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SOME MONOTONICITY PROPERTIES OF GENERALIZED ELLIPTIC INTEGRALS WITH APPLICATIONS

MIAO-KUN WANG, YU-MING CHU* AND SONG-LIANG QIU

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Abstract. In this paper, we present some monotonicity theorems involving the generalized elliptic integrals $\mathscr{K}_a(r)$ and $\mathscr{E}_a(r)$, and find an asymptotic property of $\mathscr{K}_a(r)$ when $r \to 1$. As applications, some well known results are improved.

1. Introduction

For real numbers a, b and c with $c \neq 0, -1, -2, \cdots$, the Gaussian hypergeometric function is defined by

$$F(a,b;c;x) = {}_2F_1(a,b;c;x) = \sum_{n=0}^{\infty} \frac{(a,n)(b,n)}{(c,n)} \frac{x^n}{n!}, \quad |x| < 1. \tag{1.1}$$

Here $(a,0) = 1$ for $a \neq 0$ and (a,n) denotes the shifted factorial function

$$(a,n) = a(a+1)(a+2)(a+3)\cdots(a+n-1)$$

for $n = 1, 2, \cdots$. It is well known that $F(a,b;c;x)$ has many important applications, and many classes of special functions in mathematical physics are particular or limiting cases of this function. For these, and for properties of $F(a,b;c;x)$ see [2, 7, 14–16, 19, 20, 24].

For $r \in (0,1)$, $a \in (0,1)$ and $r' = \sqrt{1-r^2}$, the generalized elliptic integrals are defined by

$$\begin{cases} \mathscr{K}_a = \mathscr{K}_a(r) = \pi F(a, 1-a; 1; r^2)/2, \\ \mathscr{K}_a' = \mathscr{K}_a'(r) = \mathscr{K}_a(r'), \\ \mathscr{K}_a(0) = \pi/2, \ \mathscr{K}_a(1) = \infty \end{cases} \tag{1.2}$$

and

$$\begin{cases} \mathscr{E}_a = \mathscr{E}_a(r) = \pi F(a-1, 1-a; 1; r^2)/2, \\ \mathscr{E}_a' = \mathscr{E}_a'(r) = \mathscr{E}_a(r'), \\ \mathscr{E}_a(0) = \pi/2, \ \mathscr{E}_a(1) = [\sin(\pi a)]/[2(1-a)], \end{cases} \tag{1.3}$$

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* Corresponding author.

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which arise from the Schwarz-Christoffel transformation of the upper half-plane onto a parallelogram. In particular, when $a = 1/2$, the functions $\mathscr{K}_a(r)$ and $\mathscr{E}_a(r)$ reduce to $\mathscr{K}(r)$ and $\mathscr{E}(r)$, respectively, which are the complete elliptic integrals of the first and second kind [1, 4, 18]. By symmetry of (1.2), we assume that $a \in (0, 1/2]$ in the sequel.

Recently, the Gaussian hypergeometric function and generalized elliptic integrals have been the subject of intensive research [5, 6, 8, 10, 11, 13, 17, 21–23, 25]. In particular, the quotient of hypergeometric functions plays an important role in deriving important inequalities and identities, for example, S. Simić and M. Vuorinen [22] established Landen inequalities for zero-balanced hypergeometric function by showing the monotonicity properties of the quotient $F(a,b;a+b;x)/F(1/2,1/2;1;x)$ with $a,b>0$. In [21], the authors considered the quotient $(F(a,b;c,x)+F(a,b;c;y))/F(a,b;c;z)$ and the difference $F(a,b;c,x)+F(a,b;c;y)-F(a,b;c;z)$ for $a,b,c>0$ and $0<x<y<1$ with $z = x+y-xy$, and give best possible bounds for both expressions under various hypotheses about the parameter triple $(a,b;c)$.

In [9], Carlson and Gustafson proved that

$$\frac{\mathscr{K}(r)}{\log(4/r')} < \frac{4}{3+r^2} \tag{1.4}$$

for all $r \in (0,1)$. Later, Anderson et al. [3] conjectured that

$$\frac{9}{8+r^2} < \frac{\mathscr{K}(r)}{\log(4/r')} \tag{1.5}$$

for all $r \in (0,1)$. Inequality (1.5) was proved by Kühnau [12].

The purpose of this paper is to establish the new monotonicity properties of the generalized elliptic integrals $\mathscr{K}_a(r)$ and $\mathscr{E}_a(r)$, present the generalization form of inequality (1.4), and improve inequality (1.5).

2. Lemmas

In order to establish our main results we need several formulas and Lemmas, which we present in this section.

Throughout this paper, we denote $r' = \sqrt{1-r^2}$ for $0<r<1$. Let

$$R(a,b) = -2\gamma - \Psi(a) - \Psi(b),\ R(a) = R(a, 1-a),$$

where γ is the Euler-Mascheroni constant defined by

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \log n\right) = 0.577215\ldots$$

and

$$\Psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}, \quad \Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt, \quad x>0.$$

The following formulas were presented in [18]:

$$\frac{d\mathscr{K}_a}{dr} = \frac{2(1-a)}{rr'^2}(\mathscr{E}_a - r'^2\mathscr{K}_a), \quad \frac{d\mathscr{E}_a}{dr} = \frac{2(a-1)}{r}(\mathscr{K}_a - \mathscr{E}_a),$$

$$\frac{d}{dr}(\mathscr{K}_a - \mathscr{E}_a) = \frac{2(1-a)r\mathscr{E}_a}{r'^2}, \quad \frac{d}{dr}(\mathscr{E}_a - r'^2\mathscr{K}_a) = 2ar\mathscr{K}_a.$$

LEMMA 2.1. [4, Theorem 1.25] *For* $-\infty < a < b < \infty$, *let* $f, g : [a,b] \to \mathbb{R}$ *be continuous on* $[a,b]$, *and be differentiable on* (a,b), *let* $g'(x) \neq 0$ *on* (a,b). *If* $f'(x)/g'(x)$ *is increasing (decreasing) on* (a,b), *then so are*

$$\frac{f(x)-f(a)}{g(x)-g(a)} \quad \text{and} \quad \frac{f(x)-f(b)}{g(x)-g(b)}.$$

If $f'(x)/g'(x)$ *is strictly monotone, then the monotonicity in the conclusion is also strict.*

The following Lemma 2.2 follows from [1, Lemma 5.2(1), (3)–(4), (6) and Lemma 5.4(1)] and [15, Theorem 2.3].

LEMMA 2.2. *Let* $a \in (0, 1/2]$, *then*

(1) $(\mathscr{E}_a - r'^2\mathscr{K}_a)/(r^2)$ *is strictly increasing from* $(0,1)$ *onto* $(\pi a/2, [\sin(\pi a)]/[2(1-a)])$;

(2) $(\mathscr{K}_a - \mathscr{E}_a)/(r^2\mathscr{K}_a)$ *is strictly increasing from* $(0,1)$ *onto* $(1-a, 1)$;

(3) $(\mathscr{E}_a - r'^2\mathscr{K}_a)/(r^2\mathscr{K}_a)$ *is strictly decreasing from* $(0,1)$ *onto* $(0,a)$;

(4) $r'^2(\mathscr{K}_a - \mathscr{E}_a)/(r^2\mathscr{E}_a)$ *is strictly decreasing from* $(0,1)$ *onto* $(0, 1-a)$;

(5) $r'^c\mathscr{K}_a$ *is strictly decreasing from* $(0,1)$ *onto* $(0, \pi/2)$ *if and only if* $c \geqslant 2a(1-a)$;

(6) $\mathscr{K}_a/\left[\log\left(e^{R(a)/2}/r'\right)\right]$ *is strictly decreasing from* $(0,1)$ *onto* $(\sin(\pi a), \pi/R(a))$.

3. Main Results

THEOREM 3.1. *Let* $a \in (0, 1/2]$, *then*

(1) $f(r) \equiv [(1-a)(\mathscr{E}_a - r'^2\mathscr{K}_a) - ar'^2(\mathscr{K}_a - \mathscr{E}_a)]/r^4$ *is strictly increasing from* $(0,1)$ *onto* $(\pi a(1-a)(2-a)/4, [\sin(\pi a)]/2)$;

(2) $g(r) \equiv \{[\sin(\pi a)]/[2(1-a)] - (\mathscr{E}_a - r'^2\mathscr{K}_a)\}/\{r'^2[R(a)/2 - \log r']\}$ *is strictly decreasing from* $(0,1)$ *onto* $(a\sin(\pi a), [\sin(\pi a)]/[R(a)(1-a)])$;

(3) $h(r) \equiv [(1-a)\mathscr{K}_a + a\mathscr{E}_a - \pi/2]/[\log(1/r') - r^2/2]$ *is strictly increasing from* $(0,1)$ *onto* $(\pi a(1-a)^2(2-a)/2, (1-a)\sin(\pi a))$;

(4) *$F(r) \equiv \mathscr{E}_a - r'^{2(1-a)}\mathscr{K}_a$ is strictly increasing from $(0,1)$ onto $(0,[\sin(\pi a)]/[2(1-a)])$;*

(5) *$G(r) \equiv 2\mathscr{E}_a - r'^2\mathscr{K}_a$ is strictly increasing in $(0,1)$ if and only if $a = 1/2$;*

(6) *$H(r) \equiv (1+r')\mathscr{K}_a$ is strictly increasing in $(0,1)$ if and only if $a = 1/2$.*

Proof. For part (1), clearly $f(1^-) = [\sin(\pi a)]/2$. Let $f_1(r) = (1-a)(\mathscr{E}_a - r'^2\mathscr{K}_a) - ar'^2(\mathscr{K}_a - \mathscr{E}_a)$ and $f_2(r) = r^4$, then $f_1(0) = f_2(0) = 0$, $f(r) = f_1(r)/f_2(r)$ and

$$\frac{f_1{}'(r)}{f_2{}'(r)} = \frac{a(2-a)}{2}\frac{\mathscr{K}_a - \mathscr{E}_a}{r^2}. \tag{3.1}$$

From equation (3.1), Lemma 2.1, Lemma 2.2(2) and l'Hôpital's rule we clearly see that $f(r)$ is strictly increasing in $(0,1)$ and $f(0^+) = \pi a(1-a)(2-a)/4$.

For part (2), clearly $g(0^+) = [\sin(\pi a)]/[(1-a)R(a)]$. Let $g_1(r) = [\sin(\pi a)]/[2(1-a)] - (\mathscr{E}_a - r'^2\mathscr{K}_a)$ and $g_2(r) = r'^2[R(a)/2 - \log r']$, then $g_1(1^-) = g_2(1^-) = 0$, $g(r) = g_1(r)/g_2(r)$ and

$$\frac{g_1{}'(r)}{g_2{}'(r)} = \frac{2a}{2\left[\log\left(e^{R(a)/2}/r'\right)\right]/\mathscr{K}_a - 1/\mathscr{K}_a}. \tag{3.2}$$

It follows from (3.2), Lemma 2.1 and Lemma 2.2(6) together with l'Hôpital's rule that $g(r)$ is strictly decreasing in $(0,1)$ and $g(1^-) = a\sin(\pi a)$.

For part (3), let $h_1(r) = (1-a)\mathscr{K}_a + a\mathscr{E}_a - \pi/2$ and $h_2(r) = \log(1/r') - r^2/2$, then $h_1(0) = h_2(0) = 0$, $h(r) = h_1(r)/h_2(r)$ and

$$\frac{h_1{}'(r)}{h_2{}'(r)} = 2(1-a)\frac{(1-a)(\mathscr{E}_a - r'^2\mathscr{K}_a) - ar'^2(\mathscr{K}_a - \mathscr{E}_a)}{r^4}. \tag{3.3}$$

From (3.3), part (1) and Lemma 2.1 we know that $h(r)$ is strictly increasing in $(0,1)$. Making use of l'Hôpital's rule we conclude that $h(0^+) = \pi a(1-a)^2(2-a)/2$ and $h(1^-) = (1-a)\sin(\pi a)$.

For part (4), clearly $F(0^+) = 0$ and $F(1^-) = [\sin(\pi a)]/[2(1-a)]$. By differentiation one has

$$\begin{aligned} F'(r) &= 2(1-a)\left[-\frac{\mathscr{K}_a - \mathscr{E}_a}{r} + rr'^{-2a}\mathscr{K}_a - r'^{-2a}\frac{\mathscr{E}_a - r'^2\mathscr{K}_a}{r}\right] \\ &= \frac{2(1-a)(1-r'^{2a})(\mathscr{K}_a - \mathscr{E}_a)}{rr'^{2a}} > 0. \end{aligned} \tag{3.4}$$

Therefore, the monotonicity of $F(r)$ follows from inequality (3.4).

For part (5), by differentiation we have

$$G'(r) = r\mathscr{K}_a G_1(r), \tag{3.5}$$

where

$$G_1(r) = 2a - 2(1-a)\frac{\mathcal{K}_a - \mathcal{E}_a}{r^2\mathcal{K}_a}. \tag{3.6}$$

From (3.6) and Lemma 2.2(2) we clearly see that $G_1(r)$ is strictly decreasing from $(0,1)$ onto $(4a-2, 2(-a^2+3a-1))$. Then part (5) follows from (3.5) and the range of $G_1(r)$.

For part (6), simple computation leads to

$$H'(r) = rr'^{-3/2}H_1(r), \tag{3.7}$$

where

$$H_1(r) = 2(1-a)\frac{\mathcal{E}_a - r'^2\mathcal{K}_a}{r^2}(r'^{1/2} + r'^{-1/2}) - r'^{1/2}\mathcal{K}_a. \tag{3.8}$$

It is well known that the function $r \mapsto r + 1/r$ is strictly decreasing in $(0,1)$. Then from (3.8), and Lemma 2.2(1) and (5) we conclude that $H_1(r)$ is strictly increasing from $(0,1)$ onto $(-\pi(1-2a)^2/2, +\infty)$. Therefore, part (6) follows from (3.7) and the range of $H_1(r)$. □

THEOREM 3.2. *Let $a \in (0, 1/2]$, then*

$$J(r) \equiv \frac{r'^2\mathcal{K}_a(r)}{\mathcal{K}_a(r)/\sin(\pi a) - \log(e^{R(a)/2}/r')}$$

is strictly decreasing from $(0,1)$ onto $([\sin(\pi a)]/[a(1-a)], \pi[\sin(\pi a)]/[\pi - R(a)\sin(\pi a)])$. Moreover, the inequality

$$\frac{\sin(\pi a)}{\alpha + (1-\alpha)r^2} < \frac{\mathcal{K}_a(r)}{\log(e^{R(a)/2}/r')} < \frac{\sin(\pi a)}{\beta + (1-\beta)r^2} \tag{3.9}$$

holds for all $r \in (0,1)$ with the best possible constants $\alpha = [\sin(\pi a)]R(a)/\pi$ and $\beta = a^2 - a + 1$.

Proof. Let $J_1(r) = r'^2\mathcal{K}_a(r)$ and $J_2(r) = [\mathcal{K}_a(r)/\sin(\pi a)] - \log(e^{R(a)/2}/r')$, then $J_1(1^-) = J_2(1^-) = 0$, $J(r) = J_1(r)/J_2(r)$ and

$$\frac{J_1'(r)}{J_2'(r)} = \frac{-2r\mathcal{K}_a + 2(1-a)(\mathcal{E}_a - r'^2\mathcal{K}_a)/r}{[2(1-a)(\mathcal{E}_a - r'^2\mathcal{K}_a)/(rr'^2\sin(\pi a))] - r/r'^2}$$
$$= \frac{r'^2[2\mathcal{K}_a - 2(1-a)(\mathcal{E}_a - r'^2\mathcal{K}_a)/r^2]}{1 - 2(1-a)(\mathcal{E}_a - r'^2\mathcal{K}_a)/[r^2\sin(\pi a)]}. \tag{3.10}$$

Denote $J_3(r) = r'^2[2\mathcal{K}_a - 2(1-a)(\mathcal{E}_a - r'^2\mathcal{K}_a)/r^2]$ and $J_4(r) = 1 - 2(1-a)(\mathcal{E}_a - r'^2\mathcal{K}_a)/[r^2\sin(\pi a)]$. Then $J_3(1^-) = J_4(1^-) = 0$, $J_1'(r)/J_2'(r) = J_3(r)/J_4(r)$ and

$$\frac{J_3'(r)}{J_4'(r)} = \frac{(1-a)(1+r^2)(\mathcal{K}_a - \mathcal{E}_a) + r^2(2a - 1 - a^2r'^2)\mathcal{K}_a}{[(1-a)/\sin(\pi a)][a(\mathcal{K}_a - \mathcal{E}_a) - (1-a)(\mathcal{E}_a - r'^2\mathcal{K}_a)]}. \tag{3.11}$$

Let $J_5(r)=(1-a)(1+r^2)(\mathscr{K}_a-\mathscr{E}_a)+r^2(2a-1-a^2r'^2)\mathscr{K}_a$ and $J_6(r)=[(1-a)/\sin(\pi a)][a(\mathscr{K}_a-\mathscr{E}_a)-(1-a)(\mathscr{E}_a-r'^2\mathscr{K}_a)]$, then $J_5(0)=J_6(0)=0$, $J_3'(r)/J_4'(r)=J_5(r)/J_6(r)$ and

$$\frac{J_5'(r)}{J_6'(r)}=\frac{(a^2-a+1)r'^2(\mathscr{K}_a-\mathscr{E}_a)\sin(\pi a)}{a(1-a)(\mathscr{E}_a-r'^2\mathscr{K}_a)}+\frac{r^2\sin(\pi a)}{a(1-a)^2(\mathscr{E}_a-r'^2\mathscr{K}_a)}[a^2(a+1)r'^2\mathscr{K}_a+(1-a)\mathscr{E}_a]. \tag{3.12}$$

It follows from (3.12), Lemma 2.2(1) and (4)-(5) that $J_5'(r)/J_6'(r)$ is strictly decreasing in $(0,1)$. Then equations (3.10) and (3.11) together with Lemma 2.1 lead to the conclusion that $J(r)$ is strictly decreasing in $(0,1)$. Moreover, Making use of l'Hôpital's rule we get

$$\lim_{r\to 0^+}J(r)=\frac{\pi\sin(\pi a)}{\pi-R(a)\sin(\pi a)} \tag{3.13}$$

and

$$\lim_{r\to 1^-}J(r)=\frac{\sin(\pi a)}{a(1-a)}. \tag{3.14}$$

Therefore, Theorem 3.2 follows from (3.13) and (3.14) together with the monotonicity of $J(r)$. □

Taking $a=1/2$ in Theorem 3.2, then we get inequality (1.4) and improve inequality (1.5) as follows.

COROLLARY 3.3. *Inequality*

$$\frac{1}{\alpha^*+(1-\alpha^*)r^2}<\frac{\mathscr{K}(r)}{\log(4/r')}<\frac{1}{\beta^*+(1-\beta^*)r^2} \tag{3.15}$$

holds for all $r\in(0,1)$ *with the best possible constants* $\alpha^*=4(\log 2)/\pi$ *and* $\beta^*=3/4$.

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Miao-Kun Wang
School of Mathematics and Computation Science
Hunan City University
Yiyang 413000, China
e-mail: wmk000@126.com

Yu-Ming Chu
School of Mathematics and Computation Science
Hunan City University
Yiyang 413000, China
e-mail: chuyuming@hutc.zj.cn

Song-Liang Qiu
Department of Mathematics, Zhejiang Sci-Tech University
Hangzhou 310018, China
e-mail: sl_qiu@zstu.edu.cn