{\nu}_{a+}f(t)\,dt. \tag{2.1}$$

Proof. We give two proofs of the assertion (a). The proofs of the assertions (b), (c), (d) follows the same lines. Let $\nu > \gamma \geqslant 0$. Then using the Fubini theorem (note that $f^{(n)} \in L^1[a,b]$ and J^{α}_{a+} is a bounded operator on $L^1[a,b]$; see for example [7, Lemma 2.1]), change of variables and the definition of the beta function, we have

$$\begin{aligned}
&\frac{1}{\Gamma(\nu-\gamma)} \int_a^x (x-y)^{\nu-\gamma-1} D^{\nu}_{a+}f(y)\,dy \\
&= \frac{1}{\Gamma(\nu-\gamma)\Gamma(n-\nu)} \int_{y=a}^x \int_{t=a}^y (x-y)^{\nu-\gamma-1}(y-t)^{n-\nu-1} f^{(n)}(t)\,dt\,dy \\
&= \frac{1}{\Gamma(\nu-\gamma)\Gamma(n-\nu)} \int_{t=a}^x f^{(n)}(t) \int_{y=t}^x (x-y)^{\nu-\gamma-1}(y-t)^{n-\nu-1}\,dy\,dt \\
&= \frac{1}{\Gamma(\nu-\gamma)\Gamma(n-\nu)} \int_a^x f^{(n)}(t) \int_0^{x-t} u^{\nu-\gamma-1}(x-u-t)^{n-\nu-1}\,du\,dt \\
&= \frac{1}{\Gamma(\nu-\gamma)\Gamma(n-\nu)} \int_a^x f^{(n)}(t) \int_0^1 (x-t)^{n-\gamma-1} v^{\nu-\gamma-1}(1-v)^{n-\nu-1}\,dv\,dt \\
&= \frac{B(\nu-\gamma, n-\nu)}{\Gamma(\nu-\gamma)\Gamma(n-\nu)} \int_a^x (x-t)^{n-\gamma-1} f^{(n)}(t)\,dt \\
&= \frac{1}{\Gamma(n-\gamma)} \int_a^x (x-t)^{n-\gamma-1} f^{(n)}(t)\,dt\,.
\end{aligned}$$

Hence, for $\nu > \gamma \geqslant 0$ we have

$$\frac{1}{\Gamma(\nu-\gamma)} \int_a^x (x-y)^{\nu-\gamma-1} D^{\nu}_{a+}f(y)\,dy = \frac{1}{\Gamma(n-\gamma)} \int_a^x (x-t)^{n-\gamma-1} f^{(n)}(t)\,dt\,.$$

Case 1. Let $[\nu]=[\gamma]$, that is $n=m$. Then (2.1) follows with no boundary conditions.

Case 2. Let $[\nu]>[\gamma]$. Then $[\nu]>\gamma$ also, and therefore $n-\gamma-1>0$. Using integration by parts, it follows

$$\begin{aligned}
&\frac{1}{\Gamma(n-\gamma)}\int_a^x (x-t)^{n-\gamma-1} f^{(n)}(t)\,dt\\
&=\frac{1}{\Gamma(n-\gamma)}\left[(x-t)^{n-\gamma-1} f^{(n-1)}(t)\Big|_a^x+(n-\gamma-1)\int_a^x (x-t)^{n-\gamma-2} f^{(n-1)}(t)\,dt\right]\\
&=\left|f^{(n-1)}(a)=0\right|\\
&=\frac{1}{\Gamma(n-\gamma-1)}\int_a^x (x-t)^{n-\gamma-2} f^{(n-1)}(t)\,dt\,.
\end{aligned}$$

Case 2a. Let $[\nu]=[\gamma]+1$. Then $m=n-1$ and with boundary condition $f^{(n-1)}(a)=0$ follows (2.1).

Case 2b. Let $[\nu]>[\gamma]+1$. Then $n-\gamma-2>0$ and

$$\begin{aligned}
&\frac{1}{\Gamma(n-\gamma-1)}\int_a^x (x-t)^{n-\gamma-2} f^{(n-1)}(t)\,dt\\
&=\frac{1}{\Gamma(n-\gamma-1)}\left[(x-t)^{n-\gamma-2} f^{(n-2)}(t)\Big|_a^x+(n-\gamma-2)\int_a^x (x-t)^{n-\gamma-3} f^{(n-2)}(t)\,dt\right]\\
&=\frac{1}{\Gamma(n-\gamma-2)}\int_a^x (x-t)^{n-\gamma-3} f^{(n-2)}(t)\,dt\,.
\end{aligned}$$

Case 2b.1. Let $[\nu]=[\gamma]+2$. Then $m=n-2$ and, with boundary conditions $f^{(n-1)}(a)=f^{(n-2)}(a)=0$, the proof of (2.1) is complete.

Case 2b.2. Continuing in this way, in the last step, when $m=n-(n-m)$, we have that with boundary conditions $f^{(n-1)}(a)=\cdots=f^{(m)}(a)=0$ follows (2.1).

We will also prove equality (2.1) using the Laplace transform. Let $\nu>\gamma\geqslant 0$ and $f^{(i)}(a)=0$ for $i=m,m+1,\dots,n-1$. By Lemma 1.1 we have

$$J_{a+}^{\nu-\gamma}D_{a+}^{\nu}f=J_{a+}^{\nu-\gamma}J_{a+}^{n-\nu}f^{(n)}=J_{a+}^{n-\gamma}f^{(n)}\,.$$

Set $g=f^{(m)}$. Now (2.1) can be written as

$$J_{a+}^{m-\gamma}g(x)=J_{a+}^{n-\gamma}g^{(n-m)}(x), \tag{2.2}$$

where $x\in[a,b]$ and $g(a)=g'(a)=\cdots=g^{(n-m-1)}(a)=0$. Define auxiliary function $h:[0,\infty)\to\mathbb{R}$ with

$$h(x)=\begin{cases} g(x+a)\,,\ x\in[0,b-a]\\ \sum_{k=0}^{n-m}\frac{g^{(k)}(b)}{k!}(x-b+a)^k\,,\ x\geqslant b-a\end{cases}. \tag{2.3}$$

Obviously $h(0) = h'(0) = \cdots = h^{(n-m-1)}(0) = 0$. Also h has polynomial growth at ∞, so the Laplace transform of h exists. Using simple substitution the identity (2.2) is equivalent to the identity

$$\frac{1}{\Gamma(m-\gamma)} \int_0^x (x-t)^{m-\gamma-1} h(t)\,dt = \frac{1}{\Gamma(n-\gamma)} \int_0^x (x-t)^{n-\gamma-1} h^{(n-m)}(t)\,dt. \tag{2.4}$$

Using standard properties of the Laplace transform we have

$$\begin{aligned} &\mathscr{L}\left(\frac{1}{\Gamma(m-\gamma)} \int_0^x (x-t)^{m-\gamma-1} h(t)\,dt\right)(s) \\ &= \frac{1}{\Gamma(m-\gamma)} \mathscr{L}\left(x^{m-\gamma-1}\right)(s) \mathscr{L}(h)(s) = s^{\gamma-m} \mathscr{L}(h)(s). \end{aligned} \tag{2.5}$$

On the other hand we have

$$\begin{aligned} &\mathscr{L}\left(\frac{1}{\Gamma(n-\gamma)} \int_0^x (x-t)^{n-\gamma-1} h^{(n-m)}(t)\,dt\right)(s) \\ &= \frac{1}{\Gamma(n-\gamma)} \mathscr{L}\left(x^{n-\gamma-1}\right)(s) \mathscr{L}\left(h^{(n-m)}\right)(s) \\ &= s^{\gamma-n} \cdot s^{n-m} \mathscr{L}(h)(s) = s^{\gamma-m} \mathscr{L}(h)(s). \end{aligned} \tag{2.6}$$

Using (2.5) and (2.6) it follows that both sides of (2.4) have the same Laplace transform, we conclude that equality holds in (2.4) for every $x \geqslant 0$ (see [11, Theorem 6.3]). □

Next theorem gives us a composition identity for the right-sided Caputo fractional derivatives.

THEOREM 2.2. *Let* $\nu > \gamma \geqslant 0$, $n = [\nu] + 1$, $m = [\gamma] + 1$ *and* $f \in AC^k[a,b]$, $k = n$ *if* $\nu \notin \mathbb{N}_0$ *and* $k = n-1$ *if* $\nu \in \mathbb{N}_0$. *Let* $D^{\nu}_{b-} f, D^{\gamma}_{b-} f \in L^1[a,b]$. *Suppose that one of the following conditions holds:*

(a) $\nu, \gamma \notin \mathbb{N}_0$ *and* $f^{(i)}(b) = 0$ *for* $i = m, \ldots, n-1$.

(b) $\nu \in \mathbb{N}$, $\gamma \notin \mathbb{N}_0$ *and* $f^{(i)}(b) = 0$ *for* $i = m, \ldots, n-2$.

(c) $\nu \notin \mathbb{N}$, $\gamma \in \mathbb{N}_0$ *and* $f^{(i)}(b) = 0$ *for* $i = m-1, \ldots, n-1$.

(d) $\nu \in \mathbb{N}$, $\gamma \in \mathbb{N}_0$ *and* $f^{(i)}(b) = 0$ *for* $i = m-1, \ldots, n-2$.

Then

$$D^{\gamma}_{b-} f(x) = \frac{1}{\Gamma(\nu-\gamma)} \int_x^b (t-x)^{\nu-\gamma-1} D^{\nu}_{b-} f(t)\,dt. \tag{2.7}$$

Proof. Let $\nu > \gamma \geqslant 0$ and $f^{(i)}(b) = 0$ for $i = m, m+1, \ldots, n-1$. Then

$$\begin{aligned}
&\frac{1}{\Gamma(\nu-\gamma)}\int_x^b (y-x)^{\nu-\gamma-1} D_{b-}^{\nu} f(y)\,dy \\
&= \frac{(-1)^n}{\Gamma(\nu-\gamma)\,\Gamma(n-\nu)}\int_{y=x}^b\int_{t=y}^b (y-x)^{\nu-\gamma-1}(t-y)^{n-\nu-1} f^{(n)}(t)\,dt\,dy \\
&= \frac{(-1)^n}{\Gamma(\nu-\gamma)\,\Gamma(n-\nu)}\int_{t=x}^b f^{(n)}(t)\int_{y=x}^t (y-x)^{\nu-\gamma-1}(t-y)^{n-\nu-1}\,dy\,dt \\
&= \frac{(-1)^n}{\Gamma(\nu-\gamma)\,\Gamma(n-\nu)}\int_x^b f^{(n)}(t)\int_0^{t-x} u^{\nu-\gamma-1}(t-u-x)^{n-\nu-1}\,du\,dt \\
&= \frac{(-1)^n}{\Gamma(\nu-\gamma)\,\Gamma(n-\nu)}\int_x^b f^{(n)}(t)\int_0^1 (t-x)^{n-\gamma-1} v^{\nu-\gamma-1}(1-v)^{n-\nu-1}\,dv\,dt \\
&= \frac{(-1)^n B(\nu-\gamma, n-\nu)}{\Gamma(\nu-\gamma)\,\Gamma(n-\nu)}\int_x^b (t-x)^{n-\gamma-1} f^{(n)}(t)\,dt \\
&= \frac{(-1)^n}{\Gamma(n-\gamma)}\int_x^b (t-x)^{n-\gamma-1} f^{(n)}(t)\,dt\,.
\end{aligned}$$

Hence, for $\nu > \gamma \geqslant 0$ we have

$$\frac{1}{\Gamma(\nu-\gamma)}\int_x^b (y-x)^{\nu-\gamma-1} D_{b-}^{\nu} f(y)\,dy = \frac{(-1)^n}{\Gamma(n-\gamma)}\int_x^b (t-x)^{n-\gamma-1} f^{(n)}(t)\,dt\,.$$

Case 1. Let $[\nu] = [\gamma]$, that is $n = m$. Then (2.7) follows with no boundary conditions.

Case 2. Let $[\nu] > [\gamma]$. Then $[\nu] > \gamma$ also, and therefore $n-\gamma-1 > 0$. Using integration by parts, it follows

$$\begin{aligned}
&\frac{(-1)^n}{\Gamma(n-\gamma)}\int_x^b (t-x)^{n-\gamma-1} f^{(n)}(t)\,dt \\
&= \frac{(-1)^n}{\Gamma(n-\gamma)}\left[(t-x)^{n-\gamma-1} f^{(n-1)}(t)\Big|_x^b - (n-\gamma-1)\int_x^b (t-x)^{n-\gamma-2} f^{(n-1)}(t)\,dt\right] \\
&= \frac{(-1)^{n-1}}{\Gamma(n-\gamma-1)}\int_x^b (t-x)^{n-\gamma-2} f^{(n-1)}(t)\,dt\,.
\end{aligned}$$

Case 2a. Let $[\nu] = [\gamma]+1$. Then $m = n-1$ and with boundary condition $f^{(n-1)}(b) = 0$ follows (2.7).

Case 2b. Let $[\nu] > [\gamma]+1$. Then $n-\gamma-2 > 0$ and

$$\begin{aligned}
&\frac{(-1)^{n-1}}{\Gamma(n-\gamma-1)}\int_x^b (t-x)^{n-\gamma-2} f^{(n-1)}(t)\,dt \\
&= \frac{(-1)^{n-1}}{\Gamma(n-\gamma-1)}\left[(t-x)^{n-\gamma-2} f^{(n-2)}(t)\Big|_x^b - (n-\gamma-2)\int_x^b (t-x)^{n-\gamma-3} f^{(n-2)}(t)\,dt\right] \\
&= \frac{(-1)^{n-2}}{\Gamma(n-\gamma-2)}\int_x^b (t-x)^{n-\gamma-3} f^{(n-2)}(t)\,dt\,.
\end{aligned}$$

Case 2b.1. Let $[\nu] = [\gamma] + 2$. Then $m = n-2$ and, with boundary conditions $f^{(n-1)}(b) = f^{(n-2)}(b) = 0$, the proof of (ii) is complete.

Hence, by induction, in the last step, that is $m = n-(n-m)$, we have boundary conditions $f^{(n-1)}(b) = \cdots = f^{(m)}(b) = 0$ and proved equality (2.7). □

3. Opial-type inequalities

Here we present Opial-type inequalities involving the Caputo fractional derivatives and they improve those from [2, Section 16.3.1]. The following theorems are based on [3] and [4] where this was done for the Riemann-Liouville fractional derivatives.

THEOREM 3.1. *Suppose that the assumptions of Theorem 2.1 hold. Suppose also* $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\nu > \gamma + \frac{1}{q}$ *and* $D^{\nu}_{a+} f \in L^q[a,b]$. *Then*

$$\int_a^x \left|D^{\gamma}_{a+} f(w)\right| \left|D^{\nu}_{a+} f(w)\right| dw \leqslant C_1(x) \left(\int_a^x \left|D^{\nu}_{a+} f(w)\right|^q dw \right)^{\frac{2}{q}}, \text{ a.e. } x \in [a,b], \quad (3.1)$$

where

$$C_1(x) = \frac{(x-a)^{r+\frac{2}{p}}}{2^{\frac{1}{q}} \Gamma(r+1)(rp+1)^{\frac{1}{p}} (rp+2)^{\frac{1}{p}}}, \quad r = \nu - \gamma - 1.$$

Inequality (3.1) is sharp for $\nu = \gamma + 1$ *where equality is attained for a function* f *such that* $D^{\nu}_{a+} f(t) = 1$ *for every* $t \in [a,b]$ *(for example* $f(t) = \frac{(t-a)^{\nu}}{\Gamma(\nu+1)}$*).*

We give a proof of the following theorem which is the right-sided version of Theorem 3.1. A proof of Theorem 3.1 is analogous.

THEOREM 3.2. *Suppose that the assumptions of Theorem 2.2 hold. Suppose also* $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\nu > \gamma + \frac{1}{q}$ *and* $D^{\nu}_{b-} f \in L^q[a,b]$. *Then*

$$\int_x^b \left|D^{\gamma}_{b-} f(w)\right| \left|D^{\nu}_{b-} f(w)\right| dw \leqslant C_2(x) \left(\int_x^b \left|D^{\nu}_{b-} f(w)\right|^q dw \right)^{\frac{2}{q}}, \text{ a.e. } x \in [a,b], \quad (3.2)$$

where

$$C_2(x) = \frac{(b-x)^{r+\frac{2}{p}}}{2^{\frac{1}{q}} \Gamma(r+1)(rp+1)^{\frac{1}{p}} (rp+2)^{\frac{1}{p}}}, \quad r = \nu - \gamma - 1.$$

Inequality (3.2) is sharp for $\nu = \gamma + 1$ *where equality is attained for a function* f *such that* $D^{\nu}_{b-} f(t) = 1$ *for every* $t \in [a,b]$ *(for example* $f(t) = \frac{(b-t)^{\nu}}{\Gamma(\nu+1)}$*).*

Proof. Using Theorem 2.2, the triangle inequality and Hölder inequality we have:

$$\begin{aligned}\left|D_{b-}^{\gamma} f(w)\right| &\leqslant \frac{1}{\Gamma(\nu-\gamma)} \int_w^b (t-w)^{\nu-\gamma-1} \left|D_{b-}^{\nu} f(t)\right| dt \\ &\leqslant \frac{1}{\Gamma(\nu-\gamma)} \left(\int_w^b (t-w)^{pr} dt\right)^{\frac{1}{p}} \left(\int_w^b \left|D_{b-}^{\nu} f(t)\right|^q dt\right)^{\frac{1}{q}} \\ &= \frac{1}{\Gamma(\nu-\gamma)} \frac{(b-w)^{r+\frac{1}{p}}}{(pr+1)^{\frac{1}{p}}} \left(\int_w^b \left|D_{b-}^{\nu} f(t)\right|^q dt\right)^{\frac{1}{q}}. \end{aligned} \tag{3.3}$$

Set $z(w) = \int_w^b \left|D_{b-}^{\nu} f(t)\right|^q dt$. Obviously $z'(w) = -\left|D_{b-}^{\nu} f(w)\right|^q$. Using (3.3) it follows:

$$\left|D_{b-}^{\gamma} f(w)\right| \left|D_{b-}^{\nu} f(w)\right| \leqslant \frac{1}{\Gamma(\nu-\gamma)} \frac{(b-w)^{r+\frac{1}{p}}}{(pr+1)^{\frac{1}{p}}} z^{\frac{1}{q}}(w) \left(-z'(w)\right)^{\frac{1}{q}}. \tag{3.4}$$

Using again the Hölder inequality and simple integration we have

$$\begin{aligned}&\int_x^b (b-w)^{r+\frac{1}{p}} z^{\frac{1}{q}}(w) \left(-z'(w)\right)^{\frac{1}{q}} dw \\ &\leqslant \left(\int_x^b (b-w)^{pr+1} dw\right)^{\frac{1}{p}} \left(\int_x^b z(w) \left(-z'(w)\right) dw\right)^{\frac{1}{q}} \\ &= \frac{(b-x)^{r+\frac{2}{p}}}{(pr+2)^{\frac{1}{p}}} \left(\frac{1}{2} \int_x^b \left|D_{b-}^{\nu} f(t)\right|^q dt\right)^{\frac{2}{q}}. \end{aligned} \tag{3.5}$$

Using (3.4) and (3.5) we obviously obtain (3.2).

Using equality condition in Hölder's inequality it follows that equality holds in (3.3) if and only if $\left|D_{b-}^{\nu} f(t)\right|^q = C(t-w)^{pr}$ and $D_{b-}^{\nu}(t) \geqslant 0$ for some $C \geqslant 0$ $D_{b-}^{\nu}(t) \geqslant 0$ for every $t \in [x,b]$. This implies $r = 0$ or $\nu = \gamma+1$. Direct verification shows that for a function f for which $D_{b-}^{\nu} f = 1$ equality holds also in (3.5). □

COROLLARY 3.3. *Suppose that the assumptions of Theorem 3.1 and Theorem 3.2 hold. Suppose also $1 < q \leqslant 2$. Then*

$$\begin{aligned}&\int_a^{\frac{a+b}{2}} \left|D_{a+}^{\gamma} f(w)\right| \left|D_{a+}^{\nu} f(w)\right| dw + \int_{\frac{a+b}{2}}^b \left|D_{b-}^{\gamma} f(w)\right| \left|D_{b-}^{\nu} f(w)\right| dw \\ &\leqslant C_{1,2}\left(\frac{a+b}{2}\right) \left(\int_a^{\frac{a+b}{2}} \left|D_{a+}^{\nu} f(w)\right|^q dw + \int_{\frac{a+b}{2}}^b \left|D_{b-}^{\nu} f(w)\right|^q dw\right)^{\frac{2}{q}}, \end{aligned} \tag{3.6}$$

where $C_{1,2}\left(\frac{a+b}{2}\right) = C_1\left(\frac{a+b}{2}\right) = C_2\left(\frac{a+b}{2}\right)$. Inequality (3.6) is sharp for $q = 2$ and $\nu = \gamma+1$.

Proof. Inequality (3.6) is a simple consequence of inequalities (3.1), (3.2) and elementary inequality $(x+y)^{\alpha} \geqslant x^{\alpha} + y^{\alpha}$ which holds for $\alpha \geqslant 1$. □

REMARK 3.4. For $\nu = 1$, $\gamma = 0$ and $q = 2$ inequality (3.6) implies classical Opial's inequality (see for example [1]). Boundary conditions $f(a) = f(b) = 0$ follow from (d) case in Theorems 2.1 and 2.2.

Next theorem is motivated by Pang-Agarwal's extension (see [10, Theorem 1.1]) of an inequality proved by Fink for ordinary derivatives (see [6]).

THEOREM 3.5. *Let $\nu > \gamma_2 \geqslant \gamma_1 + 1 \geqslant 1$, $n = [\nu] + 1$, $m_1 = [\gamma_1] + 1$, $m_2 = [\gamma_2] + 1$, $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and $f \in AC^k[a,b]$, $k = n$ if $\nu \notin \mathbb{N}_0$ and $k = n-1$ if $\nu \in \mathbb{N}_0$. Let $D^{\nu}_{a+}f \in L^q[a,b]$. Suppose that one of the following conditions holds:*

(a) ν, $\gamma_1 \notin \mathbb{N}_0$ and $f^{(i)}(a) = 0$ for $i = m_1, \ldots, n-1$.

(b) $\nu \in \mathbb{N}$, $\gamma_1 \notin \mathbb{N}_0$ and $f^{(i)}(a) = 0$ for $i = m_1, \ldots, n-2$.

(c) $\nu \notin \mathbb{N}$, $\gamma_1 \in \mathbb{N}_0$ and $f^{(i)}(a) = 0$ for $i = m_1 - 1, \ldots, n-1$.

(d) $\nu \in \mathbb{N}$, $\gamma_1 \in \mathbb{N}_0$ and $f^{(i)}(a) = 0$ for $i = m_1 - 1, \ldots, n-2$.

Then

$$\int_a^x \left|D^{\gamma_1}_{a+}f(\tau)\right| \left|D^{\gamma_2}_{a+}f(\tau)\right| d\tau \leqslant C_3(x) \left(\int_a^x \left|D^{\nu}_{a+}f(\tau)\right|^q d\tau\right)^{\frac{2}{q}}, \tag{3.7}$$

where

$$C_3(x) = \frac{(x-a)^{r_1+r_2+1+\frac{2}{p}}}{2^{\frac{1}{q}}\Gamma(r_1+1)\Gamma(r_2+2)\left[p(r_2+1)+1\right]^{\frac{1}{p}}\left[p(r_1+r_2+1)+2\right]^{\frac{1}{p}}},$$

where $r_i = \nu - \gamma_i - 1$, $i = 1,2$.

Inequality (3.7) is sharp for $\gamma_2 = \gamma_1 + 1$.

Proof. First we have $r_1 - r_2 - 1 \geqslant 0$ since $\gamma_2 \geqslant \gamma_1 + 1$. Let $a \leqslant t \leqslant s \leqslant x$. The following estimation is proved in [6]):

$$\int_a^x \left[(\tau-t)_+^{r_1}(\tau-s)_+^{r_2} + (\tau-s)_+^{r_1}(\tau-t)_+^{r_2}\right] d\tau \leqslant \frac{1}{(\nu-\gamma_2)}(x-t)^{r_1}(x-s)^{r_2+1}, \tag{3.8}$$

where $(\tau - t)_+^{\alpha}$ is define by

$$(\tau-t)_+^{\alpha} = \begin{cases} (\tau-t)^{\alpha}, & \text{if } a \leqslant t < \tau \leqslant x, \\ 0 & , \text{if } a \leqslant \tau \leqslant t \leqslant x. \end{cases}$$

In the following calculation we abbreviate

$$c_1 := \left(\Gamma(\nu-\gamma_2)\Gamma(\nu-\gamma_1)\right)^{-1}, \quad c_2 := \left(\Gamma(\nu-\gamma_2+1)\Gamma(\nu-\gamma_1)\right)^{-1},$$

$$c_3 := (\nu-\gamma_2)p + 1, \quad \varepsilon := 2\nu - \gamma_1 - \gamma_2 - 1 + \frac{1}{p}.$$

Let $a \leqslant \tau \leqslant x,\ j=1,2$. Using Theorem 2.1 (notice that conditions (a)-(d) ensure that analogous conditions hold also for γ_2) we have:

$$(D_{a+}^{\gamma_j} f)(\tau) = \frac{1}{\Gamma(\nu-\gamma_j)} \int_a^x (\tau - t)_+^{r_i} (D_{a+}^{\nu} f)(t)\, dt\,.$$

Using this representation, the auxiliary inequality (3.8), and Hölder's inequality, we obtain

$$\begin{aligned}
&\int_a^x |(D_{a+}^{\gamma_1} f)(\tau)||(D_{a+}^{\gamma_2} f)(\tau)|\, d\tau \\
&\leqslant c_1 \int_a^x \Big(\int_a^x |(D_{a+}^{\nu} f)(t)|(\tau-t)_+^{r_1}\, dt \Big) \Big(\int_a^x |(D_{a+}^{\nu} f)(s)|(\tau-s)_+^{r_2}\, ds \Big)\, d\tau \qquad (3.9)\\
&= c_1 \int_a^x |(D_{a+}^{\nu} f)(t)| \Big(\int_t^x |(D_{a+}^{\nu} f)(s)| \\
&\qquad \cdot \Big(\int_a^x [(\tau-t)_+^{r_1}(\tau-s)_+^{r_2} + (\tau-s)_+^{r_1}(\tau-t)_+^{r_2}]\, d\tau \Big)\, ds \Big)\, dt \\
&\leqslant c_2 \int_a^x |(D_{a+}^{\nu} f)(t)| \Big(\int_t^x |(D_{a+}^{\nu} f)(s)|(x-t)^{r_1}(x-s)^{r_2+1}\, ds \Big)\, dt \qquad (3.10)\\
&\leqslant c_2 \int_a^x |(D_{a+}^{\nu} f)(t)|(x-t)^{r_1} \Big(\int_t^x |(D_{a+}^{\nu} f)(s)|^q\, ds \Big)^{\frac{1}{q}} \Big(\int_t^x (x-s)^{p(r_2+1)}\, ds \Big)^{\frac{1}{p}}\, dt \\
&\qquad\qquad (3.11)\\
&= c_2 c_3^{-1/p} \int_a^x |(D_{a+}^{\nu} f)(t)|(x-t)^{\varepsilon} \Big(\int_t^x |(D_{a+}^{\nu} f)(s)|^q\, ds \Big)^{\frac{1}{q}}\, dt \\
&\leqslant c_2 c_3^{-1/p} \Big(\int_a^x |(D_{a+}^{\nu} f)(t)|^q \Big(\int_t^x |(D_{a+}^{\nu} f)(s)|^q\, ds \Big)\, dt \Big)^{\frac{1}{q}} \Big(\int_a^x (x-t)^{\varepsilon p}\, dt \Big)^{\frac{1}{p}} \qquad (3.12)\\
&= c_2 c_3^{-1/p} (\varepsilon p+1)^{-1/p} (x-a)^{(\varepsilon p+1)/p} \Big(\frac{1}{2} \Big(\int_a^x |(D_{a+}^{\nu} f)(t)|^q\, dt \Big)^2 \Big)^{\frac{1}{q}}.
\end{aligned}$$

It is obvious that in the case $\gamma_2 = \gamma_1 + 1$ equality holds in (3.8). Using equality condition for Hölder's inequality equality holds in (3.11) for a function f for which $\big(D_{a+}^{\nu} f(s)\big)^q = (x-s)^{p(r_2+1)}$. A straightforward calculation shows that for this function equality also holds (3.12). Equality in (3.9) in this case is obvious. □

We remark here that under suitable assumptions on a function f (for example when $D_{a+}^{\nu} f$ is continuous on $[a,b]$) we have

$$\lim_{\gamma_2 \to \nu - 0} D_{a+}^{\gamma_2} f(\tau) = \lim_{\gamma_2 \to \nu - 0} \frac{1}{\Gamma(\nu-\gamma_2)} \int_a^{\tau} (\tau-t)^{\nu-\gamma_2-1} D_{a+}^{\nu} f(t) dt,$$

(see [5, Section 3]), so we can formally compare estimations obtained in Theorem 3.1 and Theorem 3.5. This is interesting because inequalities (3.1) and (3.7) are obtained in a different way. Setting $\nu = \gamma_2$ and $\gamma_1 = \gamma$ in $C_3(x)$, we have

$$C_3(x) = \frac{(x-a)^{r+\frac{2}{p}}}{2^{\frac{1}{q}} \Gamma(r+1)[pr+2]^{\frac{1}{p}}},\ r = \nu - \gamma - 1,$$

so obviously $C_1(x) < C_3(x)$ for $r > 0$, so estimation in Theorem 3.1 is better than estimation in Theorem 3.5.

The following right-sided version of Theorem 3.5 can be proven analogously.

THEOREM 3.6. *Let* $\nu > \gamma_2 \geqslant \gamma_1 + 1 \geqslant 1$, $n = [\nu] + 1$, $m_1 = [\gamma_1] + 1$, $m_2 = [\gamma_2] + 1$, $p, q > 1$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$ *and* $f \in AC^k[a,b]$, $k = n$ *if* $\nu \notin \mathbb{N}_0$ *and* $k = n-1$ *if* $\nu \in \mathbb{N}_0$. *Let* $D^{\nu}_{b-} f \in L^q[a,b]$. *Suppose that one of the following conditions holds:*

(a) ν, $\gamma_1 \notin \mathbb{N}_0$ *and* $f^{(i)}(b) = 0$ *for* $i = m_1, \ldots, n-1$.

(b) $\nu \in \mathbb{N}$, $\gamma_1 \notin \mathbb{N}_0$ *and* $f^{(i)}(b) = 0$ *for* $i = m_1, \ldots, n-2$.

(c) $\nu \notin \mathbb{N}$, $\gamma_1 \in \mathbb{N}_0$ *and* $f^{(i)}(b) = 0$ *for* $i = m_1 - 1, \ldots, n-1$.

(d) $\nu \in \mathbb{N}$, $\gamma_1 \in \mathbb{N}_0$ *and* $f^{(i)}(b) = 0$ *for* $i = m_1 - 1, \ldots, n-2$.

Then

$$\int_x^b \left|D^{\gamma_1}_{b-} f(\tau)\right| \left|D^{\gamma_2}_{b-} f(\tau)\right| d\tau \leqslant C_4(x) \left(\int_x^b \left|D^{\nu}_{b-} f(\tau)\right|^q d\tau \right)^{\frac{2}{q}}, \tag{3.13}$$

where

$$C_4(x) = \frac{(b-x)^{r_1 + r_2 + 1 + \frac{2}{p}}}{2^{\frac{1}{q}} \Gamma(r_1+1)\Gamma(r_2+2) \left[p(r_2+1)+1\right]^{\frac{1}{p}} \left[p(r_1+r_2+1)+2\right]^{\frac{1}{p}}},$$

where $r_i = \nu - \gamma_i - 1$, $i = 1, 2$.

Inequality (3.13) is sharp for $\gamma_2 = \gamma_1 + 1$.

A generalization of the classical Opial inequality in this setting also holds. The proof is analogous to the proof of Corollary 3.3.

COROLLARY 3.7. *Suppose that the assumptions of Theorem 3.5 and Theorem 3.6 hold. Suppose also* $1 < q \leqslant 2$. *Then*

$$\int_a^{\frac{a+b}{2}} \left|D^{\gamma_1}_{a+} f(w)\right| \left|D^{\gamma_2}_{a+} f(w)\right| dw + \int_{\frac{a+b}{2}}^b \left|D^{\gamma_1}_{b-} f(w)\right| \left|D^{\gamma_2}_{b-} f(w)\right| dw$$

$$\leqslant C_{3,4}\left(\frac{a+b}{2}\right) \left(\int_a^{\frac{a+b}{2}} \left|D^{\nu}_{a+} f(w)\right|^q dw + \int_{\frac{a+b}{2}}^b \left|D^{\nu}_{b-} f(w)\right|^q dw \right)^{\frac{2}{q}}, \tag{3.14}$$

where $C_{3,4}\left(\frac{a+b}{2}\right) = C_3\left(\frac{a+b}{2}\right) = C_4\left(\frac{a+b}{2}\right)$. *Inequality (3.14) is sharp for* $q = 2$ *and* $\gamma_2 = \gamma_1 + 1$.

To complete studying various types of Opial's inequalities with two fractional derivatives on the left-hand side of an inequality, we present the following theorem which is improvement of [2, Theorem 16.21].

THEOREM 3.8. *Let* $\nu > \gamma+1$, $\gamma \geqslant 0$, $p,q > 1$, $\frac{1}{p}+\frac{1}{q}=1$. *Let* $D^{\nu}_{a+}f \in L^q[a,b]$. *Suppose that the assumptions of Theorem 2.1 hold. Then*

$$\int_a^x \left|D^{\gamma}_{a+}f(w)\right| \left|D^{\gamma+1}_{a+}f(w)\right| dw \leqslant C_5(x)\left(\int_a^x \left|D^{\nu}_{a+}f(w)\right|^q dw\right)^{\frac{2}{q}}, \text{ a.e. } x\in[a,b], \tag{3.15}$$

where

$$C_5(x) = \frac{(x-a)^{2r+\frac{2}{p}}}{2\Gamma^2(r+1)(rp+1)^{\frac{2}{p}}},\ r = \nu-\gamma-1.$$

Inequality (3.15) is sharp and equality in (3.15) is attained for a function f *for which* $D^{\nu}_{a+}f(t) = (x-t)^{\frac{rp}{q}}$.

Proof. Using Theorem 2.1 we have

$$\left|D^{\gamma}_{a+}f(w)\right| \leqslant \frac{1}{\Gamma(r+1)}\int_a^w (w-t)^r \left|D^{\nu}_{a+}f(t)\right| dt := U(w).$$

Using again Theorem 2.1 and since $r>0$ we have

$$\left|D^{\gamma+1}_{a+}f(w)\right| \leqslant \frac{1}{\Gamma(r)}\int_a^w (w-t)^{r-1} \left|D^{\nu}_{a+}f(t)\right| dt := U'(w).$$

By Hölder's inequality, we have

$$\begin{aligned}
&\int_a^x |(D^{\gamma}_{a+}f)(\omega)||(D^{\gamma+1}_{a+}f)(\omega)|\,d\omega \\
&\leqslant \int_a^x U(\omega)U'(\omega)\,d\omega = \frac{1}{2}U^2(x) \\
&= \frac{1}{2\left(\Gamma(r+1)\right)^2}\left(\int_a^x (x-t)^r |(D^{\nu}_{a+}f)(t)|\,dt\right)^2 \\
&\leqslant \frac{1}{2\left(\Gamma(r+1)\right)^2}\left(\int_a^x (x-t)^{rp}\,dt\right)^{\frac{2}{p}}\left(\int_a^x |(D^{\nu}_{a+}f)(t)|^q\,dt\right)^{\frac{2}{q}} \\
&= \frac{1}{2\left(\Gamma(r+1)\right)^2}\frac{(x-a)^{\frac{2rp+2}{p}}}{(rp+1)^{\frac{2}{p}}}\left(\int_a^x |(D^{\nu}_{a+}f)(t)|^q\,dt\right)^{\frac{2}{q}}.
\end{aligned}$$

Arguing the equality case is as in previous theorems. □

Although Theorem 3.8 and Theorem 3.5 for $\gamma_2 = \gamma_1+1$ are proved by different methods, they give the same (the best possible) estimation.

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Maja Andrić
Faculty of Civil Engineering and Architecture
University of Split
Split, Croatia
e-mail: maja.andric@gradst.hr

Josip Pečarić
Faculty of Textile Technology
University of Zagreb
Zagreb, Croatia
e-mail: pecaric@element.hr

Ivan Perić
Faculty of Food Technology and Biotechnology
University of Zagreb
Zagreb, Croatia
e-mail: iperic@pbf.hr